GENERAL THREE POINTS INEQUALITIES FOR WEIGHTED RIEMANN-STIELTJES INTEGRAL

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Abstract. In this paper we provide amongst others some simple error bounds in approximating the weighted Riemann-Stieltjes integral $\int_a^b f(t)\,g(t)\,dv(t)$ by the use of three points formula

$$f(b)\int_c^b g(s)\,dv(s)+f(a)\int_a^d g(s)\,dv(s)-f(x)\int_c^d g(t)\,dv(t)$$

where $x, c, d\in[a,b]$, $g, v:[a,b]\to\mathbb{C}$ under bounded variation and Lipschitzian assumptions for the function f and such that the involved Riemann-Stieltjes integrals exist.

1. Introduction

Assume that $u, f:[a,b]\to\mathbb{C}$ are bounded. If the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists, we write for simplicity, like in [1, p. 142] that $f\in\mathcal{R}_{\mathbb{C}}(u,[a,b])$, or $\mathcal{R}_{\mathbb{C}}(u)$ when the interval is implicitly known. If the functions u, f are real valued, then we write $f\in\mathcal{R}(u,[a,b])$, or $\mathcal{R}(u)$.

In order to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the use of a three points formula, namely to establish bounds for the *error functional*

$$\begin{aligned}T\Theta(f,u;a,b,x,\alpha):=&(1-\alpha)\{[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)\}\\&+\alpha[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t),\end{aligned}$$

where $\alpha\in[0,1]$ and $x\in[a,b]$, under bounded variation assumptions for the functions u and f and such that the involved Riemann-Stieltjes integral exists, in the recent paper [26] we have obtained the following result:

Theorem 1. *Let $f, u:[a,b]\to\mathbb{C}$ and $x\in[a,b]$ are such that $f\in\mathcal{R}_{\mathbb{C}}(u,[a,b])$. If f and u are of bounded variation, then*

$$\begin{aligned}(1.1)\quad &|T\Theta(f,u;a,b,x,\alpha)|\\&\le\alpha\left[\int_a^x\left(\bigvee_a^t(u)\right)d\left(\bigvee_a^t(f)\right)+\int_x^b\left(\bigvee_t^b(u)\right)d\left(\bigvee_a^t(f)\right)\right]\\&\quad+(1-\alpha)\left[\int_a^x\left(\bigvee_t^x(u)\right)d\left(\bigvee_a^t(f)\right)+\int_x^b\left(\bigvee_x^t(u)\right)d\left(\bigvee_a^t(f)\right)\right]\end{aligned}$$

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$$\leq \max\{\alpha, 1-\alpha\} \left(\bigvee_a^x (u) \bigvee_a^x (f) + \bigvee_x^b (u) \bigvee_x^b (f) \right)$$

$$\leq \frac{1}{2} \max\{\alpha, 1-\alpha\} \begin{cases} \left(\bigvee_a^b (u) + \left| \bigvee_a^x (u) - \bigvee_x^b (u) \right| \right) \bigvee_a^b (f) \\ \\ \left(\bigvee_a^b (f) + \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right) \bigvee_a^b (u) \end{cases}$$

$$\leq \max\{\alpha, 1-\alpha\} \bigvee_a^b (u) \bigvee_a^b (f).$$

In [27] we also obtained the following result in the case of Lipschitzian integrands:

Theorem 2. *Let $f, u : [a,b] \to \mathbb{C}$ and $x \in [a,b]$. If f is Lipschitzian with the constant $L > 0$, namely*

$$|f(t) - f(s)| \leq L|t-s| \text{ for all } t, \ s \in [a,b]$$

and u is of bounded variation, then $f \in \mathcal{R}_{\mathbb{C}}(u,[a,b])$ and

$$\begin{aligned} (1.2) \quad & |T\Theta(f,u;a,b,x,\alpha)| \\ & \leq \alpha L \left[\int_a^x \left(\bigvee_a^t (u) \right) dt + \int_x^b \left(\bigvee_t^b (u) \right) dt \right] \\ & + (1-\alpha) L \left[\int_a^x \left(\bigvee_t^x (u) \right) dt + \int_x^b \left(\bigvee_x^t (u) \right) dt \right] \\ & \leq \max\{\alpha, 1-\alpha\} L \left((x-a) \bigvee_a^x (u) + (b-x) \bigvee_x^b (u) \right) \\ & \leq \frac{1}{2} \max\{\alpha, 1-\alpha\} L \begin{cases} \left(\bigvee_a^b (u) + \left| \bigvee_a^x (u) - \bigvee_x^b (u) \right| \right) (b-a) \\ \\ (b-a+|2x-a-b|) \bigvee_a^b (u) \end{cases} \\ & \leq \max\{\alpha, 1-\alpha\} L (b-a) \bigvee_a^b (u). \end{aligned}$$

For various bounds on the error functional

$$D(f,u;a,b) := \int_a^b f(t)\, du(t) - \frac{1}{b-a}[u(b)-u(a)] \cdot \int_a^b f(t)\, dt$$

where f and u belong to different classes of function for which the Riemann-Stieltjes integral exists, see [22], [21], [20], and [8] and the references therein.

Bounds for the functional

$$\Theta(f,u;a,b,x) := \int_a^b f(t)\, du(t) - f(x)[u(b)-u(a)]$$

can be found in [15], [16] and [8], while for the functional

$$T(f,u;a,b,x) := \int_a^b f(t)\,du(t) - [u(b)-u(x)]\,f(b) - [u(x)-u(a)]\,f(a)$$

they may be found in [28], [8], [3] and [2]. The details are omitted.

In this paper we provide some simple error bounds in approximating the weighted Riemann-Stieltjes integral $\int_a^b f(t)\,g(t)\,dv(t)$ by the use of various general three points formulae out of which we mention the following one

$$f(b)\int_c^b g(s)\,dv(s) + f(a)\int_a^d g(s)\,dv(s) - f(x)\int_c^d g(t)\,dv(t),$$

where $x,\ c,\ d \in [a,b]$, $g,\ v : [a,b] \to \mathbb{C}$ under bounded variation and Lipschitzian assumptions for the function f and such that the involved Riemann-Stieltjes integral exist.

2. Some Preliminary Facts

The following properties of Riemann-Stieltjes integral are well know, [1, p. 158-159]:

Lemma 1. *Assume that $f \in \mathcal{R}_{\mathbb{C}}(\alpha,[a,b])$ and $f \in \mathcal{R}_{\mathbb{C}}(\alpha,[a,b])$ where $\alpha \in \mathcal{BV}_{\mathbb{C}}[a,b]$, namely of bounded variation on $[a,b]$. Define*

$$F(x) := \int_a^x f(t)\,d\alpha(t)$$

and

$$G(x) := \int_a^x g(t)\,d\alpha(t)$$

where $x \in [a,b]$.

Then $f \in \mathcal{R}_{\mathbb{C}}(G,[a,b])$, $g \in \mathcal{R}_{\mathbb{C}}(F,[a,b])$, $fg \in \mathcal{R}_{\mathbb{C}}(\alpha,[a,b])$ and we have

$$\int_a^b f(t)\,g(t)\,d\alpha(t) = \int_a^b f(t)\,dG(t) = \int_a^b g(t)\,dF(t).$$

If $c \in [a,b]$ and consider the integral $\int_c^t g(s)\,dv(s)$ that is assumed to exist for any $t \in [a,b]$, then

$$\int_c^t g(s)\,dv(s) = \int_a^t g(s)\,dv(s) - \int_a^c g(s)\,dv(s) \tag{2.1}$$

for any $t \in [a,b]$.

Indeed if $t \in [c,b]$, then (2.1) is obvious. If $t \in [a,c]$, then

$$\int_a^c g(s)\,dv(s) = \int_a^t g(s)\,dv(s) + \int_t^c g(s)\,dv(s),$$

which also gives (2.1).

We start with the following simple fact:

Lemma 2. *Let* $f,\ g,\ v:[a,b]\to\mathbb{C},\ \lambda,\ \mu\in\mathbb{C}$ *and* $x,\ c,\ d\in[a,b]$. *If* $g,\ fg\in\mathcal{R}_{\mathbb{C}}(v,[a,b])$, *then*

$$\left(\int_d^b g(s)\,dv(s)-\mu\right)f(b)+\left(\lambda+\int_a^c g(t)\,dv(t)\right)f(a)+\left(\int_c^d g(t)\,dv(t)+\mu-\lambda\right)f(x)-\int_a^b f(t)\,g(t)\,dv(t)=\int_a^x\left(\int_c^t g(s)\,dv(s)-\lambda\right)df(t)+\int_x^b\left(\int_d^t g(s)\,dv(s)-\mu\right)df(t).\tag{2.2}$$

In particular, for $\mu=\lambda$ *we have*

$$\left(\int_d^b g(s)\,dv(s)-\lambda\right)f(b)+\left(\lambda+\int_a^c g(t)\,dv(t)\right)f(a)+f(x)\int_c^d g(t)\,dv(t)-\int_a^b f(t)\,g(t)\,dv(t)=\int_a^x\left(\int_c^t g(s)\,dv(s)-\lambda\right)df(t)+\int_x^b\left(\int_d^t g(s)\,dv(s)-\lambda\right)df(t).\tag{2.3}$$

Proof. Assume that $x,\ c,\ d\in[a,b]$. Using the integration by parts formula for the Riemann-Stieltjes integral and Lemma 1, we have

$$\begin{aligned}\int_a^x\left(\int_c^t g(s)\,dv(s)-\lambda\right)df(t)&=\left(\int_c^t g(t)\,dv(t)-\lambda\right)f(t)\Big|_a^x-\int_a^x f(t)\,d\left(\int_c^t g(s)\,dv(s)-\lambda\right)\\&=\left(\int_c^x g(t)\,dv(t)-\lambda\right)f(x)-\left(\int_c^a g(t)\,dv(t)-\lambda\right)f(a)-\int_a^x f(t)\,g(t)\,dv(t)\\&=\left(\int_c^x g(t)\,dv(t)-\lambda\right)f(x)+\left(\lambda+\int_a^c g(t)\,dv(t)\right)f(a)-\int_a^x f(t)\,g(t)\,dv(t).\end{aligned}\tag{2.4}$$

In a similar way,

$$\begin{aligned}\int_x^b\left(\int_d^t g(s)\,dv(s)-\mu\right)df(t)&=\left(\int_d^t g(s)\,dv(s)-\mu\right)f(t)\Big|_x^b-\int_x^b f(t)\,d\left(\int_d^t g(s)\,dv(s)-\mu\right)\\&=\left(\int_d^b g(s)\,dv(s)-\mu\right)f(b)-\left(\int_d^x g(s)\,dv(s)-\mu\right)f(x)-\int_x^b f(t)\,g(t)\,dv(t)\\&=\left(\int_d^b g(s)\,dv(s)-\mu\right)f(b)+\left(\mu-\int_d^x g(s)\,dv(s)\right)f(x)-\int_x^b f(t)\,g(t)\,dv(t).\end{aligned}\tag{2.5}$$

If we add (2.4) and (2.5), we get

$$\int_a^x \left(\int_c^t g(s)\,dv(s) - \lambda\right) df(t) + \int_x^b \left(\int_d^t g(s)\,dv(s) - \mu\right) df(t)$$
$$= \left(\int_c^x g(t)\,dv(t) - \lambda\right) f(x) + \left(\lambda + \int_a^c g(t)\,dv(t)\right) f(a)$$
$$+ \left(\int_d^b g(s)\,dv(s) - \mu\right) f(b) + \left(\mu - \int_d^x g(s)\,dv(s)\right) f(x) - \int_a^b f(t)\,g(t)\,dv(t),$$

which is equivalent to the desired result (2.2) □

If we take $d = c$ above, we get:

Corollary 1. *Let $f,\ g,\ v,\ f : [a,b] \to \mathbb{C},\ \lambda,\ \mu \in \mathbb{C}$ and $x,\ c \in [a,b]$. If $g,\ fg \in \mathcal{R}_{\mathbb{C}}(v,[a,b])$, then*

$$\left(\int_c^b g(s)\,dv(s) - \mu\right) f(b) + \left(\lambda + \int_a^c g(t)\,dv(t)\right) f(a) \tag{2.6}$$
$$+ (\mu - \lambda) f(x) - \int_a^b f(t)\,g(t)\,dv(t)$$
$$= \int_a^x \left(\int_c^t g(s)\,dv(s) - \lambda\right) df(t) + \int_x^b \left(\int_c^t g(s)\,dv(s) - \mu\right) df(t).$$

In particular, for $\mu = \lambda$ we have

$$\left(\int_c^b g(s)\,dv(s) - \lambda\right) f(b) + \left(\lambda + \int_a^c g(t)\,dv(t)\right) f(a) \tag{2.7}$$
$$- \int_a^b f(t)\,g(t)\,dv(t) = \int_a^b \left(\int_c^t g(s)\,dv(s) - \lambda\right) df(t).$$

Remark 1. *If we take $g(t) = 1$, $t \in [a,b]$ in (2.2) and (2.3), then we get*

$$(v(b) - v(d) - \mu) f(b) + (\lambda + v(c) - v(a)) f(a) \tag{2.8}$$
$$+ (v(d) + \mu - \lambda - v(c)) f(x) - \int_a^b f(t)\,dv(t)$$
$$= \int_a^x (v(t) - v(c) - \lambda)\,df(t) + \int_x^b (v(t) - v(d) - \mu)\,df(t).$$

If we take $\beta = v(d) + \mu$ and $\gamma = \lambda + v(c)$, then by (2.8) we get

$$(v(b) - \beta) f(b) + (\gamma - v(a)) f(a) + (\beta - \gamma) f(x) - \int_a^b f(t)\,dv(t) \tag{2.9}$$
$$= \int_a^x (v(t) - \gamma)\,df(t) + \int_x^b (v(t) - \beta)\,df(t).$$

In particular, for $\beta = \gamma$ we get by (2.9) that

$$(v(b) - \beta) f(b) + (\beta - v(a)) f(a) - \int_a^b f(t)\,dv(t) = \int_a^b (v(t) - \beta)\,df(t). \tag{2.10}$$

Remark 2. *If we take $c=b$ and $d=a$ in Lemma 2, then we get*

$$\begin{aligned}(2.11)\quad &\left(\int_a^b g(s)\,dv(s)-\mu\right)f(b)+\left(\lambda+\int_a^b g(t)\,dv(t)\right)f(a)\\ &+\left(\mu-\lambda-\int_a^b g(t)\,dv(t)\right)f(x)-\int_a^b f(t)\,g(t)\,dv(t)\\ &=\int_x^b\left(\int_a^t g(s)\,dv(s)-\mu\right)df(t)-\int_a^x\left(\int_t^b g(s)\,dv(s)+\lambda\right)df(t).\end{aligned}$$

In particular, for $\mu=\lambda$ we obtain

$$\begin{aligned}(2.12)\quad &\left(\int_a^b g(s)\,dv(s)-\lambda\right)f(b)+\left(\lambda+\int_a^b g(t)\,dv(t)\right)f(a)\\ &-f(x)\int_a^b g(t)\,dv(t)-\int_a^b f(t)\,g(t)\,dv(t)\\ &=\int_x^b\left(\int_a^t g(s)\,dv(s)-\lambda\right)df(t)-\int_a^x\left(\int_t^b g(s)\,dv(s)+\lambda\right)df(t).\end{aligned}$$

If we take $c=a$ and $d=b$ in Lemma 2, then we get

$$\begin{aligned}(2.13)\quad &\lambda f(a)-\mu f(b)+(\mu-\lambda)f(x)-\int_a^b f(t)\,g(t)\,dv(t)\\ &=\int_a^x\left(\int_a^t g(s)\,dv(s)-\lambda\right)df(t)-\int_x^b\left(\int_t^b g(s)\,dv(s)+\mu\right)df(t).\end{aligned}$$

In particular, for $\mu=\lambda$ we obtain

$$\begin{aligned}(2.14)\quad &\lambda f(a)-\lambda f(b)-\int_a^b f(t)\,g(t)\,dv(t)\\ &=\int_a^x\left(\int_a^t g(s)\,dv(s)-\lambda\right)df(t)-\int_x^b\left(\int_t^b g(s)\,dv(s)+\lambda\right)df(t).\end{aligned}$$

Remark 3. *If we take $g(t)=1$, $t\in[a,b]$ in (2.11), then we get*

$$\begin{aligned}(2.15)\quad &(v(b)-v(a)-\mu)f(b)+(\lambda+v(b)-v(a))f(a)\\ &+(\mu-\lambda-v(b)+v(a))f(x)-\int_a^b f(t)\,dv(t)\\ &=\int_x^b(v(t)-v(a)-\mu)\,df(t)-\int_a^x(v(b)-v(t)+\lambda)\,df(t).\end{aligned}$$

If this equality we take $\mu=\lambda$, then we get

$$\begin{aligned}(2.16)\quad &(v(b)-v(a)-\lambda)f(b)+(\lambda+v(b)-v(a))f(a)\\ &-(v(b)-v(a))f(x)-\int_a^b f(t)\,dv(t)\\ &=\int_x^b(v(t)-v(a)-\lambda)\,df(t)-\int_a^x(v(b)-v(t)+\lambda)\,df(t).\end{aligned}$$

Remark 4. *If we take* $\mu = -\int_a^d g(s)\,dv(s)$ *and* $\lambda = \int_c^b g(t)\,dv(t)$ *in (2.2), then we get*

$$\begin{aligned}
&\int_a^b g(s)\,dv(s)\,(f(b)+f(a)) \\
&\quad+\left(\int_c^x g(t)\,dv(t) - \int_d^x g(s)\,dv(s) - \int_a^d g(s)\,dv(s) - \int_c^b g(t)\,dv(t)\right) f(x) \\
&\qquad\qquad - \int_a^b f(t)\,g(t)\,dv(t) \\
&\qquad = \int_a^x \left(\int_c^t g(s)\,dv(s) - \int_c^b g(t)\,dv(t)\right) df(t) \\
&\qquad\qquad + \int_x^b \left(\int_d^t g(s)\,dv(s) + \int_a^d g(s)\,dv(s)\right) df(t),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&[f(b)+f(a)-f(x)]\int_a^b g(s)\,dv(s) - \int_a^b f(t)\,g(t)\,dv(t) \\
&\qquad = \int_x^b \left(\int_a^t g(s)\,dv(s)\right) df(t) - \int_a^x \left(\int_t^b g(s)\,dv(s)\right) df(t).
\end{aligned} \tag{2.17}$$

If we take $\lambda = \int_c^d g(t)\,dv(t)$ *and* $\mu = -\int_c^d g(t)\,dv(t)$, *then by (2.2) we get*

$$\begin{aligned}
&\left(\int_d^b g(s)\,dv(s) + \int_c^d g(t)\,dv(t)\right) f(b) \\
&\qquad + \left(\int_c^d g(t)\,dv(t) + \int_a^c g(t)\,dv(t)\right) f(a) \\
&\quad+\left(\int_c^x g(t)\,dv(t) - \int_d^x g(s)\,dv(s) - \int_c^d g(t)\,dv(t) - \int_c^d g(t)\,dv(t)\right) f(x) \\
&\qquad\qquad - \int_a^b f(t)\,g(t)\,dv(t) \\
&\qquad = \int_a^x \left(\int_c^t g(s)\,dv(s) - \int_c^d g(t)\,dv(t)\right) df(t) \\
&\qquad\qquad + \int_x^b \left(\int_d^t g(s)\,dv(s) + \int_c^d g(t)\,dv(t)\right) df(t),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}(2.18)\quad & f(b)\int_c^b g(s)\,dv(s)+f(a)\int_a^d g(t)\,dv(t)-f(x)\int_c^d g(t)\,dv(t)\\ &\qquad -\int_a^b f(t)\,g(t)\,dv(t)\\ &=\int_a^x\left(\int_d^t g(s)\,dv(s)\right)df(t)+\int_x^b\left(\int_c^t g(s)\,dv(s)\right)df(t).\end{aligned}$$

Remark 5. *If we take* $\mu=\int_d^b g(s)\,dv(s)$ *and* $\lambda=-\int_a^c g(t)\,dv(t)$ *in (2.2), then we get*

$$\begin{aligned}&\left(\int_c^x g(t)\,dv(t)+\int_x^d g(s)\,dv(s)+\int_d^b g(s)\,dv(s)+\int_a^c g(t)\,dv(t)\right)f(x)\\ &\qquad -\int_a^b f(t)\,g(t)\,dv(t)\\ &=\int_a^x\left(\int_c^t g(s)\,dv(s)+\int_a^c g(t)\,dv(t)\right)df(t)\\ &\qquad +\int_x^b\left(\int_d^t g(s)\,dv(s)-\int_d^b g(s)\,dv(s)\right)df(t),\end{aligned}$$

which is equivalent to

$$\begin{aligned}(2.19)\quad & f(x)\int_a^b g(t)\,dv(t)-\int_a^b f(t)\,g(t)\,dv(t)\\ &=\int_a^x\left(\int_a^t g(s)\,dv(s)\right)df(t)-\int_x^b\left(\int_t^b g(s)\,dv(s)\right)df(t).\end{aligned}$$

3. Inequalities for Integrands of Bounded Variation

We have:

Theorem 3. *Assume that* $f:[a,b]\to\mathbb{C}$ *is of bounded variation,* $g,\ v:[a,b]\to\mathbb{C}$, $\lambda,\ \mu\in\mathbb{C}$ *and* $x,\ c,\ d\in[a,b]$ *are such that the Riemann-Stieltjes integrals below exist. Then*

$$\begin{aligned}(3.1)\quad & \left|\left(\int_d^b g(s)\,dv(s)-\mu\right)f(b)+\left(\lambda+\int_a^c g(t)\,dv(t)\right)f(a)\right.\\ &\qquad\left.+\left(\int_c^d g(t)\,dv(t)+\mu-\lambda\right)f(x)-\int_a^b f(t)\,g(t)\,dv(t)\right|\\ &\le\max_{t\in[a,x]}\left|\int_c^t g(s)\,dv(s)-\lambda\right|\bigvee_a^x(f)+\max_{t\in[x,b]}\left|\int_d^t g(s)\,dv(s)-\mu\right|\bigvee_x^b(f)\\ &\le\max\left\{\max_{t\in[a,x]}\left|\int_c^t g(s)\,dv(s)-\lambda\right|,\max_{t\in[x,b]}\left|\int_d^t g(s)\,dv(s)-\mu\right|\right\}\bigvee_a^b(f).\end{aligned}$$

In particular, for $\mu=\lambda$ we have

$$\begin{aligned}(3.2)\quad&\left|\left(\int_d^b g(s)\,dv(s)-\lambda\right)f(b)+\left(\lambda+\int_a^c g(t)\,dv(t)\right)f(a)\right.\\&\qquad\left.+f(x)\int_c^d g(t)\,dv(t)-\int_a^b f(t)\,g(t)\,dv(t)\right|\\&\le\max_{t\in[a,x]}\left|\int_c^t g(s)\,dv(s)-\lambda\right|\bigvee_a^x(f)+\max_{t\in[x,b]}\left|\int_d^t g(s)\,dv(s)-\lambda\right|\bigvee_x^b(f)\\&\le\max\left\{\max_{t\in[a,x]}\left|\int_c^t g(s)\,dv(s)-\lambda\right|,\max_{t\in[x,b]}\left|\int_d^t g(s)\,dv(s)-\lambda\right|\right\}\bigvee_a^b(f).\end{aligned}$$

Proof. It is well known that, if $p:[a,b]\to\mathbb{C}$ is continuous and $v:[a,b]\to\mathbb{C}$ of bounded variation, then [1]

$$(3.3)\qquad\left|\int_a^b p(t)\,dv(t)\right|\le\int_a^b|p(t)|\,d\left(\bigvee_a^t(v)\right)\le\max_{t\in[a,b]}|p(t)|\bigvee_a^b(v).$$

By using the identity (2.2) and the property (3.3) we get

$$\begin{aligned}&\left|\left(\int_d^b g(s)\,dv(s)-\mu\right)f(b)+\left(\lambda+\int_a^c g(t)\,dv(t)\right)f(a)\right.\\&\qquad\left.+\left(\int_c^d g(t)\,dv(t)+\mu-\lambda\right)f(x)-\int_a^b f(t)\,g(t)\,dv(t)\right|\\&\le\left|\int_a^x\left(\int_c^t g(s)\,dv(s)-\lambda\right)df(t)\right|+\left|\int_x^b\left(\int_d^t g(s)\,dv(s)-\mu\right)df(t)\right|\\&\le\int_a^x\left|\int_c^t g(s)\,dv(s)-\lambda\right|d\left(\bigvee_a^t(f)\right)+\int_x^b\left|\int_d^t g(s)\,dv(s)-\mu\right|d\left(\bigvee_x^t(f)\right)\\&\le\max_{t\in[a,x]}\left|\int_c^t g(s)\,dv(s)-\lambda\right|\bigvee_a^x(f)+\max_{t\in[x,b]}\left|\int_d^t g(s)\,dv(s)-\mu\right|\bigvee_x^b(f),\end{aligned}$$

which proves the first inequality in (3.1).

Observe that

$$\begin{aligned}&\max_{t\in[a,x]}\left|\int_c^t g(s)\,dv(s)-\lambda\right|\bigvee_a^x(f)+\max_{t\in[x,b]}\left|\int_d^t g(s)\,dv(s)-\mu\right|\bigvee_x^b(f)\\&\le\max\left\{\max_{t\in[a,x]}\left|\int_c^t g(s)\,dv(s)-\lambda\right|,\max_{t\in[x,b]}\left|\int_d^t g(s)\,dv(s)-\mu\right|\right\}\left(\bigvee_a^x(f)+\bigvee_x^b(f)\right)\\&=\max\left\{\max_{t\in[a,x]}\left|\int_c^t g(s)\,dv(s)-\lambda\right|,\max_{t\in[x,b]}\left|\int_d^t g(s)\,dv(s)-\mu\right|\right\}\bigvee_a^b(f),\end{aligned}$$

which proves the last part of (3.1). □

Remark 6. *If $m \in (a,b)$ is such that $\bigvee_a^m (f) = \bigvee_m^b (f)$, then under the assumptions of Theorem 3 we have the inequalities*

$$\begin{aligned} &\left| \left(\int_d^b g(s)\,dv(s) - \mu \right) f(b) + \left(\lambda + \int_a^c g(t)\,dv(t) \right) f(a) \right. \\ &\qquad \left. + \left(\int_c^d g(t)\,dv(t) + \mu - \lambda \right) f(m) - \int_a^b f(t)\,g(t)\,dv(t) \right| \\ &\qquad \le \frac{1}{2} \left[\max_{t\in[a,m]} \left| \int_c^t g(s)\,dv(s) - \lambda \right| + \max_{t\in[m,b]} \left| \int_d^t g(s)\,dv(s) - \mu \right| \right] \bigvee_a^b (f) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\left| \left(\int_d^b g(s)\,dv(s) - \lambda \right) f(b) + \left(\lambda + \int_a^c g(t)\,dv(t) \right) f(a) \right. \\ &\qquad \left. + f(m) \int_c^d g(t)\,dv(t) - \int_a^b f(t)\,g(t)\,dv(t) \right| \\ &\qquad \le \frac{1}{2} \left[\max_{t\in[a,m]} \left| \int_c^t g(s)\,dv(s) - \lambda \right| + \max_{t\in[m,b]} \left| \int_d^t g(s)\,dv(s) - \lambda \right| \right] \bigvee_a^b (f). \end{aligned} \tag{3.5}$$

Corollary 2. *With the assumptions of Theorem 3, and if $d = c$, then*

$$\begin{aligned} &\left| \left(\int_c^b g(s)\,dv(s) - \mu \right) f(b) + \left(\lambda + \int_a^c g(t)\,dv(t) \right) f(a) \right. \\ &\qquad \left. + (\mu - \lambda) f(x) - \int_a^b f(t)\,g(t)\,dv(t) \right| \\ &\le \max_{t\in[a,x]} \left| \int_c^t g(s)\,dv(s) - \lambda \right| \bigvee_a^x (f) + \max_{t\in[x,b]} \left| \int_c^t g(s)\,dv(s) - \mu \right| \bigvee_x^b (f) \\ &\qquad \le \max \left\{ \max_{t\in[a,x]} \left| \int_c^t g(s)\,dv(s) - \lambda \right|, \max_{t\in[x,b]} \left| \int_c^t g(s)\,dv(s) - \mu \right| \right\} \bigvee_a^b (f). \end{aligned} \tag{3.6}$$

In particular, for $\mu = \lambda$, we get

$$\begin{aligned} &\left| \left(\int_c^b g(s)\,dv(s) - \lambda \right) f(b) + \left(\lambda + \int_a^c g(t)\,dv(t) \right) f(a) \right. \\ &\qquad \left. - \int_a^b f(t)\,g(t)\,dv(t) \right| \\ &\le \max_{t\in[a,x]} \left| \int_c^t g(s)\,dv(s) - \lambda \right| \bigvee_a^x (f) + \max_{t\in[x,b]} \left| \int_c^t g(s)\,dv(s) - \lambda \right| \bigvee_x^b (f) \\ &\qquad \le \max \left\{ \max_{t\in[a,x]} \left| \int_c^t g(s)\,dv(s) - \lambda \right|, \max_{t\in[x,b]} \left| \int_c^t g(s)\,dv(s) - \lambda \right| \right\} \bigvee_a^b (f). \end{aligned} \tag{3.7}$$

Remark 7. *If we take $c = x$ in Corollary 2, then we get*

$$\begin{aligned}(3.8)\quad &\left|\left(\int_x^b g(s)\,dv(s)-\mu\right)f(b)+\left(\lambda+\int_a^x g(t)\,dv(t)\right)f(a)\right.\\ &\qquad\left.+(\mu-\lambda)f(x)-\int_a^b f(t)\,g(t)\,dv(t)\right|\\ &\le \max_{t\in[a,x]}\left|\int_x^t g(s)\,dv(s)-\lambda\right|\bigvee_a^x(f)+\max_{t\in[x,b]}\left|\int_x^t g(s)\,dv(s)-\mu\right|\bigvee_x^b(f)\\ &\le \max\left\{\max_{t\in[a,x]}\left|\int_x^t g(s)\,dv(s)-\lambda\right|,\max_{t\in[x,b]}\left|\int_x^t g(s)\,dv(s)-\mu\right|\right\}\bigvee_a^b(f).\end{aligned}$$

In particular, for $\mu = \lambda$, we get

$$\begin{aligned}(3.9)\quad &\left|\left(\int_x^b g(s)\,dv(s)-\lambda\right)f(b)+\left(\lambda+\int_a^x g(t)\,dv(t)\right)f(a)\right.\\ &\qquad\left.-\int_a^b f(t)\,g(t)\,dv(t)\right|\\ &\le \max_{t\in[a,x]}\left|\int_x^t g(s)\,dv(s)-\lambda\right|\bigvee_a^x(f)+\max_{t\in[x,b]}\left|\int_x^t g(s)\,dv(s)-\lambda\right|\bigvee_x^b(f)\\ &\le \max_{t\in[a,b]}\left|\int_x^t g(s)\,dv(s)-\lambda\right|\bigvee_a^b(f).\end{aligned}$$

Corollary 3. *With the assumptions of Theorem 3, and if $c = b$ and $d = a$, then*

$$\begin{aligned}(3.10)\quad &\left|\left(\int_a^b g(s)\,dv(s)-\mu\right)f(b)+\left(\lambda+\int_a^b g(t)\,dv(t)\right)f(a)\right.\\ &\qquad\left.+\left(\mu-\lambda-\int_a^b g(t)\,dv(t)\right)f(x)-\int_a^b f(t)\,g(t)\,dv(t)\right|\\ &\le \max_{t\in[a,x]}\left|\int_t^b g(s)\,dv(s)+\lambda\right|\bigvee_a^x(f)+\max_{t\in[x,b]}\left|\int_a^t g(s)\,dv(s)-\mu\right|\bigvee_x^b(f)\\ &\le \max\left\{\max_{t\in[a,x]}\left|\int_t^b g(s)\,dv(s)+\lambda\right|,\max_{t\in[x,b]}\left|\int_a^t g(s)\,dv(s)-\mu\right|\right\}\bigvee_a^b(f).\end{aligned}$$

In particular, for $\mu = \lambda$ we have

$$\left| \left(\int_a^b g(s)\, dv(s) - \lambda \right) f(b) + \left(\lambda + \int_a^b g(t)\, dv(t) \right) f(a) - \int_a^b g(t)\, dv(t)\, f(x) - \int_a^b f(t)\, g(t)\, dv(t) \right| \tag{3.11}$$
$$\leq \max_{t\in[a,x]} \left| \int_t^b g(s)\, dv(s) + \lambda \right| \bigvee_a^x (f) + \max_{t\in[x,b]} \left| \int_a^t g(s)\, dv(s) - \lambda \right| \bigvee_x^b (f)$$
$$\leq \max \left\{ \max_{t\in[a,x]} \left| \int_t^b g(s)\, dv(s) + \lambda \right|, \max_{t\in[x,b]} \left| \int_a^t g(s)\, dv(s) - \lambda \right| \right\} \bigvee_a^b (f).$$

Corollary 4. *With the assumptions of Theorem 3 and if the Riemann-Stieltjes integrals below exist, then*

$$\left| f(b) \int_c^b g(s)\, dv(s) + f(a) \int_a^d g(t)\, dv(t) - f(x) \int_c^d g(t)\, dv(t) - \int_a^b f(t)\, g(t)\, dv(t) \right| \tag{3.12}$$
$$\leq \max_{t\in[a,x]} \left| \int_d^t g(s)\, dv(s) \right| \bigvee_a^x (f) + \max_{t\in[x,b]} \left| \int_c^t g(s)\, dv(s) \right| \bigvee_x^b (f)$$
$$\leq \max \left\{ \max_{t\in[a,x]} \left| \int_d^t g(s)\, dv(s) \right|, \max_{t\in[x,b]} \left| \int_c^t g(s)\, dv(s) \right| \right\} \bigvee_a^b (f).$$

In particular, for $c = a$ and $d = b$, we have

$$\left| [f(b) + f(a) - f(x)] \int_a^b g(t)\, dv(t) - \int_a^b f(t)\, g(t)\, dv(t) \right| \tag{3.13}$$
$$\leq \max_{t\in[a,x]} \left| \int_t^b g(s)\, dv(s) \right| \bigvee_a^x (f) + \max_{t\in[x,b]} \left| \int_a^t g(s)\, dv(s) \right| \bigvee_x^b (f)$$
$$\leq \max \left\{ \max_{t\in[a,x]} \left| \int_t^b g(s)\, dv(s) \right|, \max_{t\in[x,b]} \left| \int_a^t g(s)\, dv(s) \right| \right\} \bigvee_a^b (f)$$

for $c = b$ and $d = a$, we have

$$\left| f(x) \int_a^b g(t)\, dv(t) - \int_a^b f(t)\, g(t)\, dv(t) \right| \tag{3.14}$$
$$\leq \max_{t\in[a,x]} \left| \int_a^t g(s)\, dv(s) \right| \bigvee_a^x (f) + \max_{t\in[x,b]} \left| \int_t^b g(s)\, dv(s) \right| \bigvee_x^b (f)$$
$$\leq \max \left\{ \max_{t\in[a,x]} \left| \int_a^t g(s)\, dv(s) \right|, \max_{t\in[x,b]} \left| \int_t^b g(s)\, dv(s) \right| \right\} \bigvee_a^b (f)$$

and for $c = d = x$

$$\left|f(b)\int_x^b g(s)\,dv(s) + f(a)\int_a^x g(t)\,dv(t) - \int_a^b f(t)\,g(t)\,dv(t)\right| \tag{3.15}$$
$$\leq \max_{t\in[a,x]}\left|\int_t^x g(s)\,dv(s)\right| \bigvee_a^x(f) + \max_{t\in[x,b]}\left|\int_x^t g(s)\,dv(s)\right| \bigvee_x^b(f)$$
$$\leq \max_{t\in[a,b]}\left|\int_t^x g(s)\,dv(s)\right| \bigvee_a^b(f).$$

Remark 8. *If* $m \in (a,b)$ *is such that* $\bigvee_a^m(f) = \bigvee_m^b(f)$*, then under the assumptions of Corollary 4, we have*

$$\left|f(b)\int_c^b g(s)\,dv(s) + f(a)\int_a^d g(t)\,dv(t) - f(m)\int_c^d g(t)\,dv(t) - \int_a^b f(t)\,g(t)\,dv(t)\right| \tag{3.16}$$
$$\leq \frac{1}{2}\left[\max_{t\in[a,m]}\left|\int_d^t g(s)\,dv(s)\right| + \max_{t\in[m,b]}\left|\int_c^t g(s)\,dv(s)\right|\right] \bigvee_a^b(f),$$

$$\left|[f(b)+f(a)-f(m)]\int_a^b g(t)\,dv(t) - \int_a^b f(t)\,g(t)\,dv(t)\right| \tag{3.17}$$
$$\leq \frac{1}{2}\left[\max_{t\in[a,m]}\left|\int_t^b g(s)\,dv(s)\right| + \max_{t\in[m,b]}\left|\int_a^t g(s)\,dv(s)\right|\right] \bigvee_a^b(f),$$

$$\left|f(m)\int_a^b g(t)\,dv(t) - \int_a^b f(t)\,g(t)\,dv(t)\right| \tag{3.18}$$
$$\leq \frac{1}{2}\max\left\{\max_{t\in[a,m]}\left|\int_a^t g(s)\,dv(s)\right|, \max_{t\in[m,b]}\left|\int_t^b g(s)\,dv(s)\right|\right\} \bigvee_a^b(f)$$

and

$$\left|f(b)\int_m^b g(s)\,dv(s) + f(a)\int_a^m g(t)\,dv(t) - \int_a^b f(t)\,g(t)\,dv(t)\right| \tag{3.19}$$
$$\leq \frac{1}{2}\left[\max_{t\in[a,m]}\left|\int_t^m g(s)\,dv(s)\right| + \max_{t\in[m,b]}\left|\int_m^t g(s)\,dv(s)\right|\right] \bigvee_a^b(f).$$

Using the equalities (2.9) and (2.10) one can obtain various inequalities as in the recent paper [26]. The details are omitted.

4. Inequalities for Lipschitzian Integrands

We say that the function $f : [a,b] \to \mathbb{C}$ is *Lipschitzian* with constant $L > 0$ if

$$|f(t) - f(s)| \le L|t-s| \text{ for all } t,\ s \in [a,b]. \tag{4.1}$$

Theorem 4. *Assume that $f : [a,b] \to \mathbb{C}$ is Lipschitzian with constant $L > 0$, g, $v : [a,b] \to \mathbb{C}$, λ, $\mu \in \mathbb{C}$ and x, c, $d \in [a,b]$ are such that the Riemann-Stieltjes integrals below exist. Then*

$$\begin{aligned}
&\left|\left(\int_d^b g(s)\,dv(s) - \mu\right) f(b) + \left(\lambda + \int_a^c g(t)\,dv(t)\right) f(a)\right.\\
&\qquad \left. + \left(\int_c^d g(t)\,dv(t) + \mu - \lambda\right) f(x) - \int_a^b f(t)\,g(t)\,dv(t)\right| \\
&\le L\left[\max_{t\in[a,x]}\left|\int_c^t g(s)\,dv(s) - \lambda\right|(x-a) + \max_{t\in[x,b]}\left|\int_d^t g(s)\,dv(s) - \mu\right|(b-x)\right] \\
&\le L\max\left\{\max_{t\in[a,x]}\left|\int_c^t g(s)\,dv(s) - \lambda\right|, \max_{t\in[x,b]}\left|\int_d^t g(s)\,dv(s) - \mu\right|\right\}(b-a).
\end{aligned} \tag{4.2}$$

In particular, for $\mu = \lambda$ we have

$$\begin{aligned}
&\left|\left(\int_d^b g(s)\,dv(s) - \lambda\right) f(b) + \left(\lambda + \int_a^c g(t)\,dv(t)\right) f(a)\right.\\
&\qquad \left. + f(x)\int_c^d g(t)\,dv(t) - \int_a^b f(t)\,g(t)\,dv(t)\right| \\
&\le L\left[\max_{t\in[a,x]}\left|\int_c^t g(s)\,dv(s) - \lambda\right|(x-a) + \max_{t\in[x,b]}\left|\int_d^t g(s)\,dv(s) - \lambda\right|(b-x)\right] \\
&\le L\max\left\{\max_{t\in[a,x]}\left|\int_c^t g(s)\,dv(s) - \lambda\right|, \max_{t\in[x,b]}\left|\int_d^t g(s)\,dv(s) - \lambda\right|\right\}(b-a).
\end{aligned} \tag{4.3}$$

Proof. It is well known that, if $p : [a,b] \to \mathbb{C}$ is Riemann integrable and $v : [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, then

$$\left|\int_a^b p(t)\,dv(t)\right| \le L\int_a^b |p(t)|\,dt. \tag{4.4}$$

By using the identity (2.2) and the property (4.4) we get

$$\begin{aligned}
&\left|\left(\int_d^b g(s)\,dv(s)-\mu\right)f(b)+\left(\lambda+\int_a^c g(t)\,dv(t)\right)f(a)\right.\\
&\qquad\left.+\left(\int_c^d g(t)\,dv(t)+\mu-\lambda\right)f(x)-\int_a^b f(t)\,g(t)\,dv(t)\right|\\
&\le\left|\int_a^x\left(\int_c^t g(s)\,dv(s)-\lambda\right)df(t)\right|+\left|\int_x^b\left(\int_d^t g(s)\,dv(s)-\mu\right)df(t)\right|\\
&\le L\int_a^x\left|\int_c^t g(s)\,dv(s)-\lambda\right|dt+L\int_x^b\left|\int_d^t g(s)\,dv(s)-\mu\right|dt\\
&\le L\max_{t\in[a,x]}\left|\int_c^t g(s)\,dv(s)-\lambda\right|(x-a)+\max_{t\in[x,b]}\left|\int_d^t g(s)\,dv(s)-\mu\right|(b-x),
\end{aligned}$$

which proves the first inequality in (4.2).
The rest is obvious. □

Remark 9. *For $x=\frac{a+b}{2}$ in (4.2) and (4.3) we get, under the assumptions of Theorem 4, that*

$$\begin{aligned}
(4.5)\quad&\left|\left(\int_d^b g(s)\,dv(s)-\mu\right)f(b)+\left(\lambda+\int_a^c g(t)\,dv(t)\right)f(a)\right.\\
&\qquad\left.+\left(\int_c^d g(t)\,dv(t)+\mu-\lambda\right)f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,g(t)\,dv(t)\right|\\
&\le\frac{1}{2}L\left[\max_{t\in\left[a,\frac{a+b}{2}\right]}\left|\int_c^t g(s)\,dv(s)-\lambda\right|+\max_{t\in\left[\frac{a+b}{2},b\right]}\left|\int_d^t g(s)\,dv(s)-\mu\right|\right](b-a).
\end{aligned}$$

In particular, for $\mu=\lambda$ we have

$$\begin{aligned}
(4.6)\quad&\left|\left(\int_d^b g(s)\,dv(s)-\lambda\right)f(b)+\left(\lambda+\int_a^c g(t)\,dv(t)\right)f(a)\right.\\
&\qquad\left.+f\left(\frac{a+b}{2}\right)\int_c^d g(t)\,dv(t)-\int_a^b f(t)\,g(t)\,dv(t)\right|\\
&\le\frac{1}{2}L\left[\max_{t\in\left[a,\frac{a+b}{2}\right]}\left|\int_c^t g(s)\,dv(s)-\lambda\right|+\max_{t\in\left[\frac{a+b}{2},b\right]}\left|\int_d^t g(s)\,dv(s)-\lambda\right|\right](b-a).
\end{aligned}$$

Corollary 5. *With the assumptions of Theorem 4, and if $d=c$, then*

$$\begin{aligned}(4.7)\quad &\left|\left(\int_c^b g(s)\,dv(s)-\mu\right)f(b)+\left(\lambda+\int_a^c g(t)\,dv(t)\right)f(a)\right.\\&\qquad\left.+(\mu-\lambda)f(x)-\int_a^b f(t)\,g(t)\,dv(t)\right|\\&\le\max_{t\in[a,x]}\left|\int_c^t g(s)\,dv(s)-\lambda\right|\bigvee_a^x(f)+\max_{t\in[x,b]}\left|\int_c^t g(s)\,dv(s)-\mu\right|\bigvee_x^b(f)\\&\le\max\left\{\max_{t\in[a,x]}\left|\int_c^t g(s)\,dv(s)-\lambda\right|,\max_{t\in[x,b]}\left|\int_c^t g(s)\,dv(s)-\mu\right|\right\}\bigvee_a^b(f).\end{aligned}$$

In particular, for $\mu=\lambda$, we get

$$\begin{aligned}(4.8)\quad &\left|\left(\int_c^b g(s)\,dv(s)-\lambda\right)f(b)+\left(\lambda+\int_a^c g(t)\,dv(t)\right)f(a)\right.\\&\qquad\left.-\int_a^b f(t)\,g(t)\,dv(t)\right|\\&\le L\left[\max_{t\in[a,x]}\left|\int_c^t g(s)\,dv(s)-\lambda\right|(x-a)+\max_{t\in[x,b]}\left|\int_c^t g(s)\,dv(s)-\lambda\right|(b-x)\right]\\&\le L\max\left\{\max_{t\in[a,x]}\left|\int_c^t g(s)\,dv(s)-\lambda\right|,\max_{t\in[x,b]}\left|\int_c^t g(s)\,dv(s)-\lambda\right|\right\}(b-a).\end{aligned}$$

Remark 10. *If we take $c=x$ in Corollary 5, then we get*

$$\begin{aligned}(4.9)\quad &\left|\left(\int_x^b g(s)\,dv(s)-\mu\right)f(b)+\left(\lambda+\int_a^x g(t)\,dv(t)\right)f(a)\right.\\&\qquad\left.+(\mu-\lambda)f(x)-\int_a^b f(t)\,g(t)\,dv(t)\right|\\&\le L\left[\max_{t\in[a,x]}\left|\int_x^t g(s)\,dv(s)-\lambda\right|(x-a)+\max_{t\in[x,b]}\left|\int_x^t g(s)\,dv(s)-\mu\right|(b-x)\right]\\&\le L\max\left\{\max_{t\in[a,x]}\left|\int_x^t g(s)\,dv(s)-\lambda\right|,\max_{t\in[x,b]}\left|\int_x^t g(s)\,dv(s)-\mu\right|\right\}(b-a).\end{aligned}$$

In particular, for $\mu = \lambda$, we get

$$\begin{aligned}
&\left|\left(\int_x^b g(s)\,dv(s) - \lambda\right) f(b) + \left(\lambda + \int_a^x g(t)\,dv(t)\right) f(a) - \int_a^b f(t)\,g(t)\,dv(t)\right| \\
&\leq L\left[\max_{t\in[a,x]}\left|\int_x^t g(s)\,dv(s) - \lambda\right|(x-a) + \max_{t\in[x,b]}\left|\int_x^t g(s)\,dv(s) - \lambda\right|(b-x)\right] \\
&\leq L\max_{t\in[a,b]}\left|\int_x^t g(s)\,dv(s) - \lambda\right|(b-a).
\end{aligned} \tag{4.10}$$

Corollary 6. *With the assumptions of Theorem 4, and if $c = b$ and $d = a$, then*

$$\begin{aligned}
&\left|\left(\int_a^b g(s)\,dv(s) - \mu\right) f(b) + \left(\lambda + \int_a^b g(t)\,dv(t)\right) f(a)\right. \\
&\qquad \left. + \left(\mu - \lambda - \int_a^b g(t)\,dv(t)\right) f(x) - \int_a^b f(t)\,g(t)\,dv(t)\right| \\
&\leq L\left[\max_{t\in[a,x]}\left|\int_t^b g(s)\,dv(s) + \lambda\right|(x-a) + \max_{t\in[x,b]}\left|\int_a^t g(s)\,dv(s) - \mu\right|(b-x)\right] \\
&\leq L\max\left\{\max_{t\in[a,x]}\left|\int_t^b g(s)\,dv(s) + \lambda\right|, \max_{t\in[x,b]}\left|\int_a^t g(s)\,dv(s) - \mu\right|\right\}(b-a).
\end{aligned} \tag{4.11}$$

In particular, for $\mu = \lambda$ we have

$$\begin{aligned}
&\left|\left(\int_a^b g(s)\,dv(s) - \lambda\right) f(b) + \left(\lambda + \int_a^b g(t)\,dv(t)\right) f(a)\right. \\
&\qquad \left. - f(x)\int_a^b g(t)\,dv(t) - \int_a^b f(t)\,g(t)\,dv(t)\right| \\
&\leq L\left[\max_{t\in[a,x]}\left|\int_t^b g(s)\,dv(s) + \lambda\right|(x-a) + \max_{t\in[x,b]}\left|\int_a^t g(s)\,dv(s) - \lambda\right|(b-x)\right] \\
&\leq L\max\left\{\max_{t\in[a,x]}\left|\int_t^b g(s)\,dv(s) + \lambda\right|, \max_{t\in[x,b]}\left|\int_a^t g(s)\,dv(s) - \lambda\right|\right\}(b-a).
\end{aligned} \tag{4.12}$$

Corollary 7. *With the assumptions of Theorem 4 and if the Riemann-Stieltjes integrals below exist, then*

$$\begin{aligned}(4.13)\quad &\left|f(b)\int_c^b g(s)\,dv(s)+f(a)\int_a^d g(t)\,dv(t)-f(x)\int_c^d g(t)\,dv(t)\right.\\ &\qquad\left.-\int_a^b f(t)\,g(t)\,dv(t)\right|\\ &\le L\left[\max_{t\in[a,x]}\left|\int_d^t g(s)\,dv(s)\right|(x-a)+\max_{t\in[x,b]}\left|\int_c^t g(s)\,dv(s)\right|(b-x)\right]\\ &\le L\max\left\{\max_{t\in[a,x]}\left|\int_d^t g(s)\,dv(s)\right|,\max_{t\in[x,b]}\left|\int_c^t g(s)\,dv(s)\right|\right\}(b-a).\end{aligned}$$

In particular, for $c=a$ and $d=b$, we have

$$\begin{aligned}(4.14)\quad &\left|[f(b)+f(a)-f(x)]\int_a^b g(t)\,dv(t)-\int_a^b f(t)\,g(t)\,dv(t)\right|\\ &\le \max_{t\in[a,x]}\left|\int_t^b g(s)\,dv(s)\right|\bigvee_a^x(f)+\max_{t\in[x,b]}\left|\int_a^t g(s)\,dv(s)\right|\bigvee_x^b(f)\\ &\le \max\left\{\max_{t\in[a,x]}\left|\int_t^b g(s)\,dv(s)\right|,\max_{t\in[x,b]}\left|\int_a^t g(s)\,dv(s)\right|\right\}\bigvee_a^b(f)\end{aligned}$$

for $c=b$ and $d=a$, we have

$$\begin{aligned}(4.15)\quad &\left|f(x)\int_a^b g(t)\,dv(t)-\int_a^b f(t)\,g(t)\,dv(t)\right|\\ &\le L\left[\max_{t\in[a,x]}\left|\int_a^t g(s)\,dv(s)\right|(x-a)+\max_{t\in[x,b]}\left|\int_t^b g(s)\,dv(s)\right|(b-x)\right]\\ &\le L\max\left\{\max_{t\in[a,x]}\left|\int_a^t g(s)\,dv(s)\right|,\max_{t\in[x,b]}\left|\int_t^b g(s)\,dv(s)\right|\right\}(b-a)\end{aligned}$$

and for $c=d=x$ we get

$$\begin{aligned}(4.16)\quad &\left|f(b)\int_x^b g(s)\,dv(s)+f(a)\int_a^x g(t)\,dv(t)-\int_a^b f(t)\,g(t)\,dv(t)\right|\\ &\le L\left[\max_{t\in[a,x]}\left|\int_t^x g(s)\,dv(s)\right|(x-a)+\max_{t\in[x,b]}\left|\int_x^t g(s)\,dv(s)\right|(b-x)\right]\\ &\le L\max_{t\in[a,b]}\left|\int_t^x g(s)\,dv(s)\right|(b-a).\end{aligned}$$

Remark 11. *If we take* $x=\frac{a+b}{2}$, *then under the assumptions of Corollary 7, we have*

$$\left|f(b)\int_c^b g(s)\,dv(s)+f(a)\int_a^d g(t)\,dv(t)-f\left(\frac{a+b}{2}\right)\int_c^d g(t)\,dv(t)\right.$$
$$\left.-\int_a^b f(t)\,g(t)\,dv(t)\right|$$
$$\leq\frac{1}{2}L\left[\max_{t\in\left[a,\frac{a+b}{2}\right]}\left|\int_d^t g(s)\,dv(s)\right|+\max_{t\in\left[\frac{a+b}{2},b\right]}\left|\int_c^t g(s)\,dv(s)\right|\right](b-a), \tag{4.17}$$

$$\left|\left[f(b)+f(a)-f\left(\frac{a+b}{2}\right)\right]\int_a^b g(t)\,dv(t)-\int_a^b f(t)\,g(t)\,dv(t)\right|$$
$$\leq\frac{1}{2}L\left[\max_{t\in\left[a,\frac{a+b}{2}\right]}\left|\int_t^b g(s)\,dv(s)\right|+\max_{t\in\left[\frac{a+b}{2},b\right]}\left|\int_a^t g(s)\,dv(s)\right|\right](b-a), \tag{4.18}$$

$$\left|f\left(\frac{a+b}{2}\right)\int_a^b g(t)\,dv(t)-\int_a^b f(t)\,g(t)\,dv(t)\right|$$
$$\leq\frac{1}{2}L\max\left\{\max_{t\in\left[a,\frac{a+b}{2}\right]}\left|\int_a^t g(s)\,dv(s)\right|,\max_{t\in\left[\frac{a+b}{2},b\right]}\left|\int_t^b g(s)\,dv(s)\right|\right\}(b-a) \tag{4.19}$$

and

$$\left|f(b)\int_{\frac{a+b}{2}}^b g(s)\,dv(s)+f(a)\int_a^{\frac{a+b}{2}} g(t)\,dv(t)-\int_a^b f(t)\,g(t)\,dv(t)\right|$$
$$\leq\frac{1}{2}L\left[\max_{t\in\left[a,\frac{a+b}{2}\right]}\left|\int_t^{\frac{a+b}{2}} g(s)\,dv(s)\right|+\max_{t\in\left[\frac{a+b}{2},b\right]}\left|\int_{\frac{a+b}{2}}^t g(s)\,dv(s)\right|\right](b-a). \tag{4.20}$$

5. Some Simpler Error Bounds

If $g:[a,b]\to\mathbb{C}$ is continuous and $v:[a,b]\to\mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integrals $\int_a^t g(s)\,dv(s)$ and $\int_t^b g(s)\,dv(s)$ exist for $t\in[a,b]$ and

$$\left|\int_a^t g(s)\,dv(s)\right|\leq\int_a^t |g(s)|\,d\left(\bigvee_a^s(v)\right)\leq\max_{s\in[a,t]}|g(s)|\bigvee_a^t(v)$$

and

$$\left|\int_t^b g(s)\,dv(s)\right|\leq\int_t^b |g(s)|\,d\left(\bigvee_t^s(v)\right)\leq\max_{s\in[t,b]}|g(s)|\bigvee_t^b(v),$$

which implies that

$$\max_{t\in[a,x]}\left|\int_a^t g(s)\,dv(s)\right|\leq\max_{s\in[a,x]}|g(s)|\bigvee_a^x(v)$$

and

$$\max_{t\in[x,b]}\left|\int_t^b g(s)\,dv(s)\right|\leq\max_{s\in[x,b]}|g(s)|\bigvee_x^b(v),$$

for $x \in (a,b)$.

Therefore, by (3.14) we get for $x \in (a,b)$ that

$$\left| f(x) \int_a^b g(t)\,dv(t) - \int_a^b f(t)\,g(t)\,dv(t) \right| \tag{5.1}$$
$$\leq \max_{t\in[a,x]} \left| \int_a^t g(s)\,dv(s) \right| \bigvee_a^x (f) + \max_{t\in[x,b]} \left| \int_t^b g(s)\,dv(s) \right| \bigvee_x^b (f)$$
$$\leq \max_{s\in[a,x]} |g(s)| \bigvee_a^x (v) \bigvee_a^x (f) + \max_{s\in[x,b]} |g(s)| \bigvee_x^b (v) \bigvee_x^b (f)$$
$$\leq \begin{cases} \max\left\{ \max_{s\in[a,x]} |g(s)| \bigvee_a^x (v), \max_{s\in[x,b]} |g(s)| \bigvee_x^b (v) \right\} \bigvee_a^b (f), \\ \\ \max\left\{ \max_{s\in[a,x]} |g(s)| \bigvee_a^x (f), \max_{s\in[x,b]} |g(s)| \bigvee_x^b (f) \right\} \bigvee_a^b (v) \end{cases}$$

provided f, v are of bounded variation and g is continuous and such that the integral $\int_a^b f(t)\,g(t)\,dv(t)$ exists.

If $m \in (a,b)$ is such that $\bigvee_a^m (f) = \bigvee_m^b (f)$, then from the first inequality in (5.1) we get

$$\left| f(m) \int_a^b g(t)\,dv(t) - \int_a^b f(t)\,g(t)\,dv(t) \right| \tag{5.2}$$
$$\leq \frac{1}{2} \left[\max_{t\in[a,m]} \left| \int_a^t g(s)\,dv(s) \right| + \max_{t\in[m,b]} \left| \int_t^b g(s)\,dv(s) \right| \right] \bigvee_a^b (f).$$

If $p \in (a,b)$ is such that $\bigvee_a^p (v) = \bigvee_p^b (v)$, then from the inequality (5.1) we get

$$\left| f(p) \int_a^b g(t)\,dv(t) - \int_a^b f(t)\,g(t)\,dv(t) \right| \tag{5.3}$$
$$\leq \max_{t\in[a,p]} \left| \int_a^t g(s)\,dv(s) \right| \bigvee_a^p (f) + \max_{t\in[p,b]} \left| \int_t^b g(s)\,dv(s) \right| \bigvee_p^b (f)$$
$$\leq \frac{1}{2} \left[\max_{s\in[a,p]} |g(s)| \bigvee_a^p (f) + \max_{s\in[p,b]} |g(s)| \bigvee_p^b (f) \right] \bigvee_a^b (v).$$

By (4.15) we also get for $x \in (a,b)$ that

$$\begin{aligned}(5.4)\quad & \left|f(x)\int_a^b g(t)\,dv(t) - \int_a^b f(t)\,g(t)\,dv(t)\right| \\ & \leq L\left[\max_{t\in[a,x]}\left|\int_a^t g(s)\,dv(s)\right|(x-a) + \max_{t\in[x,b]}\left|\int_t^b g(s)\,dv(s)\right|(b-x)\right] \\ & \leq L\left[\max_{s\in[a,x]}|g(s)|\bigvee_a^x(v)(x-a) + \max_{s\in[x,b]}|g(s)|\bigvee_x^b(v)(b-x)\right] \\ & \leq L\begin{cases}\max\left\{\max_{s\in[a,x]}|g(s)|\bigvee_a^x(v), \max_{s\in[x,b]}|g(s)|\bigvee_x^b(v)\right\}(b-a), \\ \\ \max\left\{\max_{s\in[a,x]}|g(s)|(x-a), \max_{s\in[x,b]}|g(s)|(b-x)\right\}\bigvee_a^b(v)\end{cases}\end{aligned}$$

provided v is of bounded variation, f is Lipschitzian with the constant $L > 0$ and g is continuous on $[a,b]$.

In particular, for $x = \frac{a+b}{2}$ we get from the first inequality in (5.4) that

$$\begin{aligned}(5.5)\quad & \left|f\left(\frac{a+b}{2}\right)\int_a^b g(t)\,dv(t) - \int_a^b f(t)\,g(t)\,dv(t)\right| \\ & \leq \frac{1}{2}L\left[\max_{t\in\left[a,\frac{a+b}{2}\right]}\left|\int_a^t g(s)\,dv(s)\right| + \max_{t\in\left[\frac{a+b}{2},b\right]}\left|\int_t^b g(s)\,dv(s)\right|\right](b-a).\end{aligned}$$

If $p \in (a,b)$ is such that $\bigvee_a^p(v) = \bigvee_p^b(v)$, then from the inequality (5.4) we get

$$\begin{aligned}(5.6)\quad & \left|f(p)\int_a^b g(t)\,dv(t) - \int_a^b f(t)\,g(t)\,dv(t)\right| \\ & \leq L\left[\max_{t\in[a,p]}\left|\int_a^t g(s)\,dv(s)\right|(p-a) + \max_{t\in[p,b]}\left|\int_t^b g(s)\,dv(s)\right|(b-p)\right] \\ & \leq \frac{1}{2}L\left[\max_{s\in[a,p]}|g(s)|(p-a) + \max_{s\in[p,b]}|g(s)|(b-p)\right]\bigvee_a^b(v).\end{aligned}$$

Similarly, by (3.15) we have for $x \in (a,b)$ that

$$
\begin{aligned}
&\left| f(b) \int_x^b g(s)\,dv(s) + f(a) \int_a^x g(t)\,dv(t) - \int_a^b f(t)\,g(t)\,dv(t) \right| \quad (5.7) \\
&\leq \max_{t\in[a,x]} \left| \int_t^x g(s)\,dv(s) \right| \bigvee_a^x (f) + \max_{t\in[x,b]} \left| \int_x^t g(s)\,dv(s) \right| \bigvee_x^b (f) \\
&\leq \max_{s\in[a,x]} |g(s)| \bigvee_a^x (v) \bigvee_a^x (f) + \max_{s\in[x,b]} |g(s)| \bigvee_x^b (v) \bigvee_x^b (f) \\
&\leq \begin{cases} \max\left\{ \max_{s\in[a,x]} |g(s)| \bigvee_a^x (v), \max_{s\in[x,b]} |g(s)| \bigvee_x^b (v) \right\} \bigvee_a^b (f) \\ \\ \max\left\{ \max_{s\in[a,x]} |g(s)| \bigvee_a^x (f), \max_{s\in[x,b]} |g(s)| \bigvee_x^b (f) \right\} \bigvee_a^b (v) \end{cases}
\end{aligned}
$$

provided f, v are of bounded variation and g is continuous and such that the integral $\int_a^b f(t)\,g(t)\,dv(t)$ exists.

If $m \in (a,b)$ is such that $\bigvee_a^m (f) = \bigvee_m^b (f)$, then from the first inequality in (5.7) we get

$$
\begin{aligned}
&\left| f(b) \int_m^b g(s)\,dv(s) + f(a) \int_a^m g(t)\,dv(t) - \int_a^b f(t)\,g(t)\,dv(t) \right| \quad (5.8) \\
&\qquad \leq \frac{1}{2} \left[\max_{t\in[a,m]} \left| \int_t^m g(s)\,dv(s) \right| + \max_{t\in[m,b]} \left| \int_m^t g(s)\,dv(s) \right| \right] \bigvee_a^b (f).
\end{aligned}
$$

If $p \in (a,b)$ is such that $\bigvee_a^p (v) = \bigvee_p^b (v)$, then from the inequality (5.1) we get

$$
\begin{aligned}
&\left| f(b) \int_p^b g(s)\,dv(s) + f(a) \int_a^p g(t)\,dv(t) - \int_a^b f(t)\,g(t)\,dv(t) \right| \quad (5.9) \\
&\leq \max_{t\in[a,p]} \left| \int_t^p g(s)\,dv(s) \right| \bigvee_a^p (f) + \max_{t\in[p,b]} \left| \int_p^t g(s)\,dv(s) \right| \bigvee_p^b (f) \\
&\qquad \leq \frac{1}{2} \left[\max_{s\in[a,p]} |g(s)| \bigvee_a^p (f) + \max_{s\in[p,b]} |g(s)| \bigvee_p^b (f) \right] \bigvee_a^b (v).
\end{aligned}
$$

From (4.16) we have

$$\begin{aligned}(5.10)\quad &\left|f(b)\int_x^b g(s)\,dv(s)+f(a)\int_a^x g(t)\,dv(t)-\int_a^b f(t)\,g(t)\,dv(t)\right|\\ &\leq L\left[\max_{t\in[a,x]}\left|\int_t^x g(s)\,dv(s)\right|(x-a)+\max_{t\in[x,b]}\left|\int_x^t g(s)\,dv(s)\right|(b-x)\right]\\ &\leq L\left[\max_{s\in[a,x]}|g(s)|\bigvee_a^x(v)\,(x-a)+\max_{s\in[x,b]}|g(s)|\bigvee_x^b(v)\,(b-x)\right]\\ &\leq L\begin{cases}\max\left\{\max_{s\in[a,x]}|g(s)|\bigvee_a^x(v),\max_{s\in[x,b]}|g(s)|\bigvee_x^b(v)\right\}(b-a),\\ \\ \max\left\{\max_{s\in[a,x]}|g(s)|(x-a),\max_{s\in[x,b]}|g(s)|(b-x)\right\}\bigvee_a^b(v)\end{cases}\end{aligned}$$

provided that v is of bounded variation, f is Lipschitzian with the constant $L>0$ and g is continuous on $[a,b]$.

In particular, for $x=\frac{a+b}{2}$ we get from the first inequality in (5.10) that

$$\begin{aligned}(5.11)\quad &\left|f(b)\int_{\frac{a+b}{2}}^b g(s)\,dv(s)+f(a)\int_a^{\frac{a+b}{2}} g(t)\,dv(t)-\int_a^b f(t)\,g(t)\,dv(t)\right|\\ &\leq\frac{1}{2}L\left[\max_{t\in\left[a,\frac{a+b}{2}\right]}\left|\int_t^{\frac{a+b}{2}} g(s)\,dv(s)\right|+\max_{t\in\left[\frac{a+b}{2},b\right]}\left|\int_{\frac{a+b}{2}}^t g(s)\,dv(s)\right|\right](b-a).\end{aligned}$$

If $p\in(a,b)$ is such that $\bigvee_a^p(v)=\bigvee_p^b(v)$, then from the inequality (5.10) we get

$$\begin{aligned}(5.12)\quad &\left|f(b)\int_p^b g(s)\,dv(s)+f(a)\int_a^p g(t)\,dv(t)-\int_a^b f(t)\,g(t)\,dv(t)\right|\\ &\leq L\left[\max_{t\in[a,p]}\left|\int_t^p g(s)\,dv(s)\right|(p-a)+\max_{t\in[p,b]}\left|\int_p^t g(s)\,dv(s)\right|(b-p)\right]\\ &\qquad\leq\frac{1}{2}L\left[\max_{s\in[a,x]}|g(s)|(p-a)+\max_{s\in[x,b]}|g(s)|(b-p)\right]\bigvee_a^b(v).\end{aligned}$$

Using the equalities (2.9) and (2.10) one can obtain various inequalities as in the recent paper [27]. The details are omitted.

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