ARITHMETIC MEAN-GEOMETRIC MEAN INEQUALITY UNDER ADDITIONAL ASSUMPTIONS

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ABSTRACT. We generalize the inequality $\frac{ab+bc+ac}{3} \le (\frac{a+b+c}{3})^2$ to n arbitrary positive real numbers and use that to obtain a non-homogenous version of the AM-GM inequality, given that their arithmetic average is the reciprocal of their harmonic average.

1. INTRODUCTION

The famous Arithmetic Mean-Geometric Mean Inequality, or simply AM-GM inequality, is perhaps the most frequent tool used in obtaining other inequalities needed in analysis or other areas of mathematics. It is stating that for arbitrary n positive real numbers $\{a_j\}_{j=1,2,..,n}$, we have

$$AM := \frac{a_1 + a_2 + \cdots + a_n}{n} \ge GM := \left(\prod_{j=1}^{n} a_j\right)^{\frac{1}{n}}. \tag{1}$$

Also, equality in (1) takes place if and only if all numbers are equal to each other. In what follows we are going to assume that $\{a_j\}_{j=1,2,..,n}$ is not a constant sequence. There are many proofs of this important inequality which are using various methods, ranging from induction to Lagrange multiplies (see [2] for a recent approach and [1] for whole collection of proofs). Let us observe that the inequality (1) is homogeneous. In this paper we are interested in a non-homogeneous version (1), of the form

$$AM \ge GM^{\alpha}, \ \alpha > 0, \tag{2}$$

under certain extra assumption on the numbers $\{a_j\}_{j=1,2,..,n}$. The extra assumption does not seem that natural, but if we introduce the Harmonic mean, defined as usual as

$$HM := \frac{n}{\sum_{j=1}^{n} \frac{1}{a_j}}$$

then we can write this extra condition in a more meaningful way:

$$AM = HM^{-1}. \tag{3}$$

Since we have $HM \le AM$ then we need to have $AM > 1$ and then $HM < 1$, otherwise $HM \le AM \le 1$ which attracts $HM = AM = 1$. In the last scenario, all numbers must be equal to one another and we excluded this situation. We may assume that $GM > 1$, otherwise (2) becomes trivial. This is the case, if $n = 2$, since $AM = HM^{-1}$ implies $GM = 1$. For this reason, we are going to assume that we have at least three numbers. If $n = 3$, let's say $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. We

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can choose a and b arbitrary positive numbers and then solve for c, since the map $f:(0,\infty)\to\mathbb{R}$, $f(x)=x-\frac{1}{x}$ is a bijection. In this case, let us show that

$$AM \geq GM^3.$$

This can be written as $a+b+c\geq 3abc$. The relation between a, b and c implies that $abc=\frac{ab+ac+bc}{a+b+c}$. Then the inequality above becomes $(a+b+c)^2\geq 3(ab+ac+bc)$ which is equivalent to $(a-b)^2+(b-c)^2+(a-c)^2\geq 0$. This insures that the inequality $AM\geq GM^3$ is true and equality takes place only if $a=b=c$. The case $n=4$ appeared as a proposed problem in [3] and that was our starting point for this note. We are interested in the following result.

Theorem 1.1. *For $n\geq 3$, if n positive real numbers $\{a_j\}_{j=1,2,..,n}$ satisfy*

$$\sum_{j=1}^{n} a_j=\sum_{j=1}^{n}\frac{1}{a_j}, \tag{4}$$

then

$$\frac{1}{n}\sum_{j=1}^{n} a_j \geq \max\left((\prod_{j=1}^{n} a_j)^{\frac{1}{n-2}}, (\prod_{j=1}^{n} a_j)^{\frac{-1}{n-2}}\right), \tag{5}$$

or

$$AM\geq \max(GM^{\frac{n}{n-2}}, GM^{\frac{-n}{n-2}}).$$

We observe that in proving (5), we may actually assume that none of the a_j are equal to 1 or $a_ia_j=1$ for some i and j in $[n]:=\{1,2,3,...,n\}$. It is interesting that for $n=4$, we found rational solutions for (3), such as $a_1=2$, $a_2=7$, $a_3=15$ and $a_4=\frac{3}{70}$, but no such solutions for $n=3$.

2. Proof of Theorem 1.1

Let us observe that (4) is invariant to the change $a_j\to 1/a_j$. As a result, we only need to prove

$$\frac{1}{n}\sum_{j=1}^{n} a_j \geq \left(\prod_{j=1}^{n} a_j\right)^{\frac{1}{n-2}}. \tag{6}$$

Let us introduce the notation: $\widehat{a_j}=\frac{1}{a_j}\prod_{k=1}^{n} a_k$. From the hypothesis we have

$$\prod_{k=1}^{n} a_k=\frac{\sum_{k=1}^{n}\widehat{a_k}}{\sum_{k=1}^{n} a_k}.$$

The (6) becomes equivalent to

$$\sum_{k=1}^{n}\widehat{a_k}\leq n\left(\frac{1}{n}\sum_{j=1}^{n} a_j\right)^{n-1}. \tag{7}$$

We observe that (7) is homogeneous. Let us show that (7) is true and independent of any other hypothesis.

Lemma 2.1. *For $n\geq 3$, if n positive real numbers $\{a_j\}_{j=1,2,..,n}$, then the inequality (7) takes place.*

To prove this, we are going to use induction. The basis case, $n = 3$, was argued in the Introduction. Let us assume the inequality (7) is true for n numbers ($n \geq 3$). Let us take $n+1$ positive numbers $\{b_j\}_{j=1,2,..,n,n+1}$. We need to prove that

$$\sum_{k=1}^{n+1} \widehat{b_k} \leq (n+1)\left(\frac{1}{n+1}\sum_{j=1}^{n+1} b_j\right)^n. \tag{8}$$

Since this is a homogeneous inequality we may assume that $\sum_{j=1}^{n+1} b_j = n+1$. So, the inequality (8) which we need to prove, becomes

$$\sum_{k=1}^{n+1} \widehat{b_k} \leq n+1. \tag{9}$$

We observe that

$$\sum_{k=1}^{n+1} \widehat{b_k} = b_1 b_2 ... b_n + b_{n+1} \sum_{k=1}^{n} \widetilde{b_k},$$

where $\widetilde{b_k}$ is $\frac{1}{b_j} \prod_{k=1}^{n} b_k$. Using the induction hypothesis we get

$$\sum_{k=1}^{n+1} \widehat{b_k} = b_1 b_2 ... b_n + b_{n+1} \sum_{k=1}^{n} \widetilde{b_k} \leq b_1 b_2 ... b_n + b_{n+1} n (\frac{1}{n}\sum_{j=1}^{n} b_j)^{n-1}.$$

Let us denote $b_{n+1} = x$. Also, using the AM-GM inequality, the above inequality can be continued as

$$\sum_{k=1}^{n+1} \widehat{b_k} \leq (\frac{1}{n}\sum_{j=1}^{n} b_j)^n + xn(\frac{1}{n}\sum_{j=1}^{n} b_j)^{n-1} = [\frac{1}{n}(n+1-x)]^n + xn[\frac{1}{n}(n+1-x)]^{n-1}.$$

In order to show (9), it is enough to prove that

$$[\frac{1}{n}(n+1-x)]^n + xn[\frac{1}{n}(n+1-x)]^{n-1} \leq n+1. \tag{10}$$

Let us then introduce the function $f(x) = [\frac{1}{n}(n+1-x)]^n + xn[\frac{1}{n}(n+1-x)]^{n-1}$ defined on $[0, n+1]$. This function can be written as

$$f(x) = \frac{n+1}{n}[\frac{1}{n}(n+1-x)]^{n-1}[(n-1)x+1].$$

If we differentiate f, we get

$$f'(x) = \frac{(n+1)(n-1)}{n}[\frac{1}{n}(n+1-x)]^{n-2}[\frac{(n+1-x)}{n} - \frac{(n-1)x+1}{n}]$$

and in factored form

$$f'(x) = \frac{(n^2-1)}{n^{n-1}}(n+1-x)^{n-2}(1-x).$$

This implies that f has a maximum at $x = 1$ on $[0, n+1]$. Since $f(1) = n+1$, we obtain (10).

References

[1] Peter S. Bullen, Handbook of means and their inequalities, Dordrecht: Kluwer Academic Foundation (2003)

[2] Marian Genčev, *On a proof of the inequality between the arithmetic and geometric means*, Amer. Math. Monthly, **125**(7), (2018), pp. 650-652

[3] Digby Smith, *Problem 4340*, Crux Mathematicorum **4**, (2018), p. 163

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