THREE POINTS INEQUALITIES FOR RIEMANN-STIELTJES INTEGRAL WITH INTEGRANDS AND INTEGRATORS OF BOUNDED VARIATION

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Abstract. In this paper we provide some simple error bounds in approximating the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the use of three points formula

$$(1-\alpha)\{[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)\}+\alpha[u(b)-u(a)]f(x)$$

where $\alpha\in[0,1]$ and $x\in[a,b]$ under bounded variation assumptions for the functions u and f and such that the involved Riemann-Stieltjes integral exists. Applications for continuous functions of selfadjoint operators and unitary operators on Hilbert spaces are also given.

1. Introduction

One can approximate the *Stieltjes integral* $\int_a^b f(t)\,du(t)$ with the following simpler quantities:

$$\frac{1}{b-a}[u(b)-u(a)]\cdot\int_a^b f(t)\,dt \qquad ([25],\ [26]) \tag{1.1}$$

$$f(x)[u(b)-u(a)] \qquad ([15],\ [16]) \tag{1.2}$$

or with

$$[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a) \qquad ([24]), \tag{1.3}$$

where $x\in[a,b]$.

In order to provide *a priory* sharp bounds for the *approximation error*, consider the functionals:

$$D(f,u;a,b):=\int_a^b f(t)\,du(t)-\frac{1}{b-a}[u(b)-u(a)]\cdot\int_a^b f(t)\,dt,$$

$$\Theta(f,u;a,b,x):=\int_a^b f(t)\,du(t)-f(x)[u(b)-u(a)]$$

and

$$T(f,u;a,b,x):=\int_a^b f(t)\,du(t)-[u(b)-u(x)]f(b)-[u(x)-u(a)]f(a).$$

If the *integrand* f is *Riemann integrable* on $[a,b]$ and the *integrator* $u:[a,b]\to\mathbb{R}$ is $L-$*Lipschitzian*, i.e.,

$$|u(t)-u(s)|\le L|t-s| \qquad \text{for each } t,s\in[a,b], \tag{1.4}$$

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then the Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and, as pointed out in [25],

$$|D(f,u;a,b)| \leq L\int_a^b \left| f(t) - \int_a^b \frac{1}{b-a} f(s)\,ds \right| dt. \tag{1.5}$$

The inequality (1.5) is sharp in the sense that the multiplicative constant $C = 1$ in front of L cannot be replaced by a smaller quantity. Moreover, if there exists the constants m, $M \in \mathbb{R}$ such that $m \leq f(t) \leq M$ for a.e. $t \in [a,b]$, then [25]

$$|D(f,u;a,b)| \leq \frac{1}{2} L(M-m)(b-a). \tag{1.6}$$

The constant $\frac{1}{2}$ is best possible in (1.6).

A different approach in the case of integrands of bounded variation were considered by the same authors in 2001, [26], where they showed that

$$|D(f,u;a,b)| \leq \max_{t\in[a,b]} \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| \bigvee_a^b (u), \tag{1.7}$$

provided that f is continuous and u is of bounded variation. Here $\bigvee_a^b (u)$ denotes the total variation of u on $[a,b]$. The inequality (1.7) is sharp.

If we assume that f is K-Lipschitzian, then [26]

$$|D(f,u;a,b)| \leq \frac{1}{2} K(b-a) \bigvee_a^b (u), \tag{1.8}$$

with $\frac{1}{2}$ the best possible constant in (1.8).

For various bounds on the error functional $D(f,u;a,b)$ where f and u belong to different classes of function for which the Stieltjes integral exists, see [21], [20], [19], and [8] and the references therein.

For the functional $\Theta(f,u;a,b,x)$ we have the bound [15]:

$$\begin{aligned} &|\Theta(f,u;a,b,x)| \\ &\leq H\left[(x-a)^r \bigvee_a^x (f) + (b-x)^r \bigvee_x^b (f)\right] \\ &\leq H\times\begin{cases} \left[(x-a)^r + (b-x)^r\right]\left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right]; \\ \left[(x-a)^{qr} + (b-x)^{qr}\right]^{\frac{1}{q}}\left[\left(\bigvee_a^x (f)\right)^p + \left(\bigvee_x^b (f)\right)^p\right]^{\frac{1}{p}} \\ \qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b (f), \end{cases} \end{aligned} \tag{1.9}$$

provided f is of bounded variation and u is of *r-H-Hölder type*, i.e.,

$$|u(t) - u(s)| \leq H|t-s|^r \qquad \text{for each } t,\ s \in [a,b], \tag{1.10}$$

with given $H > 0$ and $r \in (0,1]$.

If f is of q-K-Hölder type and u is of bounded variation, then [16]

$$|\Theta(f,u;a,b,x)| \le K\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^q \bigvee_a^b(u), \tag{1.11}$$

for any $x\in[a,b]$.

If u is monotonic nondecreasing and f of q-K-Hölder type, then the following refinement of (1.11) also holds [8]:

$$\begin{aligned}(1.12)\quad |\Theta(f,u;a,b,x)| &\le K\Bigg[(b-x)^q u(b)-(x-a)^q u(a) \\ &\qquad + q\left\{\int_a^x \frac{u(t)\,dt}{(x-t)^{1-q}}-\int_x^b \frac{u(t)\,dt}{(t-x)^{1-q}}\right\}\Bigg] \\ &\le K\left[(b-x)^q[u(b)-u(x)]+(x-a)^q[u(x)-u(a)]\right] \\ &\le K\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^q[u(b)-u(a)],\end{aligned}$$

for any $x\in[a,b]$.

If f is monotonic nondecreasing and u is of r-H-Hölder type, then [8]:

$$\begin{aligned}(1.13)\quad |\Theta(f,u;a,b,x)| & \\ &\le H\Bigg[\left[(x-a)^r-(b-x)^r\right]f(x) \\ &\qquad + r\left\{\int_a^x \frac{f(t)\,dt}{(b-t)^{1-r}}-\int_x^b \frac{f(t)\,dt}{(t-r)^{1-r}}\right\}\Bigg] \\ &\le H\left\{(b-x)^r[f(b)-f(x)]+(x-a)^r[f(x)-f(a)]\right\} \\ &\le H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r[f(b)-f(a)],\end{aligned}$$

for any $x\in[a,b]$.

The error functional $T(f,u;a,b,x)$ satisfies similar bounds, see [24], [8], [3] and [2] and the details are omitted.

Motivated by the above results, in this paper we provide some simple ways to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the use of three points formula, namely we establish bounds for the error functional

$$\begin{aligned}T\Theta(f,u;a,b,x,\alpha) &:= (1-\alpha)\left\{[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)\right\} \\ &\qquad + \alpha[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)\end{aligned}$$

where $\alpha\in[0,1]$ and $x\in[a,b]$, under bounded variation assumptions for the functions u and f and such that the involved Riemann-Stieltjes integral exists. Applications for continuous functions of selfadjoint operators and unitary operators on Hilbert spaces are also given.

2. Inequalities for Integrands of Bounded Variation

Assume that $u,\ f:[a,b]\to\mathbb{C}$. If the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists, we write for simplicity, like in [1, p. 142] that $f\in\mathcal{R}_{\mathbb{C}}(u,[a,b])$, or $\mathcal{R}_{\mathbb{C}}(u)$ when the interval is implicitly known. If the functions $u,\ f$ are real valued, then we write $f\in\mathcal{R}(u,[a,b])$, or $\mathcal{R}(u)$.

We start with the following identity of interest.

Lemma 1. *Let $f,\ u:[a,b]\to\mathbb{C}$ and $x\in[a,b]$ such that $f\in\mathcal{R}_{\mathbb{C}}(u,[a,b])$. Then for any $\gamma,\ \mu\in\mathbb{C}$,*

$$[u(b)-\mu]f(b)+[\gamma-u(a)]f(a)+(\mu-\gamma)f(x)-\int_a^b f(t)\,du(t)=\int_a^x[u(t)-\gamma]\,df(t)+\int_x^b[u(t)-\mu]\,df(t). \tag{2.1}$$

In particular, for $\mu=\gamma$ we have

$$[u(b)-\gamma]f(b)+[\gamma-u(a)]f(a)-\int_a^b f(t)\,du(t)=\int_a^b[u(t)-\gamma]\,df(t). \tag{2.2}$$

Proof. Using integration by parts rule for the Riemann-Stieltjes integral, we have

$$\int_a^x[u(t)-\gamma]\,df(t)=[u(x)-\gamma]f(x)-[u(a)-\gamma]f(a)-\int_a^x f(t)\,du(t)$$

and

$$\int_x^b[u(t)-\mu]\,df(t)=[u(b)-\mu]f(b)-[u(x)-\mu]f(x)-\int_x^b f(t)\,du(t)$$

for any $x\in[a,b]$.

If we add these two equalities, we get

$$\begin{aligned}\int_a^x[u(t)-\gamma]\,df(t)&+\int_x^b[u(t)-\mu]\,df(t)\\&=[u(b)-\mu]f(b)+[\gamma-u(a)]f(a)+[\mu-u(x)]f(x)\\&\quad+[u(x)-\gamma]f(x)-\int_a^x f(t)\,du(t)-\int_x^b f(t)\,du(t)\\&=[u(b)-\mu]f(b)+[\gamma-u(a)]f(a)+(\mu-\gamma)f(x)-\int_a^b f(t)\,du(t)\end{aligned}$$

for any $x\in[a,b]$, which proves the desired equality (2.1). □

If in (2.1) we take $\gamma=\alpha u(a)+(1-\alpha)u(x)$ and $\mu=(1-\alpha)u(x)+\alpha u(b)$ where $x\in[a,b]$ and $\alpha\in[0,1]$ we get

$$\begin{aligned}&(1-\alpha)\{[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)\}\\&\qquad+\alpha[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)\\&\quad=\int_a^x[u(t)-\alpha u(a)-(1-\alpha)u(x)]\,df(t)\\&\qquad\qquad+\int_x^b[u(t)-(1-\alpha)u(x)-\alpha u(b)]\,df(t).\end{aligned} \tag{2.3}$$

In particular, for $x=\frac{a+b}{2}$, we get

$$(1-\alpha)\left\{\left[u(b)-u\left(\frac{a+b}{2}\right)\right]f(b)+\left[u\left(\frac{a+b}{2}\right)-u(a)\right]f(a)\right\}+\alpha[u(b)-u(a)]f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,du(t)=\int_a^{\frac{a+b}{2}}\left[u(t)-\alpha u(a)-(1-\alpha)u\left(\frac{a+b}{2}\right)\right]df(t)+\int_{\frac{a+b}{2}}^b\left[u(t)-(1-\alpha)u\left(\frac{a+b}{2}\right)-\alpha u(b)\right]df(t). \tag{2.4}$$

If in this equality, we take $\alpha=1$, we get the Montgomery type identity

$$[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)=\int_a^x[u(t)-u(a)]\,df(t)+\int_x^b[u(t)-u(b)]\,df(t), \tag{2.5}$$

for $x\in[a,b]$, which was obtained for the first time by the author in [15].

In particular, for $x=\frac{a+b}{2}$, we get

$$[u(b)-u(a)]f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,du(t)=\int_a^{\frac{a+b}{2}}[u(t)-u(a)]\,df(t)+\int_{\frac{a+b}{2}}^b[u(t)-u(b)]\,df(t). \tag{2.6}$$

If in (2.3) we take $\alpha=\frac{1}{2}$, we get

$$\frac{1}{2}\{[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)+[u(b)-u(a)]f(x)\}-\int_a^b f(t)\,du(t)=\int_a^x\left[u(t)-\frac{u(a)+u(x)}{2}\right]df(t)+\int_x^b\left[u(t)-\frac{u(x)+u(b)}{2}\right]df(t) \tag{2.7}$$

for $x\in[a,b]$ and in particular

$$\frac{1}{2}\left\{\left[u(b)-u\left(\frac{a+b}{2}\right)\right]f(b)+\left[u\left(\frac{a+b}{2}\right)-u(a)\right]f(a)+[u(b)-u(a)]f\left(\frac{a+b}{2}\right)\right\}-\int_a^b f(t)\,du(t)=\int_a^{\frac{a+b}{2}}\left[u(t)-\frac{u(a)+u\left(\frac{a+b}{2}\right)}{2}\right]df(t)+\int_{\frac{a+b}{2}}^b\left[u(t)-\frac{u\left(\frac{a+b}{2}\right)+u(b)}{2}\right]df(t). \tag{2.8}$$

If in (2.3) we take $\alpha = 0$, then we get

$$
\begin{aligned}
(2.9)\quad [u(b)-u(x)] f(b) + [u(x)-u(a)] f(a) - \int_a^b f(t)\,du(t) \\
= \int_a^b [u(t)-u(x)]\,df(t)
\end{aligned}
$$

for $x \in [a,b]$, and, in particular,

$$
\begin{aligned}
(2.10)\quad \left[u(b)-u\left(\frac{a+b}{2}\right)\right] f(b) + \left[u\left(\frac{a+b}{2}\right)-u(a)\right] f(a) - \int_a^b f(t)\,du(t) \\
= \int_a^b \left[u(t)-u\left(\frac{a+b}{2}\right)\right] df(t).
\end{aligned}
$$

We have the following result:

Theorem 1. *Let $f,\ u : [a,b] \to \mathbb{C}$ and $x \in [a,b]$ such that $f \in \mathcal{R}_{\mathbb{C}}(u,[a,b])$. If f and u are of bounded variation, then*

$$
\begin{aligned}
(2.11)\quad & |T\Theta(f,u;a,b,x,\alpha)| \\
& \le \alpha \left[\int_a^x \left(\bigvee_a^t (u)\right) d\left(\bigvee_a^t (f)\right) + \int_x^b \left(\bigvee_t^b (u)\right) d\left(\bigvee_a^t (f)\right)\right] \\
& + (1-\alpha)\left[\int_a^x \left(\bigvee_t^x (u)\right) d\left(\bigvee_a^t (f)\right) + \int_x^b \left(\bigvee_x^t (u)\right) d\left(\bigvee_a^t (f)\right)\right] \\
& \le \max\{\alpha,1-\alpha\}\left(\bigvee_a^x (u)\bigvee_a^x (f) + \bigvee_x^b (u)\bigvee_x^b (f)\right) \\
& \le \frac{1}{2}\max\{\alpha,1-\alpha\}\begin{cases} \left(\bigvee_a^b (u) + \left|\bigvee_a^x (u) - \bigvee_x^b (u)\right|\right)\bigvee_a^b (f) \\ \left(\bigvee_a^b (f) + \left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right)\bigvee_a^b (u) \end{cases} \\
& \le \max\{\alpha,1-\alpha\}\bigvee_a^b (u)\bigvee_a^b (f).
\end{aligned}
$$

In particular, we have for $x=\frac{a+b}{2}$ *that*

$$\left|T\Theta\left(f,u;a,b,\frac{a+b}{2},\alpha\right)\right| \tag{2.12}$$
$$\le \alpha\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_a^t(u)\right)d\left(\bigvee_a^t(f)\right)+\int_{\frac{a+b}{2}}^b\left(\bigvee_t^b(u)\right)d\left(\bigvee_a^t(f)\right)\right]$$
$$+(1-\alpha)\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_t^{\frac{a+b}{2}}(u)\right)d\left(\bigvee_a^t(f)\right)+\int_{\frac{a+b}{2}}^b\left(\bigvee_{\frac{a+b}{2}}^t(u)\right)d\left(\bigvee_a^t(f)\right)\right]$$
$$\le \max\{\alpha,1-\alpha\}\left(\bigvee_a^{\frac{a+b}{2}}(u)\bigvee_a^{\frac{a+b}{2}}(f)+\bigvee_{\frac{a+b}{2}}^b(u)\bigvee_{\frac{a+b}{2}}^b(f)\right)$$
$$\le \frac{1}{2}\max\{\alpha,1-\alpha\}\begin{cases}\left(\bigvee_a^b(u)+\left|\bigvee_a^{\frac{a+b}{2}}(u)-\bigvee_{\frac{a+b}{2}}^b(u)\right|\right)\bigvee_a^b(f)\\ \\ \left(\bigvee_a^b(f)+\left|\bigvee_a^{\frac{a+b}{2}}(f)-\bigvee_{\frac{a+b}{2}}^b(f)\right|\right)\bigvee_a^b(u)\end{cases}$$
$$\le \max\{\alpha,1-\alpha\}\bigvee_a^b(u)\bigvee_a^b(f).$$

Proof. It is well known that, if $p:[a,b]\to\mathbb{C}$ is continuous and $v:[a,b]\to\mathbb{C}$ of bounded variation, then

$$\left|\int_a^b p(t)\,dv(t)\right|\le\int_a^b|p(t)|\,d\left(\bigvee_a^t(v)\right)\le\max_{t\in[a,b]}|p(t)|\bigvee_a^b(v). \tag{2.13}$$

By utilising (2.3), we have for $x\in(a,b)$ and $\alpha\in[0,1]$ that

$$\Big|(1-\alpha)\{[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)\} \tag{2.14}$$
$$+\alpha[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)\Big|$$
$$\le\left|\int_a^x[u(t)-\alpha u(a)-(1-\alpha)u(x)]\,df(t)\right|$$
$$+\left|\int_x^b[u(t)-(1-\alpha)u(x)-\alpha u(b)]\,df(t)\right|$$

$$\le\int_a^x|[u(t)-\alpha u(a)-(1-\alpha)u(x)]|\,d\left(\bigvee_a^t(f)\right) \tag{2.15}$$
$$+\int_x^b|[u(t)-(1-\alpha)u(x)-\alpha u(b)]|\,d\left(\bigvee_x^t(f)\right)=:B(f,u,x;\alpha).$$

Since u is of bounded variation, hence

$$\begin{aligned} \left|\left[u(t)-\alpha u(a)-(1-\alpha)u(x)\right]\right| &= \left|\left[\alpha(u(t)-u(a))+(1-\alpha)(u(t)-u(x))\right]\right| \\ &\leq \alpha\left|u(t)-u(a)\right|+(1-\alpha)\left|u(x)-u(t)\right| \\ &\leq \alpha\bigvee_a^t(u)+(1-\alpha)\bigvee_t^x(u) \end{aligned}$$

and

$$\begin{aligned} \left|\left[u(t)-(1-\alpha)u(x)-\alpha u(b)\right]\right| &= \left|\left[(1-\alpha)(u(t)-u(x))+\alpha(u(t)-u(b))\right]\right| \\ &\leq (1-\alpha)\left|u(t)-u(x)\right|+\alpha\left|u(b)-u(t)\right| \\ &\leq (1-\alpha)\bigvee_x^t(u)+\alpha\bigvee_t^b(u) \end{aligned}$$

for $x,\, t\in[a,b]$ and $\alpha\in[0,1]$.

Therefore

$$\begin{aligned} &\int_a^x \left|\left[u(t)-\alpha u(a)-(1-\alpha)u(x)\right]\right| d\left(\bigvee_a^t(f)\right) \\ &\leq \int_a^x\left[\alpha\bigvee_a^t(u)+(1-\alpha)\bigvee_t^x(u)\right]d\left(\bigvee_a^t(f)\right) \\ &= \alpha\int_a^x\left(\bigvee_a^t(u)\right)d\left(\bigvee_a^t(f)\right)+(1-\alpha)\int_a^x\left(\bigvee_t^x(u)\right)d\left(\bigvee_a^t(f)\right) \end{aligned}$$

and

$$\begin{aligned} &\int_x^b \left|\left[u(t)-(1-\alpha)u(x)-\alpha u(b)\right]\right| d\left(\bigvee_x^t(f)\right) \\ &\leq \int_x^b\left[(1-\alpha)\bigvee_x^t(u)+\alpha\bigvee_t^b(u)\right]d\left(\bigvee_x^t(f)\right) \\ &= (1-\alpha)\int_x^b\left(\bigvee_x^t(u)\right)d\left(\bigvee_x^t(f)\right)+\alpha\int_x^b\left(\bigvee_t^b(u)\right)d\left(\bigvee_x^t(f)\right) \\ &= (1-\alpha)\int_x^b\left(\bigvee_x^t(u)\right)d\left(\bigvee_a^t(f)-\bigvee_a^x(f)\right) \\ &+\alpha\int_x^b\left(\bigvee_t^b(u)\right)d\left(\bigvee_a^t(f)-\bigvee_a^x(f)\right) \\ &= (1-\alpha)\int_x^b\left(\bigvee_x^t(u)\right)d\left(\bigvee_a^t(f)\right)+\alpha\int_x^b\left(\bigvee_t^b(u)\right)d\left(\bigvee_a^t(f)\right) \end{aligned}$$

for $x\in[a,b]$ and $\alpha\in[0,1]$.

If we add these two inequalities, we get

$$
\begin{aligned}
B(f,u,x;\alpha) &\leq \alpha \int_a^x \left(\bigvee_a^t (u) \right) d \left(\bigvee_a^t (f) \right) + (1-\alpha) \int_a^x \left(\bigvee_t^x (u) \right) d \left(\bigvee_a^t (f) \right) \\
&\quad + (1-\alpha) \int_x^b \left(\bigvee_x^t (u) \right) d \left(\bigvee_a^t (f) \right) + \alpha \int_x^b \left(\bigvee_t^b (u) \right) d \left(\bigvee_a^t (f) \right) \\
&= \alpha \left[\int_a^x \left(\bigvee_a^t (u) \right) d \left(\bigvee_a^t (f) \right) + \int_x^b \left(\bigvee_t^b (u) \right) d \left(\bigvee_a^t (f) \right) \right] \\
&\quad + (1-\alpha) \left[\int_a^x \left(\bigvee_t^x (u) \right) d \left(\bigvee_a^t (f) \right) + \int_x^b \left(\bigvee_x^t (u) \right) d \left(\bigvee_a^t (f) \right) \right] \\
&=: B(f,u,x;\alpha)
\end{aligned}
$$

for $x \in [a,b]$ and $\alpha \in [0,1]$.

This prove the first inequality in (2.11).

Observe that

$$
\begin{aligned}
B(f,u,x;\alpha) &\leq \max\{\alpha, 1-\alpha\} \\
&\quad \times \left\{ \int_a^x \left(\bigvee_a^t (u) \right) d \left(\bigvee_a^t (f) \right) + \int_x^b \left(\bigvee_t^b (u) \right) d \left(\bigvee_a^t (f) \right) \right. \\
&\quad \left. + \int_a^x \left(\bigvee_t^x (u) \right) d \left(\bigvee_a^t (f) \right) + \int_x^b \left(\bigvee_x^t (u) \right) d \left(\bigvee_a^t (f) \right) \right\} \\
&= \max\{\alpha, 1-\alpha\} \\
&\quad \times \left[\int_a^x \left(\bigvee_a^t (u) + \bigvee_t^x (u) \right) d \left(\bigvee_a^t (f) \right) + \int_x^b \left(\bigvee_x^t (u) + \bigvee_t^b (u) \right) d \left(\bigvee_a^t (f) \right) \right] \\
&= \max\{\alpha, 1-\alpha\} \left[\bigvee_a^x (u) \int_a^x d \left(\bigvee_a^t (f) \right) + \bigvee_x^b (u) \int_x^b d \left(\bigvee_a^t (f) \right) \right] \\
&= \max\{\alpha, 1-\alpha\} \left(\bigvee_a^x (u) \bigvee_a^x (f) + \bigvee_x^b (u) \bigvee_x^b (f) \right)
\end{aligned}
$$

for $x \in [a,b]$ and $\alpha \in [0,1]$.

This proves the second inequality in (2.11).

The last part is obvious. □

Corollary 1. *Assume that f and u are as in Theorem 1.*

(i) *If* $m\in[a,b]$ *is such that* $\bigvee_a^m(f)=\bigvee_m^b(f)$, *then*

$$|T\Theta(f,u;a,b,m,\alpha)| \le \alpha\left[\int_a^m\left(\bigvee_a^t(u)\right)d\left(\bigvee_a^t(f)\right)+\int_m^b\left(\bigvee_t^b(u)\right)d\left(\bigvee_a^t(f)\right)\right] + (1-\alpha)\left[\int_a^m\left(\bigvee_t^m(u)\right)d\left(\bigvee_a^t(f)\right)+\int_m^b\left(\bigvee_m^t(u)\right)d\left(\bigvee_a^t(f)\right)\right] \le \frac{1}{2}\max\{\alpha,1-\alpha\}\bigvee_a^b(u)\bigvee_a^b(f). \tag{2.16}$$

(ii) *If* $p\in[a,b]$ *is such that* $\bigvee_a^p(u)=\bigvee_p^b(u)$, *then*

$$|T\Theta(f,u;a,b,p,\alpha)| \le \alpha\left[\int_a^p\left(\bigvee_a^t(u)\right)d\left(\bigvee_a^t(f)\right)+\int_p^b\left(\bigvee_t^b(u)\right)d\left(\bigvee_a^t(f)\right)\right] + (1-\alpha)\left[\int_a^p\left(\bigvee_t^p(u)\right)d\left(\bigvee_a^t(f)\right)+\int_p^b\left(\bigvee_p^t(u)\right)d\left(\bigvee_a^t(f)\right)\right] \le \frac{1}{2}\max\{\alpha,1-\alpha\}\bigvee_a^b(u)\bigvee_a^b(f). \tag{2.17}$$

If we take $\alpha=1$ in Theorem 1 we get the Ostrowski type inequalities

$$|\Theta(f,u;a,b,x)| \le \int_a^x\left(\bigvee_a^t(u)\right)d\left(\bigvee_a^t(f)\right)+\int_x^b\left(\bigvee_t^b(u)\right)d\left(\bigvee_a^t(f)\right) \tag{2.18}$$

$$\le \bigvee_a^x(u)\bigvee_a^x(f)+\bigvee_x^b(u)\bigvee_x^b(f) \le \frac{1}{2}\begin{cases}\left(\bigvee_a^b(u)+\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right)\bigvee_a^b(f)\\[2mm] \left(\bigvee_a^b(f)+\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right)\bigvee_a^b(u)\end{cases} \le \bigvee_a^b(u)\bigvee_a^b(f),$$

for any $x\in[a,b]$.

In particular, for $x=\frac{a+b}{2}$, we get

$$
\begin{aligned}
(2.19)\quad & \left|\Theta\left(f,u;a,b,\frac{a+b}{2}\right)\right| \\
&\le \int_a^{\frac{a+b}{2}}\left(\bigvee_a^t(u)\right)d\left(\bigvee_a^t(f)\right)+\int_{\frac{a+b}{2}}^b\left(\bigvee_t^b(u)\right)d\left(\bigvee_a^t(f)\right) \\
&\le \bigvee_a^{\frac{a+b}{2}}(u)\bigvee_a^{\frac{a+b}{2}}(f)+\bigvee_{\frac{a+b}{2}}^b(u)\bigvee_{\frac{a+b}{2}}^b(f) \\
&\le \frac{1}{2}\begin{cases}\left(\bigvee_a^b(u)+\left|\bigvee_a^{\frac{a+b}{2}}(u)-\bigvee_{\frac{a+b}{2}}^b(u)\right|\right)\bigvee_a^b(f) \\[2ex] \left(\bigvee_a^b(f)+\left|\bigvee_a^{\frac{a+b}{2}}(f)-\bigvee_{\frac{a+b}{2}}^b(f)\right|\right)\bigvee_a^b(u)\end{cases} \le \bigvee_a^b(u)\bigvee_a^b(f).
\end{aligned}
$$

If $m\in[a,b]$ is such that $\bigvee_a^m(f)=\bigvee_m^b(f)$, then

$$
\begin{aligned}
(2.20)\quad & \left|\Theta(f,u;a,b,m)\right| \\
&\le \int_a^m\left(\bigvee_a^t(u)\right)d\left(\bigvee_a^t(f)\right)+\int_m^b\left(\bigvee_t^b(u)\right)d\left(\bigvee_a^t(f)\right)\le\frac{1}{2}\bigvee_a^b(u)\bigvee_a^b(f).
\end{aligned}
$$

If $p\in[a,b]$ is such that $\bigvee_a^p(u)=\bigvee_p^b(u)$, then

$$
\begin{aligned}
(2.21)\quad & \left|\Theta(f,u;a,b,p)\right| \\
&\le \int_a^p\left(\bigvee_a^t(u)\right)d\left(\bigvee_a^t(f)\right)+\int_p^b\left(\bigvee_t^b(u)\right)d\left(\bigvee_a^t(f)\right)\le\frac{1}{2}\bigvee_a^b(u)\bigvee_a^b(f).
\end{aligned}
$$

If we take $\alpha=0$ in Theorem 1 we get the trapezoid type inequalities

$$
\begin{aligned}
(2.22)\quad & \left|T(f,u;a,b,x)\right| \\
&\le \int_a^x\left(\bigvee_t^x(u)\right)d\left(\bigvee_a^t(f)\right)+\int_x^b\left(\bigvee_x^t(u)\right)d\left(\bigvee_a^t(f)\right) \\
&\le \bigvee_a^x(u)\bigvee_a^x(f)+\bigvee_x^b(u)\bigvee_x^b(f) \\
&\le \frac{1}{2}\begin{cases}\left(\bigvee_a^b(u)+\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right)\bigvee_a^b(f) \\[2ex] \left(\bigvee_a^b(f)+\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right)\bigvee_a^b(u)\end{cases} \le \bigvee_a^b(u)\bigvee_a^b(f),
\end{aligned}
$$

for any $x \in [a,b]$.

In particular, for $x = \frac{a+b}{2}$, we get

$$\left| T\left(f,u;a,b,\frac{a+b}{2}\right)\right| \leq \int_a^{\frac{a+b}{2}} \left(\bigvee_t^{\frac{a+b}{2}}(u)\right) d\left(\bigvee_a^t (f)\right) + \int_{\frac{a+b}{2}}^b \left(\bigvee_{\frac{a+b}{2}}^t (u)\right) d\left(\bigvee_a^t (f)\right) \tag{2.23}$$

$$\leq \bigvee_a^{\frac{a+b}{2}}(u) \bigvee_a^{\frac{a+b}{2}}(f) + \bigvee_{\frac{a+b}{2}}^b (u) \bigvee_{\frac{a+b}{2}}^b (f)$$

$$\leq \frac{1}{2}\begin{cases} \left(\bigvee_a^b (u) + \left|\bigvee_a^{\frac{a+b}{2}}(u) - \bigvee_{\frac{a+b}{2}}^b (u)\right|\right)\bigvee_a^b (f) \\ \\ \left(\bigvee_a^b (f) + \left|\bigvee_a^{\frac{a+b}{2}}(f) - \bigvee_{\frac{a+b}{2}}^b (f)\right|\right)\bigvee_a^b (u) \end{cases} \leq \bigvee_a^b (u)\bigvee_a^b (f).$$

If $m \in [a,b]$ is such that $\bigvee_a^m (f) = \bigvee_m^b (f)$, then

$$|T(f,u;a,b,m)| \leq \int_a^m \left(\bigvee_t^m (u)\right) d\left(\bigvee_a^t (f)\right) + \int_m^b \left(\bigvee_m^t (u)\right) d\left(\bigvee_a^t (f)\right) \leq \frac{1}{2}\bigvee_a^b (u)\bigvee_a^b (f). \tag{2.24}$$

If $p \in [a,b]$ is such that $\bigvee_a^p (u) = \bigvee_p^b (u)$, then

$$|T(f,u;a,b,p)| \leq \int_a^p \left(\bigvee_t^p (u)\right) d\left(\bigvee_a^t (f)\right) + \int_p^b \left(\bigvee_p^t (u)\right) d\left(\bigvee_a^t (f)\right) \leq \frac{1}{2}\bigvee_a^b (u)\bigvee_a^b (f). \tag{2.25}$$

If we take in Theorem 1 $\alpha = \frac{1}{2}$, and consider the error functional

$$T\Theta(f,u;a,b,x) := T\Theta\left(f,u;a,b,x,\frac{1}{2}\right)$$

then we get the three point inequalities

$$|T\Theta(f,u;a,b,x)| \leq \frac{1}{2}\left[\int_a^x \left(\bigvee_a^t (u)\right) d\left(\bigvee_a^t (f)\right) + \int_x^b \left(\bigvee_t^b (u)\right) d\left(\bigvee_a^t (f)\right)\right] + \frac{1}{2}\left[\int_a^x \left(\bigvee_t^x (u)\right) d\left(\bigvee_a^t (f)\right) + \int_x^b \left(\bigvee_x^t (u)\right) d\left(\bigvee_a^t (f)\right)\right] \tag{2.26}$$

$$\leq \frac{1}{2}\left(\bigvee_a^x(u)\bigvee_a^x(f)+\bigvee_x^b(u)\bigvee_x^b(f)\right)$$

$$\leq \frac{1}{4}\begin{cases}\left(\bigvee_a^b(u)+\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right)\bigvee_a^b(f)\\ \\ \left(\bigvee_a^b(f)+\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right)\bigvee_a^b(u)\end{cases}\leq \frac{1}{2}\bigvee_a^b(u)\bigvee_a^b(f).$$

In particular, for $x=\frac{a+b}{2}$ we get the mixture of trapezoid and mid-point inequalities

$$\left|T\Theta\left(f,u;a,b,\frac{a+b}{2}\right)\right| \tag{2.27}$$

$$\leq \frac{1}{2}\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_a^t(u)\right)d\left(\bigvee_a^t(f)\right)+\int_{\frac{a+b}{2}}^b\left(\bigvee_t^b(u)\right)d\left(\bigvee_a^t(f)\right)\right]$$

$$+\frac{1}{2}\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_t^{\frac{a+b}{2}}(u)\right)d\left(\bigvee_a^t(f)\right)+\int_{\frac{a+b}{2}}^b\left(\bigvee_{\frac{a+b}{2}}^t(u)\right)d\left(\bigvee_a^t(f)\right)\right]$$

$$\leq \frac{1}{2}\left(\bigvee_a^{\frac{a+b}{2}}(u)\bigvee_a^{\frac{a+b}{2}}(f)+\bigvee_{\frac{a+b}{2}}^b(u)\bigvee_{\frac{a+b}{2}}^b(f)\right)$$

$$\leq \frac{1}{4}\begin{cases}\left(\bigvee_a^b(u)+\left|\bigvee_a^{\frac{a+b}{2}}(u)-\bigvee_{\frac{a+b}{2}}^b(u)\right|\right)\bigvee_a^b(f)\\ \\ \left(\bigvee_a^b(f)+\left|\bigvee_a^{\frac{a+b}{2}}(f)-\bigvee_{\frac{a+b}{2}}^b(f)\right|\right)\bigvee_a^b(u)\end{cases}\leq \frac{1}{2}\bigvee_a^b(u)\bigvee_a^b(f).$$

If $m\in[a,b]$ is such that $\bigvee_a^m(f)=\bigvee_m^b(f)$, then

$$\left|T\Theta(f,u;a,b,m)\right| \tag{2.28}$$

$$\leq \frac{1}{2}\left[\int_a^m\left(\bigvee_a^t(u)\right)d\left(\bigvee_a^t(f)\right)+\int_m^b\left(\bigvee_t^b(u)\right)d\left(\bigvee_a^t(f)\right)\right]$$

$$+\frac{1}{2}\left[\int_a^m\left(\bigvee_t^m(u)\right)d\left(\bigvee_a^t(f)\right)+\int_m^b\left(\bigvee_m^t(u)\right)d\left(\bigvee_a^t(f)\right)\right]\leq \frac{1}{4}\bigvee_a^b(u)\bigvee_a^b(f).$$

If $p\in[a,b]$ is such that $\bigvee_a^p(u)=\bigvee_p^b(u)$, then

$$\begin{aligned} (2.29)\quad &\left|T\Theta\left(f,u;a,b,p\right)\right| \\ &\leq \frac{1}{2}\left[\int_a^p\left(\bigvee_a^t(u)\right)d\left(\bigvee_a^t(f)\right)+\int_p^b\left(\bigvee_t^b(u)\right)d\left(\bigvee_a^t(f)\right)\right] \\ &+\frac{1}{2}\left[\int_a^p\left(\bigvee_t^p(u)\right)d\left(\bigvee_a^t(f)\right)+\int_p^b\left(\bigvee_p^t(u)\right)d\left(\bigvee_a^t(f)\right)\right]\leq\frac{1}{4}\bigvee_a^b(u)\bigvee_a^b(f). \end{aligned}$$

3. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A\in\mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda\in\mathbb{R}$ as follows

$$\varphi_\lambda(s):=\begin{cases}1, & \text{for } -\infty<s\leq\lambda,\\ 0, & \text{for } \lambda<s<+\infty.\end{cases}$$

Then for every $\lambda\in\mathbb{R}$ the operator

$$E_\lambda:=\varphi_\lambda(A) \tag{3.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [27, p. 256]:

Theorem 2 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a=\min\{\lambda\,|\lambda\in Sp(A)\}=:\min Sp(A)$ and $b=\max\{\lambda\,|\lambda\in Sp(A)\}=:\max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda\leq E_{\lambda'}$ *for* $\lambda\leq\lambda'$;
b) $E_{a-0}=0, E_b=1_H$ *and* $E_{\lambda+0}=E_\lambda$ *for all* $\lambda\in\mathbb{R}$;
c) *We have the representation*

$$A=\int_{a-0}^b\lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ there exists a unique operator $\varphi(A)\in\mathcal{B}(H)$ such that for every $\varepsilon>0$ there exists a $\delta>0$ satisfying the inequality

$$\left\|\varphi(A)-\sum_{k=1}^n\varphi\left(\lambda_k'\right)\left[E_{\lambda_k}-E_{\lambda_{k-1}}\right]\right\|\leq\varepsilon$$

whenever

$$\begin{cases}\lambda_0<a=\lambda_1<...<\lambda_{n-1}<\lambda_n=b,\\ \lambda_k-\lambda_{k-1}\leq\delta \text{ for } 1\leq k\leq n,\\ \lambda_k'\in[\lambda_{k-1},\lambda_k] \text{ for } 1\leq k\leq n\end{cases}$$

this means that

$$\varphi(A) = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda, \tag{3.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 2. *With the assumptions of Theorem 2 for A, E_λ and φ we have the representations*

$$\varphi(A)\, x = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(A)\, x, y\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_\lambda x, y\rangle \quad \text{for all } x,\ y \in H. \tag{3.3}$$

In particular,

$$\langle \varphi(A)\, x, x\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)\, x\|^2 = \int_{a-0}^{b} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 \quad \text{for all } x \in H.$$

We need the following result that provides an upper bound for the total variation of the function $\mathbb{R} \ni \lambda \mapsto \langle E_\lambda x, y\rangle \in \mathbb{C}$ on an interval $[\alpha, \beta]$, see [23].

Lemma 2. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A. Then for any x, $y \in H$ and $\alpha < \beta$ we have the inequality*

$$\left[\bigvee_{\alpha}^{\beta} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right)\right]^2 \le \langle (E_\beta - E_\alpha)\, x, x\rangle \langle (E_\beta - E_\alpha)\, y, y\rangle, \tag{3.4}$$

where $\bigvee_{\alpha}^{\beta} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right)$ denotes the total variation of the function $\left\langle E_{(\cdot)} x, y\right\rangle$ on $[\alpha, \beta]$.

Remark 1. *For $\alpha = a - \varepsilon$ with $\varepsilon > 0$ and $\beta = b$ we get from (3.4) the inequality*

$$\bigvee_{a-\varepsilon}^{b} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \le \langle (1_H - E_{a-\varepsilon})\, x, x\rangle^{1/2} \langle (1_H - E_{a-\varepsilon})\, y, y\rangle^{1/2} \tag{3.5}$$

for any x, $y \in H$.

This implies, for any x, $y \in H$, that

$$\bigvee_{a-0}^{b} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \le \|x\|\,\|y\|, \tag{3.6}$$

where $\bigvee_{a-0}^{b} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right)$ denotes the limit $\lim_{\varepsilon\to 0+} \left[\bigvee_{a-\varepsilon}^{b} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right)\right]$.

We can state the following result for functions of selfadjoint operators:

Theorem 3. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a=\min\{\lambda|\lambda\in Sp(A)\}=:\min Sp(A)$ and $b=\max\{\lambda|\lambda\in Sp(A)\}=:\max Sp(A)$. Also, assume that $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A and assume that $\varphi\in\mathcal{BV}_{\mathbb{C}}[a,b]$ and $\varphi\in\mathcal{C}_{\mathbb{C}}[a,b]$ where $[a,b]\subset\mathring{I}$ (the interior of I). Then for all $\alpha\in[0,1]$*

$$\begin{aligned}(3.7)\quad &|(1-\alpha)\{\langle(1_H-E_s)x,y\rangle\varphi(b)+\langle E_sx,y\rangle\varphi(a)\}\\&\qquad+\alpha\langle x,y\rangle\varphi(s)-\langle\varphi(A)x,y\rangle|\\&\le\frac{1}{2}\max\{\alpha,1-\alpha\}\left(\bigvee_a^b(\varphi)+\left|\bigvee_a^s(\varphi)-\bigvee_s^b(\varphi)\right|\right)\bigvee_{a-0}^b(\langle E_{(\cdot)}x,y\rangle)\\&\le\frac{1}{2}\max\{\alpha,1-\alpha\}\left(\bigvee_a^b(\varphi)+\left|\bigvee_a^s(\varphi)-\bigvee_s^b(\varphi)\right|\right)\|x\|\|y\|\end{aligned}$$

for any $x,y\in H$.

In particular, if $m\in[a,b]$ is such that $\bigvee_a^m(\varphi)=\bigvee_m^b(\varphi)$, then

$$\begin{aligned}(3.8)\quad &|(1-\alpha)\{\langle(1_H-E_m)x,y\rangle\varphi(b)+\langle E_mx,y\rangle\varphi(a)\}\\&\qquad+\alpha\langle x,y\rangle\varphi(m)-\langle\varphi(A)x,y\rangle|\\&\le\frac{1}{2}\max\{\alpha,1-\alpha\}\bigvee_a^b(\varphi)\bigvee_{a-0}^b(\langle E_{(\cdot)}x,y\rangle)\\&\le\frac{1}{2}\max\{\alpha,1-\alpha\}\bigvee_a^b(\varphi)\|x\|\|y\|\end{aligned}$$

for any $x,y\in H$.

Proof. Using the inequality (2.11) we have for $\alpha\in[0,1]$ and $s\in(a,b)$ that

$$\begin{aligned}&\Big|(1-\alpha)\{[\langle E_bx,y\rangle-\langle E_sx,y\rangle]\varphi(b)+[\langle E_sx,y\rangle-\langle E_{a-\varepsilon}x,y\rangle]\varphi(a-\varepsilon)\}\\&\qquad+\alpha[\langle E_bx,y\rangle-\langle E_{a-\varepsilon}x,y\rangle]\varphi(s)-\int_{a-\varepsilon}^b\varphi(t)\,d\langle E_tx,y\rangle\Big|\\&\le\frac{1}{2}\max\{\alpha,1-\alpha\}\left(\bigvee_{a-\varepsilon}^b(\varphi)+\left|\bigvee_{a-\varepsilon}^s(\varphi)-\bigvee_s^b(\varphi)\right|\right)\bigvee_{a-\varepsilon}^b(\langle E_{(\cdot)}x,y\rangle)\\&\le\max\{\alpha,1-\alpha\}\bigvee_{a-\varepsilon}^b(\langle E_{(\cdot)}x,y\rangle)\bigvee_{a-\varepsilon}^b(\varphi)\end{aligned}$$

for small $\varepsilon>0$ and for any $x,y\in H$.

Taking the limit over $\varepsilon\to0+$ and using the continuity of φ and the Spectral Representation Theorem, we deduce the desired result (3.7). □

Remark 2. *If we take $\alpha = 1$ in (3.7), then we get*

$$\begin{aligned}(3.9)\quad & \left|\left\langle x,y\right\rangle \varphi\left(s\right)-\left\langle \varphi\left(A\right)x,y\right\rangle\right| \\ & \leq \frac{1}{2}\left(\bigvee_{a}^{b}\left(\varphi\right)+\left|\bigvee_{a}^{s}\left(\varphi\right)-\bigvee_{s}^{b}\left(\varphi\right)\right|\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\ & \leq \frac{1}{2}\left(\bigvee_{a}^{b}\left(\varphi\right)+\left|\bigvee_{a}^{s}\left(\varphi\right)-\bigvee_{s}^{b}\left(\varphi\right)\right|\right)\left\|x\right\|\left\|y\right\|\end{aligned}$$

for $s \in (a,b)$ and, in particular, if $m \in [a,b]$ is such that $\bigvee_{a}^{m}(\varphi) = \bigvee_{m}^{b}(\varphi)$, then

$$\begin{aligned}(3.10)\quad & \left|\left\langle x,y\right\rangle \varphi\left(m\right)-\left\langle \varphi\left(A\right)x,y\right\rangle\right| \\ & \leq \frac{1}{2}\bigvee_{a}^{b}\left(\varphi\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \frac{1}{2}\bigvee_{a}^{b}\left(\varphi\right)\left\|x\right\|\left\|y\right\|\end{aligned}$$

for any $x,\ y \in H$.

If we take $\alpha = 0$ in (3.7), then we get

$$\begin{aligned}(3.11)\quad & \left|\left\langle\left(1_H - E_s\right)x,y\right\rangle \varphi\left(b\right)+\left\langle E_s x,y\right\rangle\varphi\left(a\right)-\left\langle \varphi\left(A\right)x,y\right\rangle\right| \\ & \leq \frac{1}{2}\left(\bigvee_{a}^{b}\left(\varphi\right)+\left|\bigvee_{a}^{s}\left(\varphi\right)-\bigvee_{s}^{b}\left(\varphi\right)\right|\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\ & \leq \frac{1}{2}\left(\bigvee_{a}^{b}\left(\varphi\right)+\left|\bigvee_{a}^{s}\left(\varphi\right)-\bigvee_{s}^{b}\left(\varphi\right)\right|\right)\left\|x\right\|\left\|y\right\|\end{aligned}$$

for $s \in (a,b)$ and, in particular, if $m \in [a,b]$ is as above, then

$$\begin{aligned}(3.12)\quad & \left|\left\langle\left(1_H - E_m\right)x,y\right\rangle \varphi\left(b\right)+\left\langle E_m x,y\right\rangle\varphi\left(a\right)-\left\langle \varphi\left(A\right)x,y\right\rangle\right| \\ & \leq \frac{1}{2}\bigvee_{a}^{b}\left(\varphi\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \frac{1}{2}\bigvee_{a}^{b}\left(\varphi\right)\left\|x\right\|\left\|y\right\|\end{aligned}$$

for any $x,\ y \in H$.

If we take $\alpha = \frac{1}{2}$ in (3.7), then we get

$$\begin{aligned}(3.13)\quad & \left|\frac{1}{2}\left\{\left\langle\left(1_H - E_s\right)x,y\right\rangle \varphi\left(b\right)+\left\langle E_s x,y\right\rangle\varphi\left(a\right)\right\}\right. \\ & \left. +\frac{1}{2}\left\langle x,y\right\rangle\varphi\left(s\right)-\left\langle \varphi\left(A\right)x,y\right\rangle\right| \\ & \leq \frac{1}{4}\left(\bigvee_{a}^{b}\left(\varphi\right)+\left|\bigvee_{a}^{s}\left(\varphi\right)-\bigvee_{s}^{b}\left(\varphi\right)\right|\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\ & \leq \frac{1}{4}\left(\bigvee_{a}^{b}\left(\varphi\right)+\left|\bigvee_{a}^{s}\left(\varphi\right)-\bigvee_{s}^{b}\left(\varphi\right)\right|\right)\left\|x\right\|\left\|y\right\|\end{aligned}$$

for $s \in (a,b)$ and, in particular, if $m \in [a,b]$ is as above, then

$$\begin{aligned}(3.14)\quad & \left|\frac{1}{2}\left\{\left\langle \left(1_H - E_m\right)x, y\right\rangle \varphi(b) + \left\langle E_m x, y\right\rangle \varphi(a)\right\}\right. \\ & \qquad \left. + \frac{1}{2}\left\langle x, y\right\rangle \varphi(m) - \left\langle \varphi(A)x, y\right\rangle\right| \\ & \qquad\qquad \leq \frac{1}{4}\bigvee_a^b(\varphi)\bigvee_{a-0}^b\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \frac{1}{4}\bigvee_a^b(\varphi)\left\|x\right\|\left\|y\right\|\end{aligned}$$

for any $x,\ y \in H$.

The above inequality (3.7) can produce several particular examples of interest. For example if $[a,b] \subset (0,\infty)$ and we take $\varphi(t) = \ln t$ then by (3.7) we get

$$\begin{aligned}(3.15)\quad & \left|(1-\alpha)\left\{\left\langle \left(1_H - E_s\right)x, y\right\rangle \ln b + \left\langle E_s x, y\right\rangle \ln a\right\}\right. \\ & \qquad \left. + \alpha\left\langle x, y\right\rangle \ln s - \left\langle \ln A x, y\right\rangle\right| \\ & \leq \frac{1}{2}\max\{\alpha, 1-\alpha\}\left(\ln\left(\frac{b}{a}\right) + \left|\ln\left(\frac{s^2}{ab}\right)\right|\right)\bigvee_{a-0}^b\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\ & \qquad \leq \frac{1}{2}\max\{\alpha, 1-\alpha\}\left(\ln\left(\frac{b}{a}\right) + \left|\ln\left(\frac{s^2}{ab}\right)\right|\right)\left\|x\right\|\left\|y\right\|\end{aligned}$$

for any $x,\ y \in H,\ s \in [a,b]$ and $\alpha \in [0,1]$.

In particular, for $s = \sqrt{ab} =: G(a,b)$, then by (3.15) we get

$$\begin{aligned}(3.16)\quad & \left|(1-\alpha)\left\{\left\langle \left(1_H - E_{G(a,b)}\right)x, y\right\rangle \ln b + \left\langle E_{G(a,b)} x, y\right\rangle \ln a\right\}\right. \\ & \qquad \left. + \alpha\left\langle x, y\right\rangle \frac{\ln a + \ln b}{2} - \left\langle \ln A x, y\right\rangle\right| \\ & \leq \frac{1}{2}\max\{\alpha, 1-\alpha\}\ln\left(\frac{b}{a}\right)\bigvee_{a-0}^b\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\ & \qquad \leq \frac{1}{2}\max\{\alpha, 1-\alpha\}\ln\left(\frac{b}{a}\right)\left\|x\right\|\left\|y\right\|\end{aligned}$$

for any $x,\ y \in H$ and $\alpha \in [0,1]$.

If we take $\alpha = 1$ in (3.15) and (3.16), then we get

$$\begin{aligned}(3.17)\quad & \left|\left\langle x, y\right\rangle \ln s - \left\langle \ln A x, y\right\rangle\right| \\ & \leq \frac{1}{2}\left(\ln\left(\frac{b}{a}\right) + \left|\ln\left(\frac{s^2}{ab}\right)\right|\right)\bigvee_{a-0}^b\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\ & \qquad \leq \frac{1}{2}\left(\ln\left(\frac{b}{a}\right) + \left|\ln\left(\frac{s^2}{ab}\right)\right|\right)\left\|x\right\|\left\|y\right\|\end{aligned}$$

for any $x, y \in H$, $s \in [a,b]$ and, in particular

$$\left|\langle x,y\rangle \frac{\ln a+\ln b}{2}-\langle \ln Ax,y\rangle\right| \leq \frac{1}{2}\ln\left(\frac{b}{a}\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\leq \frac{1}{2}\ln\left(\frac{b}{a}\right)\|x\|\,\|y\| \tag{3.18}$$

for any $x, y \in H$.

If we take $\alpha = 0$ in (3.15) and (3.16), then we get

$$\begin{aligned}&|\langle (1_H - E_s)\,x,y\rangle \ln b+\langle E_s x,y\rangle \ln a-\langle \ln Ax,y\rangle|\\ &\leq \frac{1}{2}\left(\ln\left(\frac{b}{a}\right)+\left|\ln\left(\frac{s^2}{ab}\right)\right|\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\ &\leq \frac{1}{2}\left(\ln\left(\frac{b}{a}\right)+\left|\ln\left(\frac{s^2}{ab}\right)\right|\right)\|x\|\,\|y\|\end{aligned} \tag{3.19}$$

for any $x, y \in H$, $s \in [a,b]$ and, in particular

$$\begin{aligned}&\left|\left\langle \left(1_H - E_{G(a,b)}\right)x,y\right\rangle \ln b+\left\langle E_{G(a,b)}x,y\right\rangle \ln a-\langle \ln Ax,y\rangle\right|\\ &\leq \frac{1}{2}\ln\left(\frac{b}{a}\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\leq \frac{1}{2}\ln\left(\frac{b}{a}\right)\|x\|\,\|y\|\end{aligned} \tag{3.20}$$

for any $x, y \in H$.

If we take $\alpha = \frac{1}{2}$ in (3.15) and (3.16), then we get

$$\begin{aligned}&\left|\frac{1}{2}\left\{\langle (1_H - E_s)\,x,y\rangle \ln b+\langle E_s x,y\rangle \ln a\right\}\right.\\ &\qquad\left.+\frac{1}{2}\langle x,y\rangle \ln s-\langle \ln Ax,y\rangle\right|\\ &\leq \frac{1}{4}\left(\ln\left(\frac{b}{a}\right)+\left|\ln\left(\frac{s^2}{ab}\right)\right|\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\ &\leq \frac{1}{4}\left(\ln\left(\frac{b}{a}\right)+\left|\ln\left(\frac{s^2}{ab}\right)\right|\right)\|x\|\,\|y\|\end{aligned} \tag{3.21}$$

for any $x, y \in H$, $s \in [a,b]$ and, in particular

$$\begin{aligned}&\left|\frac{1}{2}\left\{\left\langle \left(1_H - E_{G(a,b)}\right)x,y\right\rangle \ln b+\left\langle E_{G(a,b)}x,y\right\rangle \ln a\right\}\right.\\ &\qquad\left.+\frac{1}{2}\langle x,y\rangle \frac{\ln a+\ln b}{2}-\langle \ln Ax,y\rangle\right|\\ &\leq \frac{1}{4}\ln\left(\frac{b}{a}\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\leq \frac{1}{4}\ln\left(\frac{b}{a}\right)\|x\|\,\|y\|\end{aligned} \tag{3.22}$$

for any $x, y \in H$.

4. Applications for Unitary Operators

A unitary operator is a bounded linear operator $U : H \to H$ on a Hilbert space H satisfying

$$U^*U = UU^* = 1_H$$

where U^* is the adjoint of U, and $1_H : H \to H$ is the identity operator. This property is equivalent to the following:

(i) U preserves the inner product $\langle \cdot, \cdot \rangle$ of the Hilbert space, i.e., for all vectors x and y in the Hilbert space, $\langle Ux, Uy \rangle = \langle x, y \rangle$ and
(ii) U is surjective.

The following result is well known [27, p. 275 - p. 276]:

Theorem 4 (Spectral Representation Theorem)**.** *Let U be a unitary operator on the Hilbert space H. Then there exists a family of projections $\{P_\lambda\}_{\lambda \in [0,2\pi]}$, called the spectral family of U, with the following properties*

a) $P_\lambda \le P_{\lambda'}$ *for* $\lambda \le \lambda'$;
b) $P_0 = 0, P_{2\pi} = 1_H$ *and* $P_{\lambda+0} = P_\lambda$ *for all* $\lambda \in [0, 2\pi)$;
c) *We have the representation*

$$U = \int_0^{2\pi} \exp(i\lambda)\, dP_\lambda.$$

More generally, for every continuous complex-valued function φ defined on the unit circle $\mathcal{C}(0,1)$ there exists a unique operator $\varphi(U) \in \mathcal{B}(H)$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the inequality

$$\left\| \varphi(U) - \sum_{k=1}^{n} \varphi\left(\exp\left(i\lambda_k'\right)\right) \left[P_{\lambda_k} - P_{\lambda_{k-1}}\right] \right\| \le \varepsilon$$

whenever

$$\begin{cases} 0 = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 2\pi, \\ \lambda_k - \lambda_{k-1} \le \delta \text{ for } 1 \le k \le n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \le k \le n \end{cases}$$

this means that

$$\varphi(U) = \int_0^{2\pi} \varphi(\exp(i\lambda))\, dP_\lambda, \tag{4.1}$$

where the integral is of Riemann-Stieltjes type.

Corollary 3. *With the assumptions of Theorem 4 for U, P_λ and φ we have the representations*

$$\varphi(U)x = \int_0^{2\pi} \varphi(\exp(i\lambda))\, dP_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(U)x, y \rangle = \int_0^{2\pi} \varphi(\exp(i\lambda))\, d\langle P_\lambda x, y \rangle \quad \text{for all } x, y \in H. \tag{4.2}$$

In particular,

$$\langle \varphi(U)x, x \rangle = \int_0^{2\pi} \varphi(\exp(i\lambda))\, d\langle P_\lambda x, x \rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(U)x\|^2 = \int_0^{2\pi} |\varphi(\exp(i\lambda))|^2 \, d\|P_\lambda x\|^2 \quad \text{for all } x \in H.$$

On making use of an argument similar to the one in [23, Theorem 6], we have:

Lemma 3. *Let $\{P_\lambda\}_{\lambda\in[0,2\pi]}$ be the spectral family of the unitary operator U on the Hilbert space H. Then for any x, $y \in H$ and $0 \le \alpha < \beta \le 2\pi$ we have the inequality*

$$\bigvee_{\alpha}^{\beta}\left(\left\langle P_{(\cdot)}x, y\right\rangle\right) \le \langle (P_\beta - P_\alpha)x, x\rangle^{1/2} \langle (P_\beta - P_\alpha)y, y\rangle^{1/2}, \tag{4.3}$$

where $\bigvee_{\alpha}^{\beta}\left(\left\langle P_{(\cdot)}x, y\right\rangle\right)$ denotes the total variation of the function $\left\langle P_{(\cdot)}x, y\right\rangle$ on $[\alpha, \beta]$.

In particular,

$$\bigvee_{0}^{2\pi}\left(\left\langle P_{(\cdot)}x, y\right\rangle\right) \le \|x\|\,\|y\| \tag{4.4}$$

for any x, $y \in H$.

We have:

Theorem 5. *Let U be a unitary operator on the Hilbert space H and $\{P_\lambda\}_{\lambda\in[0,2\pi]}$ the spectral family of projections of U. Also, assume that $\varphi : \mathcal{C}(0,1) \to \mathbb{C}$ are continuous on $\mathcal{C}(0,1)$ with $\varphi(\exp(i\cdot))$ of bounded variation on $[0, 2\pi]$. If $u \in [0, 2\pi]$, then for all $\alpha \in [0, 1]$*

$$\begin{aligned}
&\left|\left[(1-\alpha)\varphi(1) + \alpha\varphi\left(e^{iu}\right)\right]\langle x, y\rangle - \langle \varphi(U)x, y\rangle\right| \\
&\quad\le \frac{1}{2}\max\{\alpha, 1-\alpha\} \bigvee_{0}^{2\pi}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\
&\quad\quad\times\left(\bigvee_{0}^{2\pi}(\varphi(\exp(i\cdot))) + \left|\bigvee_{0}^{u}(\varphi(\exp(i\cdot))) - \bigvee_{u}^{2\pi}(\varphi(\exp(i\cdot)))\right|\right) \\
&\quad\le \frac{1}{2}\max\{\alpha, 1-\alpha\}\|x\|\,\|y\| \\
&\quad\quad\times\left(\bigvee_{0}^{2\pi}(\varphi(\exp(i\cdot))) + \left|\bigvee_{0}^{u}(\varphi(\exp(i\cdot))) - \bigvee_{u}^{2\pi}(\varphi(\exp(i\cdot)))\right|\right)
\end{aligned} \tag{4.5}$$

for any x, $y \in H$.

In particular, if we take $u=\pi$, then we get

$$
\begin{aligned}
(4.6)\quad & \left|\left[(1-\alpha)\varphi(1)+\alpha\varphi(-1)\right]\langle x,y\rangle-\langle\varphi(U)x,y\rangle\right| \\
& \leq \frac{1}{2}\max\{\alpha,1-\alpha\}\bigvee_0^{2\pi}\left(\langle E_{(\cdot)}x,y\rangle\right) \\
& \times\left(\bigvee_0^{2\pi}(\varphi(\exp(i\cdot)))+\left|\bigvee_0^{\pi}(\varphi(\exp(i\cdot)))-\bigvee_{\pi}^{2\pi}(\varphi(\exp(i\cdot)))\right|\right) \\
& \leq \frac{1}{2}\max\{\alpha,1-\alpha\}\|x\|\|y\| \\
& \times\left(\bigvee_0^{2\pi}(\varphi(\exp(i\cdot)))+\left|\bigvee_0^{\pi}(\varphi(\exp(i\cdot)))-\bigvee_{\pi}^{2\pi}(\varphi(\exp(i\cdot)))\right|\right)
\end{aligned}
$$

for any $x,\ y\in H$.

The proof follows in a similar way to the one from Theorem 3 by utilising Theorem 4 and the inequality (2.11).

We observe that if φ is continuously differentiable, then the total variation can be computed in terms of the derivative, namely

$$
\begin{aligned}
\bigvee_0^{2\pi}(\varphi(\exp(i\cdot))) &= \int_0^{2\pi}\left|\frac{d\varphi(\exp(it))}{dt}\right|dt=\int_0^{2\pi}\left|\varphi'(\exp(it))\exp(it)\,i\right|dt \\
&= \int_0^{2\pi}\left|\varphi'(\exp(it))\right|dt,
\end{aligned}
$$

$$
\bigvee_0^{\pi}(\varphi(\exp(i\cdot)))=\int_0^{\pi}\left|\varphi'(\exp(it))\right|dt
$$

and

$$
\bigvee_{\pi}^{2\pi}(\varphi(\exp(i\cdot)))=\int_{\pi}^{2\pi}\left|\varphi'(\exp(it))\right|dt.
$$

If we take $\alpha=\frac{1}{2}$ in (4.6) we get

$$
\begin{aligned}
(4.7)\quad & \left|\frac{\varphi(1)+\varphi(-1)}{2}\langle x,y\rangle-\langle\varphi(U)x,y\rangle\right| \\
& \leq \frac{1}{4}\bigvee_0^{2\pi}\left(\langle E_{(\cdot)}x,y\rangle\right) \\
& \times\left(\int_0^{2\pi}\left|\varphi'(\exp(it))\right|dt+\left|\int_0^{\pi}\left|\varphi'(\exp(it))\right|dt-\int_{\pi}^{2\pi}\left|\varphi'(\exp(it))\right|dt\right|\right) \\
& \leq \frac{1}{4}\|x\|\|y\| \\
& \times\left(\int_0^{2\pi}\left|\varphi'(\exp(it))\right|dt+\left|\int_0^{\pi}\left|\varphi'(\exp(it))\right|dt-\int_{\pi}^{2\pi}\left|\varphi'(\exp(it))\right|dt\right|\right)
\end{aligned}
$$

for any $x,\ y\in H$.

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