NEW THREE POINTS INEQUALITIES FOR RIEMANN-STIELTJES INTEGRAL OF LIPSCHITZIAN INTEGRANDS AND INTEGRATORS OF BOUNDED VARIATION

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Abstract. In this paper we provide some simple error bounds in approximating the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the use of three points formula

$$(1-\alpha)\{[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)\}+\alpha[u(b)-u(a)]f(x)$$

where $\alpha\in[0,1]$ and $x\in[a,b]$ in the case that the function f is Lipschitzian and u is of bounded variation on $[a,b]$. Applications for continuous functions of selfadjoint operators and unitary operators on Hilbert spaces are also given.

1. Introduction

One can approximate the *Stieltjes integral* $\int_a^b f(t)\,du(t)$ with the *Ostrowski type rule*

$$f(x)[u(b)-u(a)] \qquad ([15],\ [16]) \tag{1.1}$$

or with the *trapezoid type rule*

$$[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a) \qquad ([24]), \tag{1.2}$$

where $x\in[a,b]$.

In order to provide *a priory* sharp bounds for the *approximation error*, consider the functionals:

$$\Theta(f,u;a,b,x):=\int_a^b f(t)\,du(t)-f(x)[u(b)-u(a)]$$

and

$$T(f,u;a,b,x):=[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)-\int_a^b f(t)\,du(t).$$

For the functional $\theta(f,u;a,b,x)$ we have the bound [15]:

$$|\theta(f,u;a,b,x)|\le H\left[(x-a)^r\bigvee_a^x(f)+(b-x)^r\bigvee_x^b(f)\right] \tag{1.3}$$

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$$\leq H\times\begin{cases}\left[(x-a)^{r}+(b-x)^{r}\right]\left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{x}(f)-\bigvee_{x}^{b}(f)\right|\right];\\[2ex] \left[(x-a)^{qr}+(b-x)^{qr}\right]^{\frac{1}{q}}\left[\left(\bigvee_{a}^{x}(f)\right)^{p}+\left(\bigvee_{x}^{b}(f)\right)^{p}\right]^{\frac{1}{p}}\\ \hfill \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{r}\bigvee_{a}^{b}(f),\end{cases}$$

provided f is of bounded variation and u is of r-H-*Hölder type*, i.e.,

$$|u(t)-u(s)|\leq H|t-s|^{r}\qquad\text{for each }\ t,s\in[a,b], \tag{1.4}$$

with given $H>0$ and $r\in(0,1]$.

If f is of q-K-Hölder type and u is of bounded variation, then [16]

$$|\theta(f,u;a,b,x)|\leq K\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{q}\bigvee_{a}^{b}(u), \tag{1.5}$$

for any $x\in[a,b]$.

If u is monotonic nondecreasing and f of q-K-Hölder type, then the following refinement of (1.5) also holds [8]:

$$\begin{aligned}|\theta(f,u;a,b,x)|&\leq K\Bigg[(b-x)^{q}u(b)-(x-a)^{q}u(a)\\&\qquad+q\left\{\int_{a}^{x}\frac{u(t)\,dt}{(x-t)^{1-q}}-\int_{x}^{b}\frac{u(t)\,dt}{(t-x)^{1-q}}\right\}\Bigg]\\&\leq K\left[(b-x)^{q}\left[u(b)-u(x)\right]+(x-a)^{q}\left[u(x)-u(a)\right]\right]\\&\leq K\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{q}\left[u(b)-u(a)\right],\end{aligned} \tag{1.6}$$

for any $x\in[a,b]$.

If f is monotonic nondecreasing and u is of r-H-Hölder type, then [8]:

$$\begin{aligned}&|\theta(f,u;a,b,x)|\\&\leq H\Bigg[\left[(x-a)^{r}-(b-x)^{r}\right]f(x)\\&\qquad+r\left\{\int_{a}^{x}\frac{f(t)\,dt}{(b-t)^{1-r}}-\int_{x}^{b}\frac{f(t)\,dt}{(t-r)^{1-r}}\right\}\Bigg]\\&\leq H\left\{(b-x)^{r}\left[f(b)-f(x)\right]+(x-a)^{r}\left[f(x)-f(a)\right]\right\}\\&\leq H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{r}\left[f(b)-f(a)\right],\end{aligned} \tag{1.7}$$

for any $x\in[a,b]$.

The error functional $T(f,u;a,b,x)$ satisfies similar bounds, see [24], [8], [3] and [2] and the details are omitted. For other approximation results for the Riemann-Stieltjes integral, see also [21], [20], [19], and [8] and the references therein.

Motivated by the above results, in this paper we provide some simple ways to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the use of three points formula, namely we establish bounds for the error functional

$$T\Theta(f,u;a,b,x,\alpha) := (1-\alpha)\{[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)\} + \alpha[u(b)-u(a)]f(x) - \int_a^b f(t)\,du(t)$$

where $\alpha \in [0,1]$ and $x \in [a,b]$ while u is of bounded variation and f is Lipschitzian on $[a,b]$. Applications for continuous functions of selfadjoint operators and unitary operators on Hilbert spaces are also given.

2. Inequalities for Lipschitzian Integrands

Assume that $u,\ f : [a,b] \to \mathbb{C}$. If the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists, we write for simplicity, like in [1, p. 142] that $f \in \mathcal{R}_{\mathbb{C}}(u,[a,b])$, or $\mathcal{R}_{\mathbb{C}}(u)$ when the interval is implicitly known. If the functions $u,\ f$ are real valued, then we write $f \in \mathcal{R}(u,[a,b])$, or $\mathcal{R}(u)$.

We start with the following identity of interest.

Lemma 1. *Let $f,\ u : [a,b] \to \mathbb{C}$ and $x \in [a,b]$ such that $f \in \mathcal{R}_{\mathbb{C}}(u,[a,b])$. Then for any $\gamma,\ \mu \in \mathbb{C}$,*

$$[u(b)-\mu]f(b)+[\gamma-u(a)]f(a)+(\mu-\gamma)f(x)-\int_a^b f(t)\,du(t) = \int_a^x [u(t)-\gamma]\,df(t)+\int_x^b [u(t)-\mu]\,df(t). \tag{2.1}$$

In particular, for $\mu=\gamma$ we have

$$[u(b)-\gamma]f(b)+[\gamma-u(a)]f(a)-\int_a^b f(t)\,du(t) = \int_a^b [u(t)-\gamma]\,df(t). \tag{2.2}$$

Proof. Using integration by parts rule for the Riemann-Stieltjes integral, we have

$$\int_a^x [u(t)-\gamma]\,df(t) = [u(x)-\gamma]f(x)-[u(a)-\gamma]f(a)-\int_a^x f(t)\,du(t)$$

and

$$\int_x^b [u(t)-\mu]\,df(t) = [u(b)-\mu]f(b)-[u(x)-\mu]f(x)-\int_x^b f(t)\,du(t)$$

for any $x \in [a,b]$.

If we add these two equalities, we get

$$\begin{aligned}\int_a^x [u(t)-\gamma]\,df(t)+\int_x^b [u(t)-\mu]\,df(t) &= [u(b)-\mu]f(b)+[\gamma-u(a)]f(a)+[\mu-u(x)]f(x) \\ &\quad + [u(x)-\gamma]f(x)-\int_a^x f(t)\,du(t)-\int_x^b f(t)\,du(t) \\ &= [u(b)-\mu]f(b)+[\gamma-u(a)]f(a)+(\mu-\gamma)f(x)-\int_a^b f(t)\,du(t)\end{aligned}$$

for any $x \in [a,b]$, which proves the desired equality (2.1). □

If in (2.1) we take $\gamma = \alpha u(a) + (1-\alpha) u(x)$ and $\mu = (1-\alpha) u(x) + \alpha u(b)$ where $x \in [a,b]$ and $\alpha \in [0,1]$ we get

$$(2.3)\quad (1-\alpha)\{[u(b)-u(x)] f(b) + [u(x)-u(a)] f(a)\} + \alpha [u(b)-u(a)] f(x) - \int_a^b f(t)\,du(t) = \int_a^x [u(t) - \alpha u(a) - (1-\alpha) u(x)]\,df(t) + \int_x^b [u(t) - (1-\alpha) u(x) - \alpha u(b)]\,df(t).$$

In particular, for $x = \frac{a+b}{2}$, we get

$$(2.4)\quad (1-\alpha)\left\{\left[u(b) - u\left(\frac{a+b}{2}\right)\right] f(b) + \left[u\left(\frac{a+b}{2}\right) - u(a)\right] f(a)\right\} + \alpha [u(b)-u(a)] f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\,du(t) = \int_a^{\frac{a+b}{2}} \left[u(t) - \alpha u(a) - (1-\alpha) u\left(\frac{a+b}{2}\right)\right] df(t) + \int_{\frac{a+b}{2}}^b \left[u(t) - (1-\alpha) u\left(\frac{a+b}{2}\right) - \alpha u(b)\right] df(t).$$

If in this equality, we take $\alpha = 1$, we get the Montgomery type identity

$$(2.5)\quad [u(b)-u(a)] f(x) - \int_a^b f(t)\,du(t) = \int_a^x [u(t)-u(a)]\,df(t) + \int_x^b [u(t)-u(b)]\,df(t),$$

for $x \in [a,b]$, which was obtained for the first time by the author in [15].

In particular, for $x = \frac{a+b}{2}$, we get

$$(2.6)\quad [u(b)-u(a)] f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\,du(t) = \int_a^{\frac{a+b}{2}} [u(t)-u(a)]\,df(t) + \int_{\frac{a+b}{2}}^b [u(t)-u(b)]\,df(t).$$

If in (2.3) we take $\alpha = \frac{1}{2}$, we get

$$(2.7)\quad \frac{1}{2}\{[u(b)-u(x)] f(b) + [u(x)-u(a)] f(a) + [u(b)-u(a)] f(x)\} - \int_a^b f(t)\,du(t) = \int_a^x \left[u(t) - \frac{u(a)+u(x)}{2}\right] df(t) + \int_x^b \left[u(t) - \frac{u(x)+u(b)}{2}\right] df(t)$$

for $x \in [a,b]$ and in particular

$$\frac{1}{2}\left\{\left[u(b)-u\left(\frac{a+b}{2}\right)\right]f(b)+\left[u\left(\frac{a+b}{2}\right)-u(a)\right]f(a) + [u(b)-u(a)]f\left(\frac{a+b}{2}\right)\right\}-\int_a^b f(t)\,du(t) = \int_a^{\frac{a+b}{2}}\left[u(t)-\frac{u(a)+u\left(\frac{a+b}{2}\right)}{2}\right]df(t)+\int_{\frac{a+b}{2}}^b\left[u(t)-\frac{u\left(\frac{a+b}{2}\right)+u(b)}{2}\right]df(t). \tag{2.8}$$

If in (2.3) we take $\alpha = 0$, then we get

$$[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)-\int_a^b f(t)\,du(t) = \int_a^b [u(t)-u(x)]\,df(t) \tag{2.9}$$

for $x \in [a,b]$, and, in particular,

$$\left[u(b)-u\left(\frac{a+b}{2}\right)\right]f(b)+\left[u\left(\frac{a+b}{2}\right)-u(a)\right]f(a)-\int_a^b f(t)\,du(t) = \int_a^b\left[u(t)-u\left(\frac{a+b}{2}\right)\right]df(t). \tag{2.10}$$

We have the following result:

Theorem 1. *Let $f,\ u : [a,b] \to \mathbb{C}$ and $x \in [a,b]$. If f is Lipschitzian with the constant $L > 0$, namely*

$$|f(t)-f(s)| \le L|t-s| \text{ for all } t,\ s \in [a,b]$$

and u are of bounded variation, then $f \in \mathcal{R}_{\mathbb{C}}(u,[a,b])$ and

$$\begin{aligned} &|T\Theta(f,u;a,b,x,\alpha)| \\ &\le \alpha L\left[\int_a^x\left(\bigvee_a^t(u)\right)dt+\int_x^b\left(\bigvee_t^b(u)\right)dt\right] \\ &+(1-\alpha)L\left[\int_a^x\left(\bigvee_t^x(u)\right)dt+\int_x^b\left(\bigvee_x^t(u)\right)dt\right] \\ &\le \max\{\alpha,1-\alpha\}L\left((x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right) \\ &\le \frac{1}{2}\max\{\alpha,1-\alpha\}L\begin{cases}\left(\bigvee_a^b(u)+\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right)(b-a) \\ \\ (b-a+|2x-a-b|)\bigvee_a^b(u)\end{cases} \\ &\le \max\{\alpha,1-\alpha\}L(b-a)\bigvee_a^b(u). \end{aligned} \tag{2.11}$$

In particular, we have for $x = \frac{a+b}{2}$ *that*

$$\left|T\Theta\left(f,u;a,b,\frac{a+b}{2},\alpha\right)\right| \le \alpha L\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_a^t(u)\right)dt + \int_{\frac{a+b}{2}}^b\left(\bigvee_t^b(u)\right)dt\right] + (1-\alpha)L\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_t^{\frac{a+b}{2}}(u)\right)dt + \int_{\frac{a+b}{2}}^b\left(\bigvee_{\frac{a+b}{2}}^t(u)\right)dt\right] \le \frac{1}{2}\max\{\alpha,1-\alpha\}(b-a)L\bigvee_a^b(u). \tag{2.12}$$

Proof. It is well known that, if $p:[a,b]\to\mathbb{C}$ is Riemann integrable and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and [1]

$$\left|\int_a^b p(t)\,dv(t)\right| \le L\int_a^b |p(t)|\,dt. \tag{2.13}$$

By utilising (2.3), we have for $x\in(a,b)$ and $\alpha\in[0,1]$ that

$$\left|(1-\alpha)\{[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)\} + \alpha[u(b)-u(a)]f(x) - \int_a^b f(t)\,du(t)\right| \tag{2.14}$$

$$\le \left|\int_a^x [u(t)-\alpha u(a)-(1-\alpha)u(x)]\,df(t)\right| + \left|\int_x^b [u(t)-(1-\alpha)u(x)-\alpha u(b)]\,df(t)\right| \tag{2.15}$$

$$\le L\int_a^x |[u(t)-\alpha u(a)-(1-\alpha)u(x)]|\,dt + L\int_x^b |[u(t)-(1-\alpha)u(x)-\alpha u(b)]|\,dt =: C(f,u,x;\alpha). \tag{2.16}$$

Since u is of bounded variation, hence

$$\begin{aligned}|[u(t)-\alpha u(a)-(1-\alpha)u(x)]| &= |[\alpha(u(t)-u(a))+(1-\alpha)(u(t)-u(x))]| \\ &\le \alpha|u(t)-u(a)|+(1-\alpha)|u(x)-u(t)| \\ &\le \alpha\bigvee_a^t(u)+(1-\alpha)\bigvee_t^x(u)\end{aligned}$$

and

$$\begin{aligned}
\left|\left[u\left(t\right)-\left(1-\alpha\right)u\left(x\right)-\alpha u\left(b\right)\right]\right| &= \left|\left[\left(1-\alpha\right)\left(u\left(t\right)-u\left(x\right)\right)+\alpha\left(u\left(t\right)-u\left(b\right)\right)\right]\right| \\
&\leq \left(1-\alpha\right)\left|u\left(t\right)-u\left(x\right)\right|+\alpha\left|u\left(b\right)-u\left(t\right)\right| \\
&\leq \left(1-\alpha\right)\bigvee_{x}^{t}\left(u\right)+\alpha\bigvee_{t}^{b}\left(u\right)
\end{aligned}$$

for x, $t\in\left[a,b\right]$ and $\alpha\in\left[0,1\right]$.

Therefore

$$\begin{aligned}
\int_{a}^{x}\left|\left[u\left(t\right)-\alpha u\left(a\right)-\left(1-\alpha\right)u\left(x\right)\right]\right|dt &\leq \int_{a}^{x}\left[\alpha\bigvee_{a}^{t}\left(u\right)+\left(1-\alpha\right)\bigvee_{t}^{x}\left(u\right)\right]dt \\
&= \alpha\int_{a}^{x}\left(\bigvee_{a}^{t}\left(u\right)\right)dt+\left(1-\alpha\right)\int_{a}^{x}\left(\bigvee_{t}^{x}\left(u\right)\right)dt
\end{aligned}$$

and

$$\begin{aligned}
\int_{x}^{b}\left|\left[u\left(t\right)-\left(1-\alpha\right)u\left(x\right)-\alpha u\left(b\right)\right]\right|dt &\leq \int_{x}^{b}\left[\left(1-\alpha\right)\bigvee_{x}^{t}\left(u\right)+\alpha\bigvee_{t}^{b}\left(u\right)\right]dt \\
&= \left(1-\alpha\right)\int_{x}^{b}\left(\bigvee_{x}^{t}\left(u\right)\right)dt+\alpha\int_{x}^{b}\left(\bigvee_{t}^{b}\left(u\right)\right)dt
\end{aligned}$$

for $x\in\left[a,b\right]$ and $\alpha\in\left[0,1\right]$.

If we add these two inequalities, we get

$$\begin{aligned}
B\left(f,u,x;\alpha\right) &\leq L\left[\alpha\int_{a}^{x}\left(\bigvee_{a}^{t}\left(u\right)\right)dt+\left(1-\alpha\right)\int_{a}^{x}\left(\bigvee_{t}^{x}\left(u\right)\right)dt\right] \\
&\quad +L\left[\left(1-\alpha\right)\int_{x}^{b}\left(\bigvee_{x}^{t}\left(u\right)\right)dt+\alpha\int_{x}^{b}\left(\bigvee_{t}^{b}\left(u\right)\right)dt\right] \\
&= \alpha L\left[\int_{a}^{x}\left(\bigvee_{a}^{t}\left(u\right)\right)dt+\int_{x}^{b}\left(\bigvee_{t}^{b}\left(u\right)\right)dt\right] \\
&\quad +\left(1-\alpha\right)L\left[\int_{a}^{x}\left(\bigvee_{t}^{x}\left(u\right)\right)dt+\int_{x}^{b}\left(\bigvee_{x}^{t}\left(u\right)\right)dt\right]=:D\left(f,u,x;\alpha\right)
\end{aligned}$$

for $x\in\left[a,b\right]$ and $\alpha\in\left[0,1\right]$.

This prove the first inequality in (2.11).

Observe that

$$\begin{aligned}
&D(f,u,x;\alpha)\leq\max\{\alpha,1-\alpha\}\\
&\times L\left\{\int_a^x\left(\bigvee_a^t(u)\right)dt+\int_x^b\left(\bigvee_t^b(u)\right)dt+\int_a^x\left(\bigvee_t^x(u)\right)dt+\int_x^b\left(\bigvee_x^t(u)\right)dt\right\}\\
&\quad=\max\{\alpha,1-\alpha\}L\left[\int_a^x\left(\bigvee_a^t(u)+\bigvee_t^x(u)\right)dt+\int_x^b\left(\bigvee_x^t(u)+\bigvee_t^b(u)\right)dt\right]\\
&\quad\quad=\max\{\alpha,1-\alpha\}L\left[\bigvee_a^x(u)\int_a^x dt+\bigvee_x^b(u)\int_x^b dt\right]\\
&\quad\quad\quad=\max\{\alpha,1-\alpha\}L\left((x-a)\bigvee_a^x(u)\bigvee_a^x(f)+(b-x)\bigvee_x^b(u)\right)
\end{aligned}$$

for $x\in[a,b]$ and $\alpha\in[0,1]$.

This proves the second inequality in (2.11).

The last part is obvious. □

Corollary 1. *Assume that f and u are as in Theorem 1.*

(i) *If we take $x=\frac{a+b}{2}$, then*

$$\begin{aligned}
&\left|T\Theta\left(f,u;a,b,\frac{a+b}{2},\alpha\right)\right|\\
&\leq\alpha L\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_a^t(u)\right)dt+\int_{\frac{a+b}{2}}^b\left(\bigvee_t^b(u)\right)dt\right]\\
&+(1-\alpha)L\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_t^{\frac{a+b}{2}}(u)\right)dt+\int_{\frac{a+b}{2}}^b\left(\bigvee_{\frac{a+b}{2}}^t(u)\right)dt\right]\\
&\leq\frac{1}{2}\max\{\alpha,1-\alpha\}(b-a)L\bigvee_a^b(u).
\end{aligned}\tag{2.17}$$

(ii) *If $p\in[a,b]$ is such that $\bigvee_a^p(u)=\bigvee_p^b(u)$, then*

$$\begin{aligned}
|T\Theta(f,u;a,b,p,\alpha)|&\leq\alpha L\left[\int_a^p\left(\bigvee_a^t(u)\right)dt+\int_p^b\left(\bigvee_t^b(u)\right)dt\right]\\
&+(1-\alpha)L\left[\int_a^p\left(\bigvee_t^p(u)\right)dt+\int_p^b\left(\bigvee_p^t(u)\right)dt\right]\\
&\leq\frac{1}{2}\max\{\alpha,1-\alpha\}(b-a)L\bigvee_a^b(u).
\end{aligned}\tag{2.18}$$

If we take $\alpha = 1$ in Theorem 1 we get the Ostrowski type inequalities

$$|\Theta(f,u;a,b,x)| \le L\left[\int_a^x \left(\bigvee_a^t (u)\right) dt + \int_x^b \left(\bigvee_t^b (u)\right) dt\right] \tag{2.19}$$

$$\le L\left[(x-a)\bigvee_a^x (u) + (b-x)\bigvee_x^b (u)\right]$$

$$\le \frac{1}{2}L\begin{cases}\left(\bigvee_a^b (u) + \left|\bigvee_a^x (u) - \bigvee_x^b (u)\right|\right)(b-a) \\ \\ (b-a+|2x-a-b|)\bigvee_a^b (u)\end{cases} \le L(b-a)\bigvee_a^b (u),$$

for any $x \in [a,b]$.

In particular, for $x = \frac{a+b}{2}$, we get

$$\left|\Theta\left(f,u;a,b,\frac{a+b}{2}\right)\right| \le L\left[\int_a^{\frac{a+b}{2}} \left(\bigvee_a^t (u)\right) dt + \int_{\frac{a+b}{2}}^b \left(\bigvee_t^b (u)\right) dt\right] \tag{2.20}$$
$$\le \frac{1}{2}(b-a)L\bigvee_a^b (u).$$

If $p \in [a,b]$ is such that $\bigvee_a^p (u) = \bigvee_p^b (u)$, then

$$|\Theta(f,u;a,b,p)| \le L\left[\int_a^p \left(\bigvee_a^t (u)\right) dt + \int_p^b \left(\bigvee_t^b (u)\right) dt\right] \tag{2.21}$$
$$\le \frac{1}{2}L(b-a)\bigvee_a^b (u).$$

If we take $\alpha = 0$ in Theorem 1 we get the trapezoid type inequalities

$$|T(f,u;a,b,x)| \tag{2.22}$$
$$\le L\left[\int_a^x \left(\bigvee_t^x (u)\right) dt + \int_x^b \left(\bigvee_x^t (u)\right) dt\right] \le L\left[(x-a)\bigvee_a^x (u) + (b-x)\bigvee_x^b (u)\right]$$

$$\le \frac{1}{2}L\begin{cases}\left(\bigvee_a^b (u) + \left|\bigvee_a^x (u) - \bigvee_x^b (u)\right|\right)(b-a) \\ \\ (b-a+|2x-a-b|)\bigvee_a^b (u)\end{cases} \le (b-a)L\bigvee_a^b (u),$$

for any $x \in [a,b]$.

In particular, for $x=\frac{a+b}{2}$, we get

$$\left|T\left(f,u;a,b,\frac{a+b}{2}\right)\right| \le L\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_t^{\frac{a+b}{2}}(u)\right)dt+\int_{\frac{a+b}{2}}^b\left(\bigvee_{\frac{a+b}{2}}^t(u)\right)dt\right]\le\frac{1}{2}(b-a)L\bigvee_a^b(u). \tag{2.23}$$

If $p\in[a,b]$ is such that $\bigvee_a^p(u)=\bigvee_p^b(u)$, then

$$|T(f,u;a,b,p)| \le L\left[\int_a^p\left(\bigvee_t^p(u)\right)dt+\int_p^b\left(\bigvee_p^t(u)\right)d\right]t\le\frac{1}{2}(b-a)L\bigvee_a^b(u). \tag{2.24}$$

If we take in Theorem 1 $\alpha=\frac{1}{2}$, and consider the error functional

$$T\Theta(f,u;a,b,x):=T\Theta\left(f,u;a,b,x,\frac{1}{2}\right)$$

then we get the three point inequalities

$$\begin{aligned}|T\Theta(f,u;a,b,x)| &\le\frac{1}{2}L\left[\int_a^x\left(\bigvee_a^t(u)\right)dt+\int_x^b\left(\bigvee_t^b(u)\right)dt\right]\\ &\quad+\frac{1}{2}L\left[\int_a^x\left(\bigvee_t^x(u)\right)dt+\int_x^b\left(\bigvee_x^t(u)\right)dt\right]\\ &\le\frac{1}{2}L\left((x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right)\\ &\le\frac{1}{4}L\begin{cases}\left(\bigvee_a^b(u)+\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right)(b-a)\\ \\ (b-a+|2x-a-b|)\bigvee_a^b(u)\end{cases}\le\frac{1}{2}(b-a)L\bigvee_a^b(u).\end{aligned} \tag{2.25}$$

In particular, for $x=\frac{a+b}{2}$ we get the mixture of trapezoid and mid-point inequalities

$$\begin{aligned}\left|T\Theta\left(f,u;a,b,\frac{a+b}{2}\right)\right| &\le\frac{1}{2}L\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_a^t(u)\right)dt+\int_{\frac{a+b}{2}}^b\left(\bigvee_t^b(u)\right)dt\right]\\ &\quad+\frac{1}{2}L\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_t^{\frac{a+b}{2}}(u)\right)dt+\int_{\frac{a+b}{2}}^b\left(\bigvee_{\frac{a+b}{2}}^t(u)\right)dt\right]\le\frac{1}{4}(b-a)L\bigvee_a^b(u).\end{aligned} \tag{2.26}$$

If $p\in[a,b]$ is such that $\bigvee_a^p(u)=\bigvee_p^b(u)$, then

$$\begin{aligned}(2.27)\quad &\left|T\Theta(f,u;a,b,p)\right|\\ &\le\frac{1}{2}L\left[\int_a^p\left(\bigvee_a^t(u)\right)d\left(\bigvee_a^t(f)\right)+\int_p^b\left(\bigvee_t^b(u)\right)d\left(\bigvee_a^t(f)\right)\right]\\ &+\frac{1}{2}L\left[\int_a^p\left(\bigvee_t^p(u)\right)d\left(\bigvee_a^t(f)\right)+\int_p^b\left(\bigvee_p^t(u)\right)d\left(\bigvee_a^t(f)\right)\right]\\ &\le\frac{1}{4}(b-a)L\bigvee_a^b(u).\end{aligned}$$

3. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A\in\mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda\in\mathbb{R}$ as follows

$$\varphi_\lambda(s):=\begin{cases}1, & \text{for } -\infty<s\le\lambda,\\ 0, & \text{for } \lambda<s<+\infty.\end{cases}$$

Then for every $\lambda\in\mathbb{R}$ the operator

$$E_\lambda:=\varphi_\lambda(A) \tag{3.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [27, p. 256]:

Theorem 2 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a=\min\{\lambda\,|\lambda\in Sp(A)\}=:\min Sp(A)$ and $b=\max\{\lambda\,|\lambda\in Sp(A)\}=:\max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) *$E_\lambda\le E_{\lambda'}$ for $\lambda\le\lambda'$;*
b) *$E_{a-0}=0, E_b=1_H$ and $E_{\lambda+0}=E_\lambda$ for all $\lambda\in\mathbb{R}$;*
c) *We have the representation*

$$A=\int_{a-0}^b\lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ there exists a unique operator $\varphi(A)\in\mathcal{B}(H)$ such that for every $\varepsilon>0$ there exists a $\delta>0$ satisfying the inequality

$$\left\|\varphi(A)-\sum_{k=1}^n\varphi(\lambda_k')\left[E_{\lambda_k}-E_{\lambda_{k-1}}\right]\right\|\le\varepsilon$$

whenever

$$\begin{cases} \lambda_0 < a = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = b, \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases}$$

this means that

$$\varphi(A) = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda, \tag{3.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 2. *With the assumptions of Theorem 2 for A, E_λ and φ we have the representations*

$$\varphi(A)x = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(A)x, y\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_\lambda x, y\rangle \quad \text{for all } x,\ y \in H. \tag{3.3}$$

In particular,

$$\langle \varphi(A)x, x\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{a-0}^{b} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 \quad \text{for all } x \in H.$$

We need the following result that provides an upper bound for the total variation of the function $\mathbb{R} \ni \lambda \mapsto \langle E_\lambda x, y\rangle \in \mathbb{C}$ on an interval $[\alpha, \beta]$, see [23].

Lemma 2. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A. Then for any x, $y \in H$ and $\alpha < \beta$ we have the inequality*

$$\left[\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]^2 \leq \langle (E_\beta - E_\alpha)x, x\rangle \langle (E_\beta - E_\alpha)y, y\rangle, \tag{3.4}$$

where $\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ denotes the total variation of the function $\left\langle E_{(\cdot)}x, y\right\rangle$ on $[\alpha, \beta]$.

Remark 1. *For $\alpha = a - \varepsilon$ with $\varepsilon > 0$ and $\beta = b$ we get from (3.4) the inequality*

$$\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \langle (1_H - E_{a-\varepsilon})x, x\rangle^{1/2} \langle (1_H - E_{a-\varepsilon})y, y\rangle^{1/2} \tag{3.5}$$

for any x, $y \in H$.

This implies, for any x, $y \in H$, that

$$\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \|x\|\,\|y\|, \tag{3.6}$$

where $\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$ *denotes the limit* $\lim_{\varepsilon\to0+}\left[\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]$.

We can state the following result for functions of selfadjoint operators:

Theorem 3. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a=\min\{\lambda\,|\lambda\in Sp(A)\}=:\min Sp(A)$ and $b=\max\{\lambda\,|\lambda\in Sp(A)\}=:\max Sp(A)$. Also, assume that $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A and assume that φ is Lipschitzian with the constant $L>0$ on $[a,b]$, where $[a,b]\subset\mathring{I}$ (the interior of I). Then for all $\alpha\in[0,1]$ and $s\in[a,b]$*

$$\begin{aligned}(3.7)\quad &\left|(1-\alpha)\{\langle(1_H-E_s)x,y\rangle\varphi(b)+\langle E_sx,y\rangle\varphi(a)\}\right.\\ &\qquad\left.+\alpha\langle x,y\rangle\varphi(s)-\langle\varphi(A)x,y\rangle\right|\\ &\le\max\{\alpha,1-\alpha\}L\left(\frac{b-a}{2}+\left|s-\frac{a+b}{2}\right|\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\ &\le\max\{\alpha,1-\alpha\}L\left(\frac{b-a}{2}+\left|s-\frac{a+b}{2}\right|\right)\|x\|\,\|y\|\end{aligned}$$

for any x, $y\in H$.

In particular,

$$\begin{aligned}(3.8)\quad &\left|(1-\alpha)\left\{\left\langle\left(1_H-E_{\frac{a+b}{2}}\right)x,y\right\rangle\varphi(b)+\left\langle E_{\frac{a+b}{2}}x,y\right\rangle\varphi(a)\right\}\right.\\ &\qquad\left.+\alpha\langle x,y\rangle\varphi\left(\frac{a+b}{2}\right)-\langle\varphi(A)x,y\rangle\right|\\ &\le\frac{1}{2}\max\{\alpha,1-\alpha\}(b-a)L\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\ &\le\frac{1}{2}\max\{\alpha,1-\alpha\}(b-a)L\|x\|\,\|y\|\end{aligned}$$

for any x, $y\in H$.

Proof. Using the inequality (2.11) we have for $\alpha\in[0,1]$ and $s\in(a,b)$ that

$$\begin{aligned}&\left|(1-\alpha)\{[\langle E_bx,y\rangle-\langle E_sx,y\rangle]\varphi(b)+[\langle E_sx,y\rangle-\langle E_{a-\varepsilon}x,y\rangle]\varphi(a-\varepsilon)\}\right.\\ &\qquad\left.+\alpha[\langle E_bx,y\rangle-\langle E_{a-\varepsilon}x,y\rangle]\varphi(s)-\int_{a-\varepsilon}^{b}\varphi(t)\,d\langle E_tx,y\rangle\right|\\ &\le\max\{\alpha,1-\alpha\}L\left(\frac{b-a+\varepsilon}{2}+\left|s-\frac{a-\varepsilon+b}{2}\right|\right)\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\end{aligned}$$

for small $\varepsilon>0$ and for any x, $y\in H$.

Taking the limit over $\varepsilon\to0+$ and using the continuity of φ and the Spectral Representation Theorem, we deduce the desired result (3.7). $\square$

Remark 2. *If we take* $\alpha = 1$ *in (3.7), then we get*

$$|\langle x,y\rangle \varphi(s) - \langle \varphi(A)x,y\rangle| \leq L\left(\frac{b-a}{2} + \left|s-\frac{a+b}{2}\right|\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq L\left(\frac{b-a}{2} + \left|s-\frac{a+b}{2}\right|\right)\|x\|\,\|y\| \tag{3.9}$$

for $s \in (a,b)$ *and, in particular,*

$$\left|\langle x,y\rangle \varphi\left(\frac{a+b}{2}\right) - \langle \varphi(A)x,y\rangle\right| \leq \frac{1}{2}(b-a)L\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \frac{1}{2}(b-a)L\|x\|\,\|y\| \tag{3.10}$$

for any $x,\ y \in H$.

If we take $\alpha = 0$ *in (3.7), then we get*

$$|\langle (1_H - E_s)x,y\rangle \varphi(b) + \langle E_s x,y\rangle \varphi(a) - \langle \varphi(A)x,y\rangle| \leq L\left(\frac{b-a}{2} + \left|s-\frac{a+b}{2}\right|\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq L\left(\frac{b-a}{2} + \left|s-\frac{a+b}{2}\right|\right)\|x\|\,\|y\| \tag{3.11}$$

for $s \in (a,b)$ *and, in particular,*

$$\left|\left\langle \left(1_H - E_{\frac{a+b}{2}}\right)x,y\right\rangle \varphi(b) + \left\langle E_{\frac{a+b}{2}}x,y\right\rangle \varphi(a) - \langle \varphi(A)x,y\rangle\right| \leq \frac{1}{2}(b-a)L\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \frac{1}{2}(b-a)L\|x\|\,\|y\| \tag{3.12}$$

for any $x,\ y \in H$.

If we take $\alpha = \frac{1}{2}$ *in (3.7), then we get*

$$\left|\frac{1}{2}\{\langle (1_H - E_s)x,y\rangle \varphi(b) + \langle E_s x,y\rangle \varphi(a)\} + \frac{1}{2}\langle x,y\rangle \varphi(s) - \langle \varphi(A)x,y\rangle\right| \leq \frac{1}{2}L\left(\frac{b-a}{2} + \left|s-\frac{a+b}{2}\right|\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \frac{1}{2}L\left(\frac{b-a}{2} + \left|s-\frac{a+b}{2}\right|\right)\|x\|\,\|y\| \tag{3.13}$$

for $s\in(a,b)$ *and, in particular,*

$$\left|\frac{1}{2}\left\{\left\langle\left(1_H-E_{\frac{a+b}{2}}\right)x,y\right\rangle\varphi(b)+\left\langle E_{\frac{a+b}{2}}x,y\right\rangle\varphi(a)\right\}\right.$$
$$\left.+\frac{1}{2}\langle x,y\rangle\varphi\left(\frac{a+b}{2}\right)-\langle\varphi(A)x,y\rangle\right|$$
$$\le\frac{1}{4}(b-a)L\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\le\frac{1}{4}(b-a)L\|x\|\|y\| \tag{3.14}$$

for any $x,\ y\in H.$

The above inequality (3.7) can produce several particular examples of interest. For example if $[a,b]\subset(0,\infty)$ and we take $\varphi(t)=\ln t$ then φ is Lipschitzian with the constant

$$L:=\max_{t\in[a,b]}\left(\frac{1}{t}\right)=\frac{1}{a}>0$$

and by (3.7) we get

$$|(1-\alpha)\{\langle(1_H-E_s)x,y\rangle\ln b+\langle E_sx,y\rangle\ln a\}$$
$$+\alpha\langle x,y\rangle\ln s-\langle\ln Ax,y\rangle|$$
$$\le\frac{1}{a}\max\{\alpha,1-\alpha\}\left(\frac{b-a}{2}+\left|s-\frac{a+b}{2}\right|\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$
$$\le\frac{1}{a}\max\{\alpha,1-\alpha\}\left(\frac{b-a}{2}+\left|s-\frac{a+b}{2}\right|\right)\|x\|\|y\| \tag{3.15}$$

for any $x,\ y\in H,\ s\in[a,b]$ and $\alpha\in[0,1]$.

In particular, by (3.8) we get

$$\left|(1-\alpha)\left\{\left\langle\left(1_H-E_{\frac{a+b}{2}}\right)x,y\right\rangle\ln b+\left\langle E_{\frac{a+b}{2}}x,y\right\rangle\ln a\right\}\right.$$
$$\left.+\alpha\langle x,y\rangle\ln\left(\frac{a+b}{2}\right)-\langle\ln Ax,y\rangle\right|$$
$$\le\frac{1}{2}\max\{\alpha,1-\alpha\}\left(\frac{b}{a}-1\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\le\frac{1}{2}\max\{\alpha,1-\alpha\}\left(\frac{b}{a}-1\right)\|x\|\|y\| \tag{3.16}$$

for any $x,\ y\in H$ and $\alpha\in[0,1]$.

If we take $\alpha=1$ in (3.15) and (3.16), then we get

$$|\langle x,y\rangle\ln s-\langle\ln Ax,y\rangle|$$
$$\le\frac{1}{a}\left(\frac{b-a}{2}+\left|s-\frac{a+b}{2}\right|\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$
$$\le\frac{1}{a}\left(\frac{b-a}{2}+\left|s-\frac{a+b}{2}\right|\right)\|x\|\|y\| \tag{3.17}$$

for any $x, y \in H$, $s \in [a,b]$ and, in particular

$$\left|\langle x,y\rangle \ln\left(\frac{a+b}{2}\right) - \langle \ln Ax, y\rangle\right| \leq \frac{1}{2}\left(\frac{b}{a}-1\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \frac{1}{2}\left(\frac{b}{a}-1\right)\|x\|\,\|y\| \tag{3.18}$$

for any $x, y \in H$.

If we take $\alpha = 0$ in (3.15) and (3.16), then we get

$$\left|\langle (1_H - E_s)x,y\rangle \ln b + \langle E_s x,y\rangle \ln a - \langle \ln Ax,y\rangle\right| \leq \frac{1}{a}\left(\frac{b-a}{2}+\left|s-\frac{a+b}{2}\right|\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \frac{1}{a}\left(\frac{b-a}{2}+\left|s-\frac{a+b}{2}\right|\right)\|x\|\,\|y\| \tag{3.19}$$

for any $x, y \in H$, $s \in [a,b]$ and, in particular

$$\left|\left\langle\left(1_H - E_{\frac{a+b}{2}}\right)x,y\right\rangle \ln b + \left\langle E_{\frac{a+b}{2}}x,y\right\rangle \ln a - \langle \ln Ax,y\rangle\right| \leq \frac{1}{2}\left(\frac{b}{a}-1\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \frac{1}{2}\left(\frac{b}{a}-1\right)\|x\|\,\|y\| \tag{3.20}$$

for any $x, y \in H$.

If we take $\alpha = \frac{1}{2}$ in (3.15) and (3.16), then we get

$$\left|\frac{1}{2}\left\{\langle (1_H - E_s)x,y\rangle \ln b + \langle E_s x,y\rangle \ln a\right\} + \frac{1}{2}\langle x,y\rangle \ln s - \langle \ln Ax,y\rangle\right| \leq \frac{1}{2a}\left(\frac{b-a}{2}+\left|s-\frac{a+b}{2}\right|\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \frac{1}{2a}\left(\frac{b-a}{2}+\left|s-\frac{a+b}{2}\right|\right)\|x\|\,\|y\| \tag{3.21}$$

for any $x, y \in H$, $s \in [a,b]$ and, in particular

$$\left|\frac{1}{2}\left\{\left\langle\left(1_H - E_{\frac{a+b}{2}}\right)x,y\right\rangle \ln b + \left\langle E_{\frac{a+b}{2}}x,y\right\rangle \ln a\right\} + \frac{1}{2}\langle x,y\rangle \ln\left(\frac{a+b}{2}\right) - \langle \ln Ax,y\rangle\right| \leq \frac{1}{4}\left(\frac{b}{a}-1\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \frac{1}{4}\left(\frac{b}{a}-1\right)\|x\|\,\|y\| \tag{3.22}$$

for any $x, y \in H$.

4. Applications for Unitary Operators

A unitary operator is a bounded linear operator $U : H \to H$ on a Hilbert space H satisfying

$$U^{*}U = UU^{*} = 1_{H}$$

where U^{*} is the adjoint of U, and $1_{H} : H \to H$ is the identity operator. This property is equivalent to the following:

(i) U preserves the inner product $\langle \cdot, \cdot \rangle$ of the Hilbert space, i.e., for all vectors x and y in the Hilbert space, $\langle Ux, Uy \rangle = \langle x, y \rangle$ and
(ii) U is surjective.

The following result is well known [27, p. 275 - p. 276]:

Theorem 4 (Spectral Representation Theorem)**.** *Let U be a unitary operator on the Hilbert space H. Then there exists a family of projections $\{P_{\lambda}\}_{\lambda \in [0,2\pi]}$, called the spectral family of U, with the following properties*

a) $P_{\lambda} \leq P_{\lambda'}$ *for* $\lambda \leq \lambda'$;
b) $P_{0} = 0, P_{2\pi} = 1_{H}$ *and* $P_{\lambda+0} = P_{\lambda}$ *for all* $\lambda \in [0, 2\pi)$;
c) *We have the representation*

$$U = \int_{0}^{2\pi} \exp(i\lambda)\, dP_{\lambda}.$$

More generally, for every continuous complex-valued function φ defined on the unit circle $\mathcal{C}(0,1)$ there exists a unique operator $\varphi(U) \in \mathcal{B}(H)$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the inequality

$$\left\| \varphi(U) - \sum_{k=1}^{n} \varphi\left(\exp\left(i\lambda_{k}'\right)\right) \left[P_{\lambda_{k}} - P_{\lambda_{k-1}}\right] \right\| \leq \varepsilon$$

whenever

$$\begin{cases} 0 = \lambda_{1} < \ldots < \lambda_{n-1} < \lambda_{n} = 2\pi, \\ \lambda_{k} - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \lambda_{k}' \in [\lambda_{k-1}, \lambda_{k}] \text{ for } 1 \leq k \leq n \end{cases}$$

this means that

$$\varphi(U) = \int_{0}^{2\pi} \varphi(\exp(i\lambda))\, dP_{\lambda}, \tag{4.1}$$

where the integral is of Riemann-Stieltjes type.

Corollary 3. *With the assumptions of Theorem 4 for U, P_{λ} and φ we have the representations*

$$\varphi(U)x = \int_{0}^{2\pi} \varphi(\exp(i\lambda))\, dP_{\lambda}x \quad \text{for all } x \in H$$

and

$$\langle \varphi(U)x, y \rangle = \int_{0}^{2\pi} \varphi(\exp(i\lambda))\, d\langle P_{\lambda}x, y \rangle \quad \text{for all } x, y \in H. \tag{4.2}$$

In particular,

$$\langle \varphi(U)x, x \rangle = \int_{0}^{2\pi} \varphi(\exp(i\lambda))\, d\langle P_{\lambda}x, x \rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(U)x\|^2=\int_0^{2\pi}|\varphi(\exp(i\lambda))|^2\,d\|P_\lambda x\|^2 \quad \text{for all } x\in H.$$

On making use of an argument similar to the one in [23, Theorem 6], we have:

Lemma 3. *Let $\{P_\lambda\}_{\lambda\in[0,2\pi]}$ be the spectral family of the unitary operator U on the Hilbert space H. Then for any $x,\ y\in H$ and $0\le\alpha<\beta\le 2\pi$ we have the inequality*

$$\bigvee_{\alpha}^{\beta}\left(\left\langle P_{(\cdot)}x,y\right\rangle\right)\le\langle(P_\beta-P_\alpha)x,x\rangle^{1/2}\langle(P_\beta-P_\alpha)y,y\rangle^{1/2}, \tag{4.3}$$

where $\bigvee_{\alpha}^{\beta}\left(\left\langle P_{(\cdot)}x,y\right\rangle\right)$ denotes the total variation of the function $\left\langle P_{(\cdot)}x,y\right\rangle$ on $[\alpha,\beta]$.

In particular,

$$\bigvee_{0}^{2\pi}\left(\left\langle P_{(\cdot)}x,y\right\rangle\right)\le\|x\|\,\|y\| \tag{4.4}$$

for any $x,\ y\in H$.

We have:

Theorem 5. *Let U be a unitary operator on the Hilbert space H and $\{P_\lambda\}_{\lambda\in[0,2\pi]}$ the spectral family of projections of U. Also, assume that $\varphi:\mathcal{C}(0,1)\to\mathbb{C}$ is continuous on $\mathcal{C}(0,1)$. If $\varphi\circ\exp(i\cdot)$ is Lipschitzian with the constant $K>0$, then for all $s\in[0,2\pi]$*

$$\begin{aligned}|[(1-\alpha)\varphi(1)+\alpha\varphi\circ\exp(is)]\langle x,y\rangle-\langle\varphi(U)x,y\rangle|&\\ \le\max\{\alpha,1-\alpha\}K(\pi+|s-\pi|)\bigvee_{0}^{2\pi}\left(\left\langle P_{(\cdot)}x,y\right\rangle\right)&\\ \le\max\{\alpha,1-\alpha\}K(\pi+|s-\pi|)\|x\|\,\|y\|&\end{aligned} \tag{4.5}$$

for any $x,\ y\in H$.

In particular,

$$\begin{aligned}|[(1-\alpha)\varphi(1)+\alpha\varphi(-1)]\langle x,y\rangle-\langle\varphi(U)x,y\rangle|&\\ \le\max\{\alpha,1-\alpha\}\pi K\bigvee_{0}^{2\pi}\left(\left\langle P_{(\cdot)}x,y\right\rangle\right)\le\max\{\alpha,1-\alpha\}\pi L\|x\|\,\|y\|&\end{aligned} \tag{4.6}$$

for any $x,\ y\in H$.

The proof follows in a similar way to the one from Theorem 3 by utilising Theorem 4 and the inequality (2.11).

We observe that, for $\alpha=\frac{1}{2}$ we get from (4.5) and (4.6) that

$$\begin{aligned}\left|\left[\frac{\varphi(1)+\varphi\circ\exp(is)}{2}\right]\langle x,y\rangle-\langle\varphi(U)x,y\rangle\right|&\\ \le\frac{1}{2}K(\pi+|s-\pi|)\bigvee_{0}^{2\pi}\left(\left\langle P_{(\cdot)}x,y\right\rangle\right)\le\frac{1}{2}K(\pi+|s-\pi|)\|x\|\,\|y\|&\end{aligned} \tag{4.7}$$

for any $x,\ y\in H$ and $s\in[0,2\pi]$.

In particular,

$$\left|\left[\frac{\varphi(1)+\varphi(-1)}{2}\right]\langle x,y\rangle-\langle\varphi(U)x,y\rangle\right|\leq\frac{1}{2}\pi K\bigvee_{0}^{2\pi}\left(\langle P_{(\cdot)}x,y\rangle\right)\leq\frac{1}{2}\pi K\|x\|\|y\| \tag{4.8}$$

for any $x,\, y\in H$.

For the real numbers $a\neq\pm1,0$ consider the function $\varphi:\mathcal{C}(0,1)\to\mathbb{C}$, $\varphi_a(z)=\frac{1}{1-az}$. Observe that

$$|\varphi_a(z)-\varphi_a(w)|=\frac{|a||z-w|}{|1-az||1-aw|} \tag{4.9}$$

for any $z,\, w\in\mathcal{C}(0,1)$.

If $z=e^{it}$ with $t\in[0,2\pi]$, then we have

$$\begin{aligned}|1-az|^2&=1-2a\operatorname{Re}(\bar{z})+a^2|z|^2=1-2a\cos t+a^2\\&\geq1-2|a|+a^2=(1-|a|)^2\end{aligned}$$

therefore

$$\frac{1}{|1-az|}\leq\frac{1}{|1-|a||}\text{ and }\frac{1}{|1-aw|}\leq\frac{1}{|1-|a||} \tag{4.10}$$

for any $z,\, w\in\mathcal{C}(0,1)$.

Utilising (4.9) and (4.10) we deduce

$$|\varphi_a(z)-\varphi_a(w)|\leq\frac{|a|}{(1-|a|)^2}|z-w| \tag{4.11}$$

for any $z,\, w\in\mathcal{C}(0,1)$, showing that the function φ_a is Lipschitzian with the constant $L_a=\frac{|a|}{(1-|a|)^2}$ on the circle $\mathcal{C}(0,1)$.

If we take $z=e^{it}$ and $w=e^{is}$ with $t,\, s\in[0,2\pi]$ in (4.11) we get

$$\left|\varphi_a\left(e^{it}\right)-\varphi_a\left(e^{is}\right)\right|\leq\frac{|a|}{(1-|a|)^2}\left|e^{it}-e^{is}\right|. \tag{4.12}$$

Since

$$\begin{aligned}\left|e^{is}-e^{it}\right|^2&=\left|e^{is}\right|^2-2\operatorname{Re}\left(e^{i(s-t)}\right)+\left|e^{it}\right|^2\\&=2-2\cos(s-t)=4\sin^2\left(\frac{s-t}{2}\right)\end{aligned}$$

for any $t,\, s\in\mathbb{R}$, hence

$$\left|e^{is}-e^{it}\right|=2\left|\sin\left(\frac{s-t}{2}\right)\right|\leq|s-t| \tag{4.13}$$

for $t,\, s\in[0,2\pi]$.

Therefore by (4.12) and (4.13) we get

$$\left|\varphi_a\left(e^{it}\right)-\varphi_a\left(e^{is}\right)\right|\leq\frac{|a|}{(1-|a|)^2}|s-t| \tag{4.14}$$

for $t,\, s\in[0,2\pi]$, which shows that $\varphi_a\left(e^{i\cdot}\right)$ is Lipschitzian with the constant $K=\frac{|a|}{(1-|a|)^2}>0$ on $[0,2\pi]$.

Using the inequality (4.8) we have for $a \neq \pm 1, 0$ that

$$\left| \frac{1}{1-a^2} \langle x, y \rangle - \left\langle (1_H - aU)^{-1} x, y \right\rangle \right| \leq \frac{1}{2}\pi \frac{|a|}{(1-|a|)^2} \bigvee_0^{2\pi} \left(\left\langle P_{(\cdot)} x, y \right\rangle \right) \leq \frac{1}{2}\pi \frac{|a|}{(1-|a|)^2} \|x\| \|y\| \tag{4.15}$$

for any $x, y \in H$, where $\{P_\lambda\}_{\lambda \in [0,2\pi]}$ is the spectral family of the unitary operator U on the Hilbert space H.

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