Conformable fractional inequalities

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Abstract

This is a long journey in the modern realm of Conformable fractional differentiation. In that setting the author presents the following types of analytic inequalities: Landau, Hilbert-Pachpatte, Ostrowski, Opial, Poincare and Sobolev inequalities. We present uniform and L_p results, involving left and right conformable fractional derivatives, as well engaging several functions. We discuss many interesting special cases.

1 Introduction

Our motivations to write this work follow. The first inspiration comes next.

Let $p \in [1, \infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$ and $f : I \to \mathbb{R}$ is twice differentiable with $f, f'' \in L_p(I)$, then $f' \in L_p(I)$.

Moreover, there exists a constant $C_p(I) > 0$ independent of f, such that

$$\|f'\|_{p,I} \leq C_p(I) \|f\|_{p,I}^{\frac{1}{2}} \|f''\|_{p,I}^{\frac{1}{2}},$$

where $\|\cdot\|_{p,I}$ is the p-norm on the interval I, see [2], [13].

The research on these inequalities started by E. Landau [20] in 1914. For the case of $p = \infty$ he proved that

$$C_\infty(\mathbb{R}_+) = 2 \text{ and } C_\infty(\mathbb{R}) = \sqrt{2},$$

are the best constants above.

In 1932, G.H. Hardy and J.E. Littlewood [16] proved above inequality for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2} \text{ and } C_2(\mathbb{R}) = 1.$$

In 1935, G.H. Hardy, E. Landau and J.E. Littlewood [17] showed that the best constant $C_p(\mathbb{R}_+)$ above satisfies the estimate

$$C_p(\mathbb{R}_+) \leq 2 \text{ for } p \in [1, \infty),$$

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which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1, \infty)$.

In fact in [14] and [18], was shown that $C_p(\mathbb{R}) \leq \sqrt{2}$.

The author in [8], studied extensively fractional Landau type inequalities involving right and left Caputo fractional derivatives.

The famous Ostrowski ([21]) inequality motivates this work and has as follows:

$$\left| \frac{1}{b-a} \int_a^b f(y)\, dy - f(x) \right| \leq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) (b-a) \|f'\|_\infty ,$$

where $f \in C^1([a,b])$, $x \in [a,b]$, and it is a sharp inequality.

Another motivation is author's next Ostrowski type fractional result, see [8], p. 44:

Let $[a,b] \subset \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$ ($\lceil \cdot \rceil$ ceiling of the number), $f \in AC^m([a,b])$ (i.e. $f^{(m-1)}$ is absolutely continuous), and $\left\| D_{x_0-}^{\alpha} f \right\|_{\infty,[a,x_0]}$, $\left\| D_{*x_0}^{\alpha} f \right\|_{\infty,[x_0,b]} < \infty$ (where $D_{x_0-}^{\alpha} f$, $D_{*x_0}^{\alpha} f$ are the right and left Caputo fractional derivatives of f of order α, respectively), $x_0 \in [a,b]$. Assume $f^{(k)}(x_0) = 0$, $k = 1, ..., m-1$.

Then

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f(x_0) \right| \leq \frac{1}{(b-a)\,\Gamma(\alpha+2)} \cdot$$

$$\left\{ \left\| D_{x_0-}^{\alpha} f \right\|_{\infty,[a,x_0]} (x_0 - a)^{\alpha+1} + \left\| D_{*x_0}^{\alpha} f \right\|_{\infty,[x_0,b]} (b - x_0)^{\alpha+1} \right\} \leq$$

$$\frac{1}{\Gamma(\alpha+2)} \max \left\{ \left\| D_{x_0-}^{\alpha} f \right\|_{\infty,[a,x_0]}, \left\| D_{*x_0}^{\alpha} f \right\|_{\infty,[x_0,b]} \right\} (b-a)^{\alpha} .$$

The author's monographs [3], [4], [5], [6], [7], [8], motivate and support greatly this work too.

Under the point of view of Conformable fractional differentiation the author scans the broad area of analytic inequalities and reveals a great variety of well-known inequalities in the Conformable fractional environment to all possible directions.

2 Main Results - I

We need

Definition 1 *([15], [19]) Let $f : [0, \infty) \to \mathbb{R}$. The conformable α-fractional derivative for $\alpha \in (0,1]$ is given by*

$$D_\alpha f(t) := \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, \tag{1}$$

$$D_\alpha f(0) = \lim_{t \to 0+} D_\alpha f(t). \tag{2}$$

If f is differentiable, then

$$D_\alpha f(t) = t^{1-\alpha} f'(t), \tag{3}$$

where f' is the usual derivative.

We define

$$D_\alpha^n f = D_\alpha^{n-1} (D_\alpha f) . \tag{4}$$

If $f : [0, \infty) \to \mathbb{R}$ is α-differentiable at $t_0 > 0$, $\alpha \in (0, 1]$, then f is continuous at t_0, see [19].

Definition 2 *([12]) Let $\alpha \in (0, 1]$ and $0 \leq a < b$. A function $f : [a, b] \to \mathbb{R}$ is α-fractional integrable on $[a, b]$ if the integral*

$$I_\alpha^a f (b) := \int_a^b f (t) \, d_\alpha t := \int_a^b f (t) \, t^{\alpha - 1} dt, \tag{5}$$

exists and is finite.

We need

Theorem 3 *([12]) (Ostrowski type inequality) Let $a, b, t \in \mathbb{R}_+$ with $0 \leq a < b$, and let $f : [a, b] \to \mathbb{R}$ be α-fractional differentiable for $\alpha \in (0, 1]$. Then*

$$\left| \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f (t) \, d_\alpha t - f (t) \right| \leq \frac{M_1}{2\alpha (b^\alpha - a^\alpha)} \left[(t^\alpha - a^\alpha)^2 + (b^\alpha - t^\alpha)^2 \right], \tag{6}$$

where

$$M_1 := \sup_{t \in (a,b)} |D_\alpha f (t)| . \tag{7}$$

Inequality (6) is sharp.

Corollary 4 *(to Theorem 3) Let $a, b \in \mathbb{R}_+$ with $0 \leq a < b$, and let $f : [a, b] \to \mathbb{R}$ be α-fractional differentiable for $\alpha \in (0, 1]$. Then*

$$\left| \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f (t) \, d_\alpha t - f (a) \right| \leq \frac{M_1}{2\alpha} (b^\alpha - a^\alpha) , \tag{8}$$

where

$$M_1 := \sup_{t \in (a,b)} |D_\alpha f (t)| .$$

We need

Theorem 5 *([10]) Let $\alpha \in (0, 1]$, and $f : [a, b] \to \mathbb{R}$, $a \geq 0$, be α-fractional differentiable on $[a, b]$. Assume that $D_\alpha f$ is continuous on $[a, b]$. Then*

$$I_\alpha^a D_\alpha f (t) = f (t) - f (a) , \quad \forall \ t \in [a, b] . \tag{9}$$

We make

Remark 6 *Let $\alpha \in (0,1]$, and any $a,b \in \mathbb{R}_+ : 0 \leq a < b$, and $D_\alpha f$ is α-fractional differentiable and continuous on every $[a,b] \subset \mathbb{R}_+$. By Corollary 4 we get*

$$\left| \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b D_\alpha f(t)\, d_\alpha t - D_\alpha f(a) \right| \leq \frac{M_2}{2\alpha} \left(b^\alpha - a^\alpha \right), \tag{10}$$

where

$$M_2 := \sup_{t \in (a,b)} \left| D_\alpha^2 f(t) \right|.$$

By Theorem 5, equivalently we have:

$$\left| \frac{\alpha}{b^\alpha - a^\alpha} \left(f(b) - f(a) \right) - D_\alpha f(a) \right| \leq \frac{M_2}{2\alpha} \left(b^\alpha - a^\alpha \right). \tag{11}$$

Hence it holds

$$\left| D_\alpha f(a) \right| - \frac{\alpha}{b^\alpha - a^\alpha} \left| f(b) - f(a) \right| \leq \frac{M_2}{2\alpha} \left(b^\alpha - a^\alpha \right). \tag{12}$$

Equivalently, we can write

$$\left| D_\alpha f(a) \right| \leq \frac{\alpha}{b^\alpha - a^\alpha} \left| f(b) - f(a) \right| + \frac{M_2}{2\alpha} \left(b^\alpha - a^\alpha \right) \leq \tag{13}$$

$$\left(\frac{\alpha}{b^\alpha - a^\alpha} \right) \left(2 \left\| f \right\|_{\infty,[0,+\infty)} \right) + \left(\frac{b^\alpha - a^\alpha}{2\alpha} \right) \left\| D_\alpha^2 f \right\|_{\infty,[0,+\infty)},$$

$\forall\ a,b \in \mathbb{R}_+ : a < b$.

Notice that the right hand side of (13) depends only on $b^\alpha - a^\alpha$. Therefore it holds

$$\left\| D_\alpha f \right\|_{\infty,[0,+\infty)} \leq \left(\frac{2\alpha}{b^\alpha - a^\alpha} \right) \left\| f \right\|_{\infty,[0,+\infty)} + \left(\frac{\left\| D_\alpha^2 f \right\|_{\infty,[0,+\infty)}}{2\alpha} \right) \left(b^\alpha - a^\alpha \right). \tag{14}$$

Set $t := b^\alpha - a^\alpha > 0$. Thus

$$\left\| D_\alpha f \right\|_{\infty,[0,+\infty)} \leq \left(\frac{2\alpha}{t} \right) \left\| f \right\|_{\infty,[0,+\infty)} + \left(\frac{\left\| D_\alpha^2 f \right\|_{\infty,[0,+\infty)}}{2\alpha} \right) t, \quad \forall\ t > 0. \tag{15}$$

Call

$$\mu := 2\alpha \left\| f \right\|_{\infty,[0,+\infty)},$$

$$\text{and} \tag{16}$$

$$\theta := \left(\frac{\left\| D_\alpha^2 f \right\|_{\infty,[0,+\infty)}}{2\alpha} \right),$$

both are greater than zero.

That is we have

$$\|D_\alpha f\|_{\infty,[0,+\infty)} \leq \frac{\mu}{t} + \theta \cdot t, \ \forall\ t > 0. \tag{17}$$

Consider the function

$$y(t) := \frac{\mu}{t} + \theta \cdot t, \ t > 0. \tag{18}$$

As in [8], pp. 80-82, y has a global minimum at

$$t_0 = \left(\frac{\mu}{\theta}\right)^{\frac{1}{2}}, \tag{19}$$

which is

$$y(t_0) = 2\sqrt{\theta\mu}. \tag{20}$$

Consequently we derive

$$y(t_0) = 2\sqrt{\|f\|_{\infty,[0,+\infty)} \|D_\alpha^2 f\|_{\infty,[0,+\infty)}}. \tag{21}$$

We have proved that

$$\|D_\alpha f\|_{\infty,[0,+\infty)} \leq 2\sqrt{\|f\|_{\infty,[0,+\infty)} \|D_\alpha^2 f\|_{\infty,[0,+\infty)}}. \tag{22}$$

We have established the following conformable fractional Landau type inequality:

Theorem 7 *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be α-fractional differentiable, $\alpha \in (0,1]$. And $D_\alpha f$ is also α-fractional differentiable and continuous on $\mathbb{R}_+$. Assume that $\|f\|_{\infty,\mathbb{R}_+}$, $\left\|D_\alpha^2 f\right\|_{\infty,\mathbb{R}_+} < \infty$. Then*

$$\|D_\alpha f\|_{\infty,\mathbb{R}_+} \leq 2\,\|f\|_{\infty,\mathbb{R}_+}^{\frac{1}{2}} \left\|D_\alpha^2 f\right\|_{\infty,\mathbb{R}_+}^{\frac{1}{2}}, \tag{23}$$

that is $\|D_\alpha f\|_{\infty,\mathbb{R}_+} < \infty$.

Note 8 *If f is differentiable then $D_\alpha f(t) = t^{1-\alpha} f'(t)$, $t > 0$, $\alpha \in (0,1]$. When $t > 0$, $t^{1-\alpha}$ is differentiable. If f is twice differentiable and $t > 0$, then we have*

$$D_\alpha^2 f(t) = D_\alpha(D_\alpha f(t)) = D_\alpha\left(t^{1-\alpha} f'(t)\right) = t^{1-\alpha}\left(t^{1-\alpha} f'(t)\right)'$$
$$= t^{1-\alpha}\left((1-\alpha)\, t^{-\alpha} f'(t) + t^{1-\alpha} f''(t)\right).$$

That is an interesting formula:

$$D_\alpha^2 f(t) = (1-\alpha)\, t^{1-2\alpha} f'(t) + t^{2(1-\alpha)} f''(t), \ t > 0. \tag{24}$$

We need

Definition 9 *Let $\alpha \in (0,1]$. We define the spaces of functions:*

$$L_{\alpha}^{p}([a,b]) := \left\{ f : [a,b] \subset \mathbb{R}_{+} \to \mathbb{R} : \int_{a}^{b} |f(t)|^{p} d_{\alpha}t < +\infty,\ p \geq 1 \right\},$$

and

$$L_{\alpha}^{p}(\mathbb{R}_{+}) := \left\{ f : \mathbb{R}_{+} \to \mathbb{R} : \int_{\mathbb{R}_{+}} |f(x)|^{p} d_{\alpha}x := \int_{\mathbb{R}_{+}} |f(x)|^{p} x^{\alpha-1} dx < +\infty,\ p \geq 1 \right\}.$$

We need the conformable fractional L_p Ostrowski type inequality:

Theorem 10 *([22]) Let $a \geq 0$, $f : [a,b] \to \mathbb{R}$ be an α-fractional differentiable function for $\alpha \in (0,1]$, $D_{\alpha}(f) \in L_{\alpha}^{p}([a,b])$; $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then for all $x \in [a,b]$, we have the inequality:*

$$\left| \frac{\alpha}{b^{\alpha} - a^{\alpha}} \int_{a}^{b} f(t) d_{\alpha}t - f(x) \right| \leq A_{\alpha}(x,q) \|D_{\alpha}(f)\|_{p}, \tag{25}$$

where

$$A_{\alpha}(x,q) = \frac{1}{(b^{\alpha} - a^{\alpha})} \left(\frac{1}{\alpha(q+1)} \left(\frac{b^{\alpha} - a^{\alpha}}{2} \right)^{\alpha(q+1)} \right)^{\frac{1}{q}} + \left| \frac{1}{\alpha} \left(x^{\alpha} - \frac{a^{\alpha} + b^{\alpha}}{2} \right) \right|^{\frac{1}{q}}. \tag{26}$$

When $x = a$, we get:

$$\left| \frac{\alpha}{b^{\alpha} - a^{\alpha}} \int_{a}^{b} f(t) d_{\alpha}t - f(a) \right| \leq A_{\alpha}(a,q) \|D_{\alpha}(f)\|_{p}, \tag{27}$$

where

$$A_{\alpha}(a,q) = \frac{1}{b^{\alpha} - a^{\alpha}} \left(\frac{1}{\alpha(q+1)} \left(\frac{b^{\alpha} - a^{\alpha}}{2} \right)^{\alpha(q+1)} \right)^{\frac{1}{q}} + \left[\frac{1}{\alpha} \left(\frac{b^{\alpha} - a^{\alpha}}{2} \right) \right]^{\frac{1}{q}}. \tag{28}$$

We need

Corollary 11 *([22]) Let $a \geq 0$, $f : [a,b] \to \mathbb{R}$ be an α-fractional differentiable function for $\alpha \in (0,1]$, $D_{\alpha}(f) \in L_{\alpha}^{p}([a,b])$, $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left| \frac{\alpha}{b^{\alpha} - a^{\alpha}} \int_{a}^{b} f(t) d_{\alpha}t - f\left(\left(\frac{a^{\alpha} + b^{\alpha}}{2} \right)^{\frac{1}{\alpha}} \right) \right| \leq \tag{29}$$

$$\frac{1}{2} \left(\frac{1}{\alpha(q+1)} \right)^{\frac{1}{2}} \left(\frac{b^{\alpha} - a^{\alpha}}{2} \right)^{\alpha\left(1+\frac{1}{q}\right)-1} \|D_{\alpha}(f)\|_{p,[a,b]}.$$

We make

Remark 12 *Here* $f : \mathbb{R}_+ \to \mathbb{R}$ *be* α*-fractional differentiable and* $0 < \alpha \leq 1$, *and* $D_\alpha f$ *is also* α*-fractional differentiable and continuous function on* $\mathbb{R}_+$, *and* $D_\alpha^2 (f) \in L_\alpha^p (\mathbb{R}_+)$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Here* $[a, b] \subset \mathbb{R}_+$.

Then, by (29), we get:

$$\left| D_\alpha f \left(\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b D_\alpha f (t) \, d_\alpha t \right| \leq \tag{30}$$

$$\frac{1}{2} \left(\frac{1}{\alpha (q+1)} \right)^{\frac{1}{2}} \left(\frac{b^\alpha - a^\alpha}{2} \right)^{\alpha \left(1 + \frac{1}{q}\right) - 1} \left\| D_\alpha^2 (f) \right\|_{p,[a,b]},$$

equivalently it holds

$$\left| D_\alpha f \left(\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) - \frac{\alpha}{b^\alpha - a^\alpha} (f (b) - f (a)) \right| \leq$$

$$\frac{1}{2} \left(\frac{1}{\alpha (q+1)} \right)^{\frac{1}{2}} \left(\frac{b^\alpha - a^\alpha}{2} \right)^{\alpha \left(1 + \frac{1}{q}\right) - 1} \left\| D_\alpha^2 (f) \right\|_{p,[a,b]}. \tag{31}$$

Hence it follows

$$\left| D_\alpha f \left(\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) \right| - \frac{\alpha}{b^\alpha - a^\alpha} |f (b) - f (a)| \leq \tag{32}$$

$$\frac{1}{2} \left(\frac{1}{\alpha (q+1)} \right)^{\frac{1}{2}} \left\| D_\alpha^2 (f) \right\|_{p,[a,b]} \left(\frac{b^\alpha - a^\alpha}{2} \right)^{\alpha \left(1 + \frac{1}{q}\right) - 1}.$$

Thus, it holds

$$\left| D_\alpha f \left(\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) \right| \leq \frac{\left(2\alpha \left\| f \right\|_{\infty, \mathbb{R}_+} \right)}{b^\alpha - a^\alpha} + \tag{33}$$

$$\left[\frac{1}{2^{\alpha \left(1 + \frac{1}{q}\right)}} \left(\frac{1}{\alpha (q+1)} \right)^{\frac{1}{q}} \left\| D_\alpha^2 (f) \right\|_{p, \mathbb{R}_+} \right] (b^\alpha - a^\alpha)^{\alpha \left(1 + \frac{1}{q}\right) - 1}$$

true, $\forall \ a, b \in \mathbb{R}_+$, $a < b$.

The right hand side of (33) depends only on $b^\alpha - a^\alpha$.

We have $a^\alpha \leq \frac{a^\alpha + b^\alpha}{2} \leq b^\alpha$, *iff* $a \leq \left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \leq b$.

From now on we assume that $|D_\alpha f|$ *is increasing (or decreasing) then*

$$|D_\alpha f (a)| \leq \left| D_\alpha f \left(\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) \right| \tag{34}$$

(or $|D_\alpha f(b)| \leq \left| D_\alpha f\left(\left(\frac{a^\alpha + b^\alpha}{2}\right)^{\frac{1}{\alpha}} \right) \right|$*).*

Therefore it holds

$$\|D_\alpha f\|_{\infty,\mathbb{R}_+} \leq \frac{\left(2\alpha \|f\|_{\infty,\mathbb{R}_+}\right)}{b^\alpha - a^\alpha} + \tag{35}$$

$$\left(\frac{1}{2^{\alpha\left(1+\frac{1}{q}\right)}} \left(\frac{1}{\alpha(q+1)} \right)^{\frac{1}{q}} \left\| D_\alpha^2 (f) \right\|_{p,\mathbb{R}_+} \right) (b^\alpha - a^\alpha)^{\alpha\left(1+\frac{1}{q}\right)-1}.$$

Set $t := b^\alpha - a^\alpha > 0$*, so that*

$$\|D_\alpha f\|_{\infty,\mathbb{R}_+} \leq \frac{\left(2\alpha \|f\|_{\infty,\mathbb{R}_+}\right)}{t} + \tag{36}$$

$$\left(\frac{1}{2^{\alpha\left(1+\frac{1}{q}\right)}} \left(\frac{1}{\alpha(q+1)} \right)^{\frac{1}{q}} \left\| D_\alpha^2 (f) \right\|_{p,\mathbb{R}_+} \right) t^{\alpha\left(1+\frac{1}{q}\right)-1}, \ \forall\, t > 0.$$

Call

$$\widetilde{\mu} := 2\alpha \|f\|_{\infty,\mathbb{R}_+},$$

$$\text{and} \tag{37}$$

$$\widetilde{\theta} := \left(\frac{1}{2^{\alpha\left(1+\frac{1}{q}\right)}} \left(\frac{1}{\alpha(q+1)} \right)^{\frac{1}{q}} \left\| D_\alpha^2 (f) \right\|_{p,\mathbb{R}_+} \right),$$

both are greater than 0*.*

From now on we consider $\alpha \in (0,1)$*, i.e.* $0 < \alpha < 1$*, thus* $\frac{1}{\alpha} > 1$*.*

We would like to have

$$0 < \alpha\left(1 + \frac{1}{q}\right) - 1 < 1 \Leftrightarrow$$

$$(0 <)\frac{1-\alpha}{\alpha} < \frac{1}{q} < \frac{2-\alpha}{\alpha}; \tag{38}$$

where $0 < \frac{1}{q} < 1$*.*

By $\alpha < 1$ *we get* $\frac{2-\alpha}{\alpha} > 1$*. Therefore* $\frac{1}{q} < \frac{2-\alpha}{\alpha}$*, always correct.*

Inequalities (38) are written, equivalently, as

$$\frac{\alpha}{2-\alpha} < q < \frac{\alpha}{1-\alpha}. \tag{39}$$

Notice that $\frac{1}{2} < \alpha < 1$ *is equivalently to* $\frac{\alpha}{1-\alpha} > 1$*.*

From now on we assume that

$$\frac{1}{2} < \alpha < 1 \quad \text{and} \quad 1 < q < \frac{\alpha}{1-\alpha}, \tag{40}$$

and it holds

$$0<\alpha\left(1+\frac{1}{q}\right)-1<1. \tag{41}$$

Next, we call

$$\widetilde{\nu}:=\alpha\left(1+\frac{1}{q}\right)-1,\ \ \widetilde{\nu}\in(0,1). \tag{42}$$

We consider the function

$$\widetilde{y}(t)=\frac{\widetilde{\mu}}{t}+\widetilde{\theta}t^{\widetilde{\nu}},\ \ t\in(0,\infty). \tag{43}$$

Next we act as in [8], pp. 80-82.

The only critical number here is

$$\widetilde{t_0}=\left(\frac{\widetilde{\mu}}{\widetilde{\nu}\widetilde{\theta}}\right)^{\frac{1}{\widetilde{\nu}+1}}, \tag{44}$$

and $\widetilde{y}$ has a global minimum at $\widetilde{t_0}$, which is

$$\widetilde{y}\left(\widetilde{t_0}\right)=\left(\widetilde{\theta}\widetilde{\mu}^{\widetilde{\nu}}\right)^{\frac{1}{\widetilde{\nu}+1}}\left(\widetilde{\nu}+1\right)\widetilde{\nu}^{-\left(\frac{\widetilde{\nu}}{\widetilde{\nu}+1}\right)}. \tag{45}$$

Thus, we have proved

$$\|D_\alpha f\|_{\infty,\mathbb{R}_+}\leq \tag{46}$$

$$\left[\left(\frac{1}{2^{\alpha\left(1+\frac{1}{q}\right)}}\left(\frac{1}{\alpha(q+1)}\right)^{\frac{1}{q}}\left\|D_\alpha^2(f)\right\|_{p,\mathbb{R}_+}\right)\left(2\alpha\|f\|_{\infty,\mathbb{R}_+}\right)^{a\left(1+\frac{1}{q}\right)-1}\right]^{\frac{1}{\alpha\left(1+\frac{1}{q}\right)}}$$

$$\left(\alpha\left(1+\frac{1}{q}\right)\right)\left(\alpha\left(1+\frac{1}{q}\right)-1\right)^{-\frac{\left(\alpha\left(1+\frac{1}{q}\right)-1\right)}{\alpha\left(1+\frac{1}{q}\right)}}.$$

We have established the following L_p conformable fractional Landau type inequality:

Theorem 13 *Here $f:\mathbb{R}_+\to\mathbb{R}$ be α-fractional differentiable with $\frac{1}{2}<\alpha<1$, and $D_\alpha f$ is also α-fractional differentiable and continuous function on $\mathbb{R}_+$, and $D_\alpha^2(f)\in L_\alpha^p(\mathbb{R}_+)$, where $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $1<q<\frac{\alpha}{1-\alpha}$. Assume $|D_\alpha f|$ is monotone and $\|f\|_{\infty,\mathbb{R}_+}<\infty$. Then*

$$\|D_\alpha f\|_{\infty,\mathbb{R}_+}\leq\left[\|f\|_{\infty,\mathbb{R}_+}^{\left(\frac{\left(\alpha\left(1+\frac{1}{q}\right)-1\right)}{\alpha\left(1+\frac{1}{q}\right)}\right)}\left\|D_\alpha^2(f)\right\|_{p,\mathbb{R}_+}^{\frac{1}{\alpha\left(1+\frac{1}{q}\right)}}\right].$$

$$\left[\frac{(2\alpha)^{a\left(1+\frac{1}{q}\right)-1}}{2^{\alpha\left(1+\frac{1}{q}\right)}(\alpha(q+1))^{\frac{1}{q}}}\right]^{\frac{1}{\alpha\left(1+\frac{1}{q}\right)}}\left(\alpha\left(1+\frac{1}{q}\right)\right)\left(\alpha\left(1+\frac{1}{q}\right)-1\right)^{-\frac{\left(\alpha\left(1+\frac{1}{q}\right)-1\right)}{\alpha\left(1+\frac{1}{q}\right)}}. \tag{47}$$

That is, $\|D_\alpha f\|_{\infty,\mathbb{R}_+}<+\infty$.

3 Main Results - II

In this section we use generalized Conformable fractional calculus.

Here we follow [1] for the basics of generalized Conformable fractional calculus, see also [19].

We need

Definition 14 *([1]) Let $a, b \in \mathbb{R}$. The left conformable fractional derivative starting from a of a function $f : [a, \infty) \to \mathbb{R}$ of order $0 < \alpha \leq 1$ is defined by*

$$\left(T_{\alpha}^{a} f\right)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon (t-a)^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{48}$$

If $\left(T_{\alpha}^{a} f\right)(t)$ exists on (a, b), then

$$\left(T_{\alpha}^{a} f\right)(a) = \lim_{t \to a+} \left(T_{\alpha}^{a} f\right)(t). \tag{49}$$

The right conformable fractional derivative of order $0 < \alpha \leq 1$ terminating at b of $f : (-\infty, b] \to \mathbb{R}$ is defined by

$$\left({}_{\alpha}^{b}T f\right)(t) = -\lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon (b-t)^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{50}$$

If $\left({}_{\alpha}^{b}T f\right)(t)$ exists on (a, b), then

$$\left({}_{\alpha}^{b}T f\right)(b) = \lim_{t \to b-} \left({}_{\alpha}^{b}T f\right)(t). \tag{51}$$

Note that if f is differentiable then

$$\left(T_{\alpha}^{a} f\right)(t) = (t-a)^{1-\alpha} f'(t), \tag{52}$$

and

$$\left({}_{\alpha}^{b}T f\right)(t) = -(b-t)^{1-\alpha} f'(t). \tag{53}$$

Denote by

$$\left(I_{\alpha}^{a} f\right)(t) = \int_{a}^{t} (x-a)^{\alpha-1} f(x)\, dx, \tag{54}$$

and

$$\left({}^{b}I_{\alpha} f\right)(t) = \int_{t}^{b} (b-x)^{\alpha-1} f(x)\, dx, \tag{55}$$

these are the left and right conformable fractional integrals of order $0 < \alpha \leq 1$.

In the higher order case we can generalize things as follows:

Definition 15 *([1]) Let $\alpha \in (n, n+1]$, and set $\beta = \alpha - n$. Then, the left conformable fractional derivative starting from a of a function $f : [a, \infty) \to \mathbb{R}$ of order α, where $f^{(n)}(t)$ exists, is defined by*

$$\left(\mathbf{T}_{\alpha}^{a} f\right)(t) = \left(T_{\beta}^{a} f^{(n)}\right)(t), \tag{56}$$

The right conformable fractional derivative of order α terminating at b of $f : (-\infty, b] \to \mathbb{R}$, where $f^{(n)}(t)$ exists, is defined by

$$\left({}_{\alpha}^{b}\mathbf{T} f\right)(t) = (-1)^{n+1} \left({}_{\beta}^{b} T f^{(n)}\right)(t). \tag{57}$$

If $\alpha = n+1$ then $\beta = 1$ and $\mathbf{T}_{n+1}^{a} f = f^{(n+1)}$.

If n is odd, then ${}_{n+1}^{b}\mathbf{T} f = -f^{(n+1)}$, and if n is even, then ${}_{n+1}^{b}\mathbf{T} f = f^{(n+1)}$.

When $n = 0$ (or $\alpha \in (0,1]$), then $\beta = \alpha$, and (56), (57) collapse to $\{$(48)-(51)$\}$, respectively.

Lemma 16 *([1]) Let $f : (a, b) \to \mathbb{R}$ be continuously differentiable and $0 < \alpha \leq 1$. Then, for all $t > a$ we have*

$$I_{\alpha}^{a} T_{\alpha}^{a} (f)(t) = f(t) - f(a). \tag{58}$$

We need

Definition 17 *(see also [1]) If $\alpha \in (n, n+1]$, then the left fractional integral of order α starting at a is defined by*

$$\left(\mathbf{I}_{\alpha}^{a} f\right)(t) = \frac{1}{n!} \int_{a}^{t} (t-x)^{n} (x-a)^{\beta-1} f(x)\, dx. \tag{59}$$

Similarly, (author's definition, see [11]) the right fractional integral of order α terminating at b is defined by

$$\left({}^{b}\mathbf{I}_{\alpha} f\right)(t) = \frac{1}{n!} \int_{t}^{b} (x-t)^{n} (b-x)^{\beta-1} f(x)\, dx. \tag{60}$$

We need

Proposition 18 *([1]) Let $\alpha \in (n, n+1]$ and $f : [a, \infty) \to \mathbb{R}$ be $(n+1)$ times continuously differentiable for $t > a$. Then, for all $t > a$ we have*

$$\mathbf{I}_{\alpha}^{a} \mathbf{T}_{\alpha}^{a} (f)(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^{k}}{k!}. \tag{61}$$

We also have

Proposition 19 *([11]) Let* $\alpha \in (n, n+1]$ *and* $f : (-\infty, b] \to \mathbb{R}$ *be* $(n+1)$ *times continuously differentiable for* $t < b$*. Then, for all* $t < b$ *we have*

$$-^{b}\mathbf{I}_{\alpha}\ {}_{\alpha}^{b}\mathbf{T}(f)(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(b)(t-b)^{k}}{k!}. \quad (62)$$

If $n = 0$ *or* $0 < \alpha \leq 1$*, then (see also [1])*

$$^{b}I_{\alpha}\ {}_{\alpha}^{b}T(f)(t) = f(t) - f(b). \quad (63)$$

In conclusion we derive

Theorem 20 *([11]) Let* $\alpha \in (n, n+1]$ *and* $f \in C^{n+1}([a, b])$*,* $n \in \mathbb{N}$*. Then*
1)

$$f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^{k}}{k!} = \frac{1}{n!} \int_{a}^{t} (t-x)^{n} (x-a)^{\beta-1} \left(\mathbf{T}_{\alpha}^{a}(f)\right)(x)\, dx, \quad (64)$$

and
2)

$$f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(b)(t-b)^{k}}{k!} = -\frac{1}{n!} \int_{t}^{b} (b-x)^{\beta-1} (x-t)^{n} \left({}_{\alpha}^{b}\mathbf{T}(f)\right)(x)\, dx, \quad (65)$$

$\forall\ t \in [a, b]$.

We need

Remark 21 *([11]) We notice the following: let* $\alpha \in (n, n+1]$ *and* $f \in C^{n+1}([a, b])$*,* $n \in \mathbb{N}$*. Then (*$\beta := \alpha - n$*,* $0 < \beta \leq 1$*)*

$$\left(\mathbf{T}_{\alpha}^{a}(f)\right)(x) = \left(T_{\beta}^{\alpha} f^{(n)}\right)(x) = (x-a)^{1-\beta} f^{(n+1)}(x), \quad (66)$$

and

$$\left({}_{\alpha}^{b}\mathbf{T}(f)\right)(x) = (-1)^{n+1} \left({}_{\beta}^{b}T f^{(n)}\right)(x) =$$

$$(-1)^{n+1}(-1)(b-x)^{1-\beta} f^{(n+1)}(x) = (-1)^{n}(b-x)^{1-\beta} f^{(n+1)}(x). \quad (67)$$

Consequently we get that

$$\left(\mathbf{T}_{\alpha}^{a}(f)\right)(x),\ \left({}_{\alpha}^{b}\mathbf{T}(f)\right)(x) \in C([a, b]).$$

Furthermore it is obvious that

$$\left(\mathbf{T}_{\alpha}^{a}(f)\right)(a) = \left({}_{\alpha}^{b}\mathbf{T}(f)\right)(b) = 0, \quad (68)$$

when $0 < \beta < 1$*, i.e. when* $\alpha \in (n, n+1)$*.*

If $f^{(k)}(a)=0,\ k=1,...,n,$ *then*

$$f(t)-f(a)=\frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\beta-1} \left(\mathbf{T}_\alpha^a(f)\right)(x)\,dx, \tag{69}$$

$\forall\ t\in[a,b]$.

If $f^{(k)}(b)=0,\ k=1,...,n,$ *then*

$$f(t)-f(b)=-\frac{1}{n!}\int_t^b (b-x)^{\beta-1}(x-t)^n \left({}_\alpha^b\mathbf{T}(f)\right)(x)\,dx, \tag{70}$$

$\forall\ t\in[a,b]$.

We make

Remark 22 *Here let* $\alpha_i\in(n_i,n_i+1],\ f_i\in C^{n_i+1}([a_i,b_i]),\ n_i\in\mathbb{Z}_+;\ \beta_i:=\alpha_i-n_i\ (0<\beta_i\leq 1),$ *where* $i=1,2$.

By definition we have

$$\left(T_{\alpha_i}^{a_i}(f_i)\right)(t_i)=\left(T_{\beta_i}^{\alpha_i}\left(f_i^{(n_i)}\right)\right)(t_i),\quad i=1,2.$$

Assume that $f_i^{(k_i)}(a_i)=0,\ k_i=0,1,...,n_i;\ i=1,2.$

Then (by (69))

$$f_i(t_i)=\frac{1}{n_i!}\int_{a_i}^{t_i}(t_i-x_i)^{n_i}(x_i-a_i)^{\beta_i-1}\left(T_{\alpha_i}^{a_i}(f_i)\right)(x_i)\,dx_i, \tag{71}$$

$\forall\ t_i\in[a_i,b_i];\ i=1,2.$

Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1,$ *then*

$$|f_i(t_i)|\leq\frac{(b_i-a_i)^{n_i}}{n_i!}\int_{a_i}^{t_i}(x_i-a_i)^{\beta_i-1}\left|\left(T_{\alpha_i}^{a_i}(f_i)\right)(x_i)\right|dx_i, \tag{72}$$

$i=1,2.$

Therefore we get

$$|f_1(t_1)|\leq\frac{(b_1-a_1)^{n_1}}{n_1!}\int_{a_1}^{t_1}(x_1-a_1)^{\beta_1-1}\left|T_{\alpha_1}^{a_1}(f_1)(x_1)\right|dx_1\leq$$

$$\frac{(b_1-a_1)^{n_1}}{n_1!}\left(\int_{a_1}^{t_1}(x_1-a_1)^{p(\beta_1-1)}dx_1\right)^{\frac{1}{p}}\left(\int_{a_1}^{t_1}\left|\left(T_{\alpha_1}^{a_1}(f_1)\right)(x_1)\right|^q dx_1\right)^{\frac{1}{q}}\leq$$

$$\frac{(b_1-a_1)^{n_1}}{n_1!}\left(\frac{(t_1-a_1)^{p(\beta_1-1)+1}}{p(\beta_1-1)+1}\right)^{\frac{1}{p}}\left\|T_{\alpha_1}^{a_1}(f_1)\right\|_{L_q([a_1,b_1])}, \tag{73}$$

under the assumption $\beta_1>\frac{1}{q}\Leftrightarrow p(\beta_1-1)+1>0.$

We have proved that

$$|f_1(t_1)| \leq \frac{(b_1-a_1)^{n_1}}{n_1!} \left(\frac{(t_1-a_1)^{p(\beta_1-1)+1}}{p(\beta_1-1)+1} \right)^{\frac{1}{p}} \left\| T_{\alpha_1}^{a_1}(f_1) \right\|_{L_q([a_1,b_1])}, \tag{74}$$

$\forall\ t_1 \in [a_1, b_1]$, *where* $\beta_1 > \frac{1}{q}$.

Similarly, by assuming $\beta_2 > \frac{1}{p}$, *we get*

$$|f_2(t_2)| \leq \frac{(b_2-a_2)^{n_2}}{n_2!} \left(\frac{(t_2-a_2)^{q(\beta_2-1)+1}}{q(\beta_2-1)+1} \right)^{\frac{1}{q}} \left\| T_{\alpha_2}^{a_2}(f_2) \right\|_{L_p([a_2,b_2])}, \tag{75}$$

$\forall\ t_2 \in [a_2, b_2]$.

Hence we have (by (74) and (75) multiplication)

$$|f_1(t_1)|\,|f_2(t_2)| \leq \left[\frac{(b_1-a_1)^{n_1}}{n_1!} \cdot \frac{(b_2-a_2)^{n_2}}{n_2!} \right]$$

$$\frac{1}{(p(\beta_1-1)+1)^{\frac{1}{p}} (q(\beta_2-1)+1)^{\frac{1}{q}}} (t_1-a_1)^{\frac{p(\beta_1-1)+1}{p}} (t_2-a_2)^{\frac{q(\beta_2-1)+1}{q}} \tag{76}$$

$$\left\| T_{\alpha_1}^{a_1}(f_1) \right\|_{L_q([a_1,b_1])} \left\| T_{\alpha_2}^{a_2}(f_2) \right\|_{L_p([a_2,b_2])} \leq$$

(using Young's inequality for $a, b \geq 0$, $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$*)*

$$\left(\frac{(b_1-a_1)^{n_1} (b_2-a_2)^{n_2}}{n_1! n_2!} \right) \frac{1}{(p(\beta_1-1)+1)^{\frac{1}{p}} (q(\beta_2-1)+1)^{\frac{1}{q}}}$$

$$\left[\frac{(t_1-a_1)^{p(\beta_1-1)+1}}{p} + \frac{(t_2-a_2)^{q(\beta_2-1)+1}}{q} \right]$$

$$\left\| T_{\alpha_1}^{a_1}(f_1) \right\|_{L_q([a_1,b_1])} \left\| T_{\alpha_2}^{a_2}(f_2) \right\|_{L_p([a_2,b_2])}, \tag{77}$$

$\forall\ t_i \in [a_i, b_i]$; $i = 1, 2$.

Therefore we can write

$$\frac{|f_1(t_1)|\,|f_2(t_2)|}{\left[\frac{(t_1-a_1)^{p(\beta_1-1)+1}}{p} + \frac{(t_2-a_2)^{q(\beta_2-1)+1}}{q} \right]} \leq$$

$$\frac{(b_1-a_1)^{n_1} (b_2-a_2)^{n_2}}{n_1! n_2! (p(\beta_1-1)+1)^{\frac{1}{p}} (q(\beta_2-1)+1)^{\frac{1}{q}}} \tag{78}$$

$$\left\| T_{\alpha_1}^{a_1}(f_1) \right\|_{L_q([a_1,b_1])} \left\| T_{\alpha_2}^{a_2}(f_2) \right\|_{L_p([a_2,b_2])},$$

$\forall\ t_i \in [a_i, b_i]$; $i = 1, 2$.

The denominator of left hand side of (78) can be zero only when both $t_1 = a_1$ *and* $t_2 = a_2$.

Therefore it holds

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(t_1)|\,|f_2(t_2)|\,dt_1dt_2}{\left[\frac{(t_1-a_1)^{p(\beta_1-1)+1}}{p}+\frac{(t_2-a_2)^{q(\beta_2-1)+1}}{q}\right]}\leq$$

$$\frac{(b_1-a_1)^{n_1+1}(b_2-a_2)^{n_2+1}\left\|T_{\alpha_1}^{a_1}(f_1)\right\|_{L_q([a_1,b_1])}\left\|T_{\alpha_2}^{a_2}(f_2)\right\|_{L_p([a_2,b_2])}}{n_1!n_2!\,(p(\beta_1-1)+1)^{\frac{1}{p}}\,(q(\beta_2-1)+1)^{\frac{1}{q}}}. \quad (79)$$

Notice here that $T_{\alpha_i}^{a_i}(f_i)\in C([a_i,b_i])$.

We have proved the left Conformable fractional Hilbert-Pachpatte inequality:

Theorem 23 *Let* $\alpha_i\in(n_i,n_i+1]$, $f_i\in C^{n_i+1}([a_i,b_i])$, $[a_i,b_i]\subset\mathbb{R}$, $n_i\in\mathbb{Z}_+$; $\beta_i:=\alpha_i-n_i$, $i=1,2$; $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. *Assume that* $f_i^{(k_i)}(a_i)=0$, $k_i=0,1,...,n_i$; $i=1,2$. *Suppose that* $\beta_1>\frac{1}{q}$ *and* $\beta_2>\frac{1}{p}$. *Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(t_1)|\,|f_2(t_2)|\,dt_1dt_2}{\left[\frac{(t_1-a_1)^{p(\beta_1-1)+1}}{p}+\frac{(t_2-a_2)^{q(\beta_2-1)+1}}{q}\right]}\leq \quad (80)$$

$$\frac{(b_1-a_1)^{n_1+1}(b_2-a_2)^{n_2+1}\left\|T_{\alpha_1}^{a_1}(f_1)\right\|_{L_q([a_1,b_1])}\left\|T_{\alpha_2}^{a_2}(f_2)\right\|_{L_p([a_2,b_2])}}{n_1!n_2!\,(p(\beta_1-1)+1)^{\frac{1}{p}}\,(q(\beta_2-1)+1)^{\frac{1}{q}}}.$$

We make

Remark 24 *Here let* $\alpha_i\in(n_i,n_i+1]$, $f_i\in C^{n_i+1}([a_i,b_i])$, $n_i\in\mathbb{Z}_+$; $\beta_i:=\alpha_i-n_i$ $(0<\beta_i\leq1)$, *where* $i=1,2$.

By definition we have

$$\left({}_{\alpha_i}^{b_i}T(f_i)\right)(t_i)=(-1)^{n_i+1}\left({}_{\beta_i}^{b_i}T\left(f_i^{(n_i)}\right)\right)(t_i),\quad i=1,2.$$

Assume that $f_i^{(k_i)}(b_i)=0$, $k_i=0,1,...,n_i$; $i=1,2$.

Then (by (70))

$$f_i(t_i)=-\frac{1}{n_i!}\int_{t_i}^{b_i}(b_i-x_i)^{\beta_i-1}(x_i-t_i)^{n_i}\left({}_{\alpha_i}^{b_i}T(f_i)\right)(x_i)\,dx_i, \quad (81)$$

$\forall\ t_i\in[a_i,b_i]$; $i=1,2$ $(\beta_i:=\alpha_i-n_i,\ 0<\beta_i<1$ *when* $\alpha_i\in(n_i,n_i+1))$.

Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, *then*

$$|f_i(t_i)|\leq\frac{(b_i-a_i)^{n_i}}{n_i!}\int_{t_i}^{b_i}(b_i-x_i)^{\beta_i-1}\left|\left({}_{\alpha_i}^{b_i}T(f_i)\right)(x_i)\right|dx_i, \quad (82)$$

$i=1,2$.

We have

$$|f_1(t_1)| \leq \frac{(b_1-a_1)^{n_1}}{n_1!} \int_{t_1}^{b_1} (b_1-x_1)^{\beta_1-1} \left| \left({}^{b_1}_{\alpha_1} T(f_1) \right)(x_1) \right| dx_1 \leq$$

$$\frac{(b_1-a_1)^{n_1}}{n_1!} \left(\int_{t_1}^{b_1} (b_1-x_1)^{p(\beta_1-1)} dx_1 \right)^{\frac{1}{p}} \left\| {}^{b_1}_{\alpha_1} T(f_1) \right\|_{L_q([a_1,b_1])} = \tag{83}$$

$$\frac{(b_1-a_1)^{n_1}}{n_1!} \left(\frac{(b_1-t_1)^{p(\beta_1-1)+1}}{p(\beta_1-1)+1} \right)^{\frac{1}{p}} \left\| {}^{b_1}_{\alpha_1} T(f_1) \right\|_{L_q([a_1,b_1])}.$$

We assume $\beta_1 > \frac{1}{q}$ *and we have proved*

$$|f_1(t_1)| \leq \frac{(b_1-a_1)^{n_1}}{n_1!} \left(\frac{(b_1-t_1)^{p(\beta_1-1)+1}}{p(\beta_1-1)+1} \right)^{\frac{1}{p}} \left\| {}^{b_1}_{\alpha_1} T(f_1) \right\|_{L_q([a_1,b_1])}, \tag{84}$$

$\forall\ t_1 \in [a_1, b_1]$.

Similarly, by assuming $\beta_2 > \frac{1}{p}$*, we get*

$$|f_2(t_2)| \leq \frac{(b_2-a_2)^{n_2}}{n_2!} \left(\frac{(b_2-t_2)^{q(\beta_2-1)+1}}{q(\beta_2-1)+1} \right)^{\frac{1}{q}} \left\| {}^{b_2}_{\alpha_2} T(f_2) \right\|_{L_p([a_2,b_2])}, \tag{85}$$

$\forall\ t_2 \in [a_2, b_2]$.

Hence it holds

$$|f_1(t_1)|\,|f_2(t_2)| \leq \left[\frac{(b_1-a_1)^{n_1}}{n_1!} \frac{(b_2-a_2)^{n_2}}{n_2!} \right]$$

$$\frac{1}{(p(\beta_1-1)+1)^{\frac{1}{p}} (q(\beta_2-1)+1)^{\frac{1}{q}}} (b_1-t_1)^{\frac{p(\beta_1-1)+1}{p}} (b_2-t_2)^{\frac{q(\beta_2-1)+1}{q}} \tag{86}$$

$$\left\| {}^{b_1}_{\alpha_1} T(f_1) \right\|_{L_q([a_1,b_1])} \left\| {}^{b_2}_{\alpha_2} T(f_2) \right\|_{L_p([a_2,b_2])} \leq$$

(using Young's inequality for $a, b \geq 0$*,* $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$*)*

$$\left(\frac{(b_1-a_1)^{n_1} (b_2-a_2)^{n_2}}{n_1! n_2!} \right) \frac{1}{(p(\beta_1-1)+1)^{\frac{1}{p}} (q(\beta_2-1)+1)^{\frac{1}{q}}}$$

$$\left[\frac{(b_1-t_1)^{p(\beta_1-1)+1}}{p} + \frac{(b_2-t_2)^{q(\beta_2-1)+1}}{q} \right] \tag{87}$$

$$\left\| {}^{b_1}_{\alpha_1} T(f_1) \right\|_{L_q([a_1,b_1])} \left\| {}^{b_2}_{\alpha_2} T(f_2) \right\|_{L_p([a_2,b_2])},$$

$\forall\ t_i \in [a_i, b_i]$; $i = 1, 2$.

Therefore we can write

$$\frac{|f_1(t_1)|\,|f_2(t_2)|}{\left[\frac{(b_1-t_1)^{p(\beta_1-1)+1}}{p}+\frac{(b_2-t_2)^{q(\beta_2-1)+1}}{q}\right]}\leq$$

$$\frac{(b_1-a_1)^{n_1}(b_2-a_2)^{n_2}}{n_1!n_2!\,(p(\beta_1-1)+1)^{\frac{1}{p}}(q(\beta_2-1)+1)^{\frac{1}{q}}} \tag{88}$$

$$\left\|{}_{\alpha_1}^{b_1}T(f_1)\right\|_{L_q([a_1,b_1])}\left\|{}_{\alpha_2}^{b_2}T(f_2)\right\|_{L_p([a_2,b_2])},$$

$\forall\ t_i\in[a_i,b_i]$; $i=1,2$.

The denominator of left hand side of (88) equals 0 *only when both* $t_1=b_1$ *and* $t_2=b_2$.

Therefore it holds

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(t_1)|\,|f_2(t_2)|\,dt_1dt_2}{\left[\frac{(b_1-t_1)^{p(\beta_1-1)+1}}{p}+\frac{(b_2-t_2)^{q(\beta_2-1)+1}}{q}\right]}\leq$$

$$\frac{(b_1-a_1)^{n_1+1}(b_2-a_2)^{n_2+1}\left\|{}_{\alpha_1}^{b_1}T(f_1)\right\|_{L_q([a_1,b_1])}\left\|{}_{\alpha_2}^{b_2}T(f_2)\right\|_{L_p([a_2,b_2])}}{n_1!n_2!\,(p(\beta_1-1)+1)^{\frac{1}{p}}(q(\beta_2-1)+1)^{\frac{1}{q}}}. \tag{89}$$

Notice here that ${}_{\alpha_i}^{b_i}T(f_i)\in C([a_i,b_i])$.

We have proved the right conformable fractional Hilbert-Pachpatte inequality:

Theorem 25 *Let* $\alpha_i\in(n_i,n_i+1]$, $f_i\in C^{n_i+1}([a_i,b_i])$, $[a_i,b_i]\subset\mathbb{R}$, $n_i\in\mathbb{Z}_+$; $\beta_i:=\alpha_i-n_i$, $i=1,2$. *Assume that* $f_i^{(k_i)}(b_i)=0$, $k_i=0,1,...,n_i$; $i=1,2$. *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, *with* $\beta_1>\frac{1}{q}$, $\beta_2>\frac{1}{p}$. *Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(t_1)|\,|f_2(t_2)|\,dt_1dt_2}{\left[\frac{(b_1-t_1)^{p(\beta_1-1)+1}}{p}+\frac{(b_2-t_2)^{q(\beta_2-1)+1}}{q}\right]}\leq$$

$$\frac{(b_1-a_1)^{n_1+1}(b_2-a_2)^{n_2+1}\left\|{}_{\alpha_1}^{b_1}T(f_1)\right\|_{L_q([a_1,b_1])}\left\|{}_{\alpha_2}^{b_2}T(f_2)\right\|_{L_p([a_2,b_2])}}{n_1!n_2!\,(p(\beta_1-1)+1)^{\frac{1}{p}}(q(\beta_2-1)+1)^{\frac{1}{q}}}. \tag{90}$$

Next we present Conformable fractional Ostrowski type inequalities:

Theorem 26 *Let* $\alpha\in(n,n+1]$, $n\in\mathbb{Z}_+$, $f\in C^{n+1}([a,b])$, $\beta:=\alpha-n$; $x_0\in[a,b]$ *be fixed. Assume* $f^{(k)}(x_0)=0$, $k=1,...,n$. *Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f(x_0)\right|\leq\frac{\Gamma(\beta)}{\Gamma(\alpha+2)(b-a)}$$

$$\left\{(x_0-a)^{\alpha+1}\left\|{}_{\alpha}^{x_0}T(f)\right\|_{\infty,[a,x_0]}+(b-x_0)^{\alpha+1}\left\|T_{\alpha}^{x_0}(f)\right\|_{\infty,[x_0,b]}\right\}. \tag{91}$$

Proof. We have (by (69))

$$f(t)-f(x_0)=\frac{1}{n!}\int_{x_0}^{t}(t-x)^n(x-x_0)^{\beta-1}\left(T_{\alpha}^{x_0}(f)\right)(x)\,dx, \tag{92}$$

$\forall\ t\in[x_0,b]$, and (by (70))

$$f(t)-f(x_0)=-\frac{1}{n!}\int_{t}^{x_0}(x_0-x)^{\beta-1}(x-t)^n\left({}_{\alpha}^{x_0}T(f)\right)(x)\,dx, \tag{93}$$

$\forall\ t\in[a,x_0]$.

We observe that

$$\begin{aligned}|f(t)-f(x_0)|\le\frac{1}{n!}\int_{x_0}^{t}(t-x)^n(x-x_0)^{\beta-1}\left|T_{\alpha}^{x_0}(f)(x)\right|dx\le\\ \frac{\left\|T_{\alpha}^{x_0}(f)\right\|_{\infty,[x_0,b]}}{n!}\int_{x_0}^{t}(t-x)^{(n+1)-1}(x-x_0)^{\beta-1}dx=\\ \frac{\left\|T_{\alpha}^{x_0}(f)\right\|_{\infty,[x_0,b]}}{n!}\frac{\Gamma(n+1)\Gamma(\beta)}{\Gamma(n+\beta+1)}(t-x_0)^{n+\beta}=\\ \frac{\left\|T_{\alpha}^{x_0}(f)\right\|_{\infty,[x_0,b]}\Gamma(\beta)}{\Gamma(n+\beta+1)}(t-x_0)^{n+\beta}.\end{aligned} \tag{94}$$

That is

$$|f(t)-f(x_0)|\le\frac{\left\|T_{\alpha}^{x_0}(f)\right\|_{\infty,[x_0,b]}\Gamma(\beta)}{\Gamma(n+\beta+1)}(t-x_0)^{n+\beta},\ \forall\ t\in[x_0,b]. \tag{95}$$

Similarly, it holds

$$\begin{aligned}|f(t)-f(x_0)|\le\frac{1}{n!}\left\|{}_{\alpha}^{x_0}T(f)\right\|_{\infty,[a,x_0]}\int_{t}^{x_0}(x_0-x)^{\beta-1}(x-t)^{(n+1)-1}dx=\\ \frac{\left\|{}_{\alpha}^{x_0}T(f)\right\|_{\infty,[a,x_0]}}{n!}\frac{\Gamma(\beta)\Gamma(n+1)}{\Gamma(\beta+n+1)}(x_0-t)^{\beta+n}=\\ \frac{\left\|{}_{\alpha}^{x_0}T(f)\right\|_{\infty,[a,x_0]}\Gamma(\beta)}{\Gamma(\beta+n+1)}(x_0-t)^{\beta+n}.\end{aligned} \tag{96}$$

That is

$$|f(t)-f(x_0)|\le\frac{\left\|{}_{\alpha}^{x_0}T(f)\right\|_{\infty,[a,x_0]}\Gamma(\beta)}{\Gamma(\beta+n+1)}(x_0-t)^{\beta+n},\ \forall\ t\in[a,x_0]. \tag{97}$$

Hence, we can write

$$\left|\frac{1}{b-a}\int_{a}^{b}f(t)\,dt-f(x_0)\right|\le$$

$$\frac{1}{b-a}\left\{\int_a^{x_0}|f(t)-f(x_0)|\,dt+\int_{x_0}^b|f(t)-f(x_0)|\,dt\right\}\leq$$

$$\frac{\Gamma(\beta)}{\Gamma(n+\beta+1)(b-a)}\left\{\left(\int_a^{x_0}(x_0-t)^{\beta+n}\,dt\right)\left\|{}_{\alpha}^{x_0}T(f)\right\|_{\infty,[a,x_0]}+\right. \tag{98}$$

$$\left.\left(\int_{x_0}^b(t-x_0)^{n+\beta}\,dt\right)\left\|T_\alpha^{x_0}(f)\right\|_{\infty,[x_0,b]}\right\}=$$

$$\frac{\Gamma(\beta)}{\Gamma(n+\beta+1)(b-a)}\left\{(x_0-a)^{\beta+n+1}\left\|{}_{\alpha}^{x_0}T(f)\right\|_{\infty,[a,x_0]}+\right.$$

$$\left.(b-x_0)^{n+\beta+1}\left\|T_\alpha^{x_0}(f)\right\|_{\infty,[x_0,b]}\right\},$$

proving (91). ■

Theorem 27 *Here all as in Theorem 26. Let* $p_1,p_2,p_3>1:\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$, *with* $\beta>\frac{1}{p_1}+\frac{1}{p_3}$. *Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f(x_0)\right|\leq$$

$$\frac{1}{(b-a)\,n!\,(p_1n+1)^{\frac{1}{p_1}}(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_2}+\frac{1}{p_3}\right)}$$

$$\left\{(b-x_0)^{\alpha+\frac{1}{p_2}+\frac{1}{p_3}}\left\|T_\alpha^{x_0}(f)\right\|_{p_3,[x_0,b]}+(x_0-a)^{\alpha+\frac{1}{p_2}+\frac{1}{p_3}}\left\|{}_{\alpha}^{x_0}T(f)\right\|_{p_3,[a,x_0]}\right\}. \tag{99}$$

Proof. By (92) we get

$$|f(t)-f(x_0)|\leq\frac{1}{n!}\int_{x_0}^t(t-x)^n(x-x_0)^{\beta-1}\left|T_\alpha^{x_0}(f)(x)\right|dx\leq$$

$$\frac{1}{n!}\left(\int_{x_0}^t(t-x)^{p_1n}\,dx\right)^{\frac{1}{p_1}}\left(\int_{x_0}^t(x-x_0)^{p_2(\beta-1)}\,dx\right)^{\frac{1}{p_2}}\left\|T_\alpha^{x_0}(f)\right\|_{p_3,[x_0,b]}=$$

$$\frac{\left\|T_\alpha^{x_0}(f)\right\|_{p_3,[x_0,b]}}{n!}\left(\frac{(t-x_0)^{p_1n+1}}{p_1n+1}\right)^{\frac{1}{p_1}}\left(\frac{(t-x_0)^{p_2(\beta-1)+1}}{p_2(\beta-1)+1}\right)^{\frac{1}{p_2}}= \tag{100}$$

$$\frac{\left\|T_\alpha^{x_0}(f)\right\|_{p_3,[x_0,b]}(t-x_0)^{n+\frac{1}{p_1}+\beta-1+\frac{1}{p_2}}}{n!\,(p_1n+1)^{\frac{1}{p_1}}(p_2(\beta-1)+1)^{\frac{1}{p_2}}}.$$

Notice that $p_2(\beta-1)+1>0$, iff $\beta>\frac{1}{p_1}+\frac{1}{p_3}$.

We have proved

$$|f(t)-f(x_0)|\leq\frac{\left\|T_\alpha^{x_0}(f)\right\|_{p_3,[x_0,b]}(t-x_0)^{n+\beta-\frac{1}{p_3}}}{n!\,(p_1n+1)^{\frac{1}{p_1}}(p_2(\beta-1)+1)^{\frac{1}{p_2}}}, \tag{101}$$

$\forall\ t \in [x_0, b]$.

Similarly, we have (by (93))

$$|f(t) - f(x_0)| \leq$$

$$\frac{1}{n!}\left(\int_t^{x_0} (x_0 - x)^{p_2(\beta-1)}\, dx\right)^{\frac{1}{p_2}} \left(\int_t^{x_0} (x - t)^{p_1 n}\, dx\right)^{\frac{1}{p_1}} \left\|{}_{\alpha}^{x_0}T(f)\right\|_{p_3,[a,x_0]} =$$

$$\frac{1}{n!}\left(\frac{(x_0 - t)^{p_2(\beta-1)+1}}{p_2(\beta-1)+1}\right)^{\frac{1}{p_2}} \left(\frac{(x_0 - t)^{p_1 n+1}}{p_1 n + 1}\right)^{\frac{1}{p_1}} \left\|{}_{\alpha}^{x_0}T(f)\right\|_{p_3,[a,x_0]} =$$

$$\frac{\left\|{}_{\alpha}^{x_0}T(f)\right\|_{p_3,[a,x_0]} (x_0 - t)^{\beta+n-\frac{1}{p_3}}}{n!\,(p_2(\beta-1)+1)^{\frac{1}{p_2}} (p_1 n + 1)^{\frac{1}{p_1}}}. \tag{102}$$

We have proved that

$$|f(t) - f(x_0)| \leq \frac{\left\|{}_{\alpha}^{x_0}T(f)\right\|_{p_3,[a,x_0]} (x_0 - t)^{\beta+n-\frac{1}{p_3}}}{n!\,(p_2(\beta-1)+1)^{\frac{1}{p_2}} (p_1 n + 1)^{\frac{1}{p_1}}}, \tag{103}$$

$\forall\ t \in [a, x_0]$, where $\beta > \frac{1}{p_1} + \frac{1}{p_3}$.

Therefore, we derive

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(x_0)\right| \leq$$

$$\frac{1}{b-a}\left\{\int_a^{x_0} |f(t) - f(x_0)|\,dt + \int_{x_0}^b |f(t) - f(x_0)|\,dt\right\} \leq$$

$$\frac{1}{n!\,(p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}} (b-a)}$$

$$\left\{\left(\int_a^{x_0} (x_0 - t)^{\beta+n-\frac{1}{p_3}}\, dt\right) \left\|{}_{\alpha}^{x_0}T(f)\right\|_{p_3,[a,x_0]}\right.$$

$$\left. + \left(\int_{x_0}^b (t - x_0)^{n+\beta-\frac{1}{p_3}}\, dt\right) \left\|T_{\alpha}^{x_0}(f)\right\|_{p_3,[x_0,b]}\right\} = \tag{104}$$

$$\frac{1}{n!\,(p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}} (b-a)} \left\{\frac{(x_0 - a)^{\beta+n+\frac{1}{p_2}+\frac{1}{p_3}}}{\left(\beta + n + \frac{1}{p_2} + \frac{1}{p_3}\right)} \left\|{}_{\alpha}^{x_0}T(f)\right\|_{p_3,[a,x_0]}\right.$$

$$\left. + \frac{(b - x_0)^{n+\beta+\frac{1}{p_2}+\frac{1}{p_3}}}{\left(\beta + n + \frac{1}{p_2} + \frac{1}{p_3}\right)} \left\|T_{\alpha}^{x_0}(f)\right\|_{p_3,[x_0,b]}\right\}, \tag{105}$$

proving (99). ■

We make

Remark 28 *Here we will discuss about generalised conformable fractional Ostrowski and Grüss type inequalities involving several functions.*

Let $\alpha \in (n, n+1]$, $n \in \mathbb{Z}_+$, $f_i \in C^{n+1}([a,b])$, $i = 1, ..., r \in \mathbb{N}$, $[a,b] \subset \mathbb{R}$, $\beta := \alpha - n$, $x_0 \in [a,b]$, *and* $f_i^{(k)}(x_0) = 0$, $k = 1, ..., n$; $i = 1, ..., r$.

If $n = 0$, *initial conditions are void, i.e.* $0 < \alpha \leq 1$.

By (69) and (70) we get that

$$f_i(t) - f_i(x_0) = \frac{1}{n!} \int_{x_0}^{t} (t-x)^n (x-x_0)^{\beta-1} \left(T_{\alpha}^{x_0}(f_i)\right)(x)\, dx, \tag{106}$$

$\forall\ t \in [x_0, b]$, *all* $i = 1, ..., r$,

and

$$f_i(t) - f_i(x_0) = -\frac{1}{n!} \int_{t}^{x_0} (x_0 - x)^{\beta-1} (x-t)^n \left({}_{\alpha}^{x_0}T(f_i)\right)(x)\, dx, \tag{107}$$

$\forall\ t \in [a, x_0]$, *all* $i = 1, ..., r$.

Multiply (106), (107) by $\prod\limits_{\substack{j=1 \\ j \neq i}}^{r} f_j(t)$ *to get*

$$\prod_{k=1}^{r} f_k(t) - \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(t)\right) f_i(x_0) =$$

$$\frac{\prod\limits_{\substack{j=1 \\ j \neq i}}^{r} f_j(t)}{n!} \int_{x_0}^{t} (t-x)^n (x-x_0)^{\beta-1} \left(T_{\alpha}^{x_0}(f_i)\right)(x)\, dx, \tag{108}$$

and

$$\prod_{k=1}^{r} f_k(t) - \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(t)\right) f_i(x_0) =$$

$$-\frac{\prod\limits_{\substack{j=1 \\ j \neq i}}^{r} f_j(t)}{n!} \int_{t}^{x_0} (x_0-x)^{\beta-1} (x-t)^n \left({}_{\alpha}^{x_0}T(f_i)\right)(x)\, dx, \tag{109}$$

$\forall\ i = 1, ..., r$.

Adding (108), (109) per set, we obtain

$$r\left(\prod_{k=1}^{r} f_k(t)\right) - \sum_{i=1}^{r} \left[\left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(t)\right) f_i(x_0)\right] =$$

$$\frac{1}{n!}\sum_{i=1}^{r}\left[\prod_{\substack{j=1\\j\neq i}}^{r}f_j(t)\int_{x_0}^{t}(t-x)^n(x-x_0)^{\beta-1}\left(T_{\alpha}^{x_0}(f_i)\right)(x)\,dx\right], \tag{110}$$

$\forall\ t\in[x_0,b]$, *and*

$$r\left(\prod_{k=1}^{r}f_k(t)\right)-\sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(t)\right)f_i(x_0)\right]=$$

$$-\frac{1}{n!}\sum_{i=1}^{r}\left[\prod_{\substack{j=1\\j\neq i}}^{r}f_j(t)\int_{t}^{x_0}(x_0-x)^{\beta-1}(x-t)^n\left({}_{\alpha}^{x_0}T(f_i)\right)(x)\,dx\right], \tag{111}$$

$\forall\ t\in[a,x_0]$.

Next we integrate (110), (111) with respect to $t\in[a,b]$. *We have*

$$r\int_{x_0}^{b}\left(\prod_{k=1}^{r}f_k(t)\right)dt-\sum_{i=1}^{r}\left[f_i(x_0)\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(t)\right)dt\right]=$$

$$\frac{1}{n!}\sum_{i=1}^{r}\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(t)\right)\left(\int_{x_0}^{t}(t-x)^n(x-x_0)^{\beta-1}\left(T_{\alpha}^{x_0}(f_i)\right)(x)\,dx\right)dt\right], \tag{112}$$

and

$$r\int_{a}^{x_0}\left(\prod_{k=1}^{r}f_k(t)\right)dt-\sum_{i=1}^{r}\left[f_i(x_0)\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(t)\right)dt\right]=$$

$$-\frac{1}{n!}\sum_{i=1}^{r}\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(t)\right)\left(\int_{t}^{x_0}(x_0-x)^{\beta-1}(x-t)^n\left({}_{\alpha}^{x_0}T(f_i)\right)(x)\,dx\right)dt\right]. \tag{113}$$

Adding (112), (113) we obtain

$$\theta(f_1,...,f_r)(x_0):=r\int_{a}^{b}\left(\prod_{k=1}^{r}f_k(t)\right)dt-\sum_{i=1}^{r}\left[f_i(x_0)\int_{a}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(t)\right)dt\right]$$

$$= \frac{1}{n!} \sum_{i=1}^{r} \left[\left[\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(t) \right) \left(\int_{x_0}^{t} (t-x)^n (x-x_0)^{\beta-1} \left(T_{\alpha}^{x_0}(f_i) \right)(x)\, dx \right) dt \right] \right.$$

$$\left. - \left[\int_{a}^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(t) \right) \left(\int_{t}^{x_0} (x_0-x)^{\beta-1} (x-t)^n \left({}_{\alpha}^{x_0}T(f_i) \right)(x)\, dx \right) dt \right] \right]. \tag{114}$$

Hence, it holds

$$|\theta(f_1, ..., f_r)(x_0)| \leq$$

$$\frac{1}{n!} \sum_{i=1}^{r} \left[\left[\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(t)| \right) \left(\int_{x_0}^{t} (t-x)^n (x-x_0)^{\beta-1} \left| T_{\alpha}^{x_0}(f_i)(x) \right| dx \right) dt \right] \right.$$

$$\left. + \left[\int_{a}^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(t)| \right) \left(\int_{t}^{x_0} (x_0-x)^{\beta-1} (x-t)^n \left| {}_{\alpha}^{x_0}T(f_i)(x) \right| dx \right) dt \right] \right] =: (*). \tag{115}$$

We notice that

$$(*) \leq \sum_{i=1}^{r} \left[\left[\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(t)| \right) \frac{\left\| T_{\alpha}^{x_0}(f_i) \right\|_{\infty,[x_0,b]} \Gamma(\beta)}{\Gamma(n+\beta+1)} (t-x_0)^{n+\beta} dt \right] + \right.$$

$$\left. + \left[\int_{a}^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(t)| \right) \frac{\left\| {}_{\alpha}^{x_0}T(f_i) \right\|_{\infty,[a,x_0]} \Gamma(\beta)}{\Gamma(\beta+n+1)} (x_0-t)^{\beta+n} dt \right] \right]. \tag{116}$$

Thus we have proved so far

$$|\theta(f_1, ..., f_r)(x_0)| \leq \frac{\Gamma(\beta)}{\Gamma(\beta+n+1)}$$

$$\sum_{i=1}^{r} \left[\left[\left\| T_{\alpha}^{x_0}(f_i) \right\|_{\infty,[x_0,b]} \int_{x_0}^{b} (t-x_0)^{n+\beta} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(t)| \right) dt \right] + \right.$$

$$\left. \left[\left\| {}_{\alpha}^{x_0}T(f_i) \right\|_{\infty,[a,x_0]} \int_{a}^{x_0} (x_0-t)^{\beta+n} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(t)| \right) dt \right] \right]. \tag{117}$$

We further notice that

$$|\theta(f_1,...,f_r)(x_0)| \leq \frac{\Gamma(\beta)}{\Gamma(\beta+n+2)}$$

$$\sum_{i=1}^{r}\left[\left[\|T_{\alpha}^{x_0}(f_i)\|_{\infty,[x_0,b]}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j\|_{\infty,[x_0,b]}\right)(b-x_0)^{n+\beta+1}\right]+\right.$$

$$\left.\left[\|_{\alpha}^{x_0}T(f_i)\|_{\infty,[a,x_0]}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j\|_{\infty,[a,x_0]}\right)(x_0-a)^{n+\beta+1}\right]\right], \tag{118}$$

which is an ∞*-Ostrowski type inequality.*

Next let $p_1,p_2,p_3>1: \frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$*, such that* $\beta>\frac{1}{p_1}+\frac{1}{p_3}$*.*

Hence we can write

$$(*) \leq \frac{1}{n!}\sum_{i=1}^{r}\left[\left[\frac{\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(t)|\right)\|T_{\alpha}^{x_0}(f_i)\|_{p_3,[x_0,b]}(t-x_0)^{n+\beta-\frac{1}{p_3}}}{(p_1n+1)^{\frac{1}{p_1}}(p_2(\beta-1)+1)^{\frac{1}{p_2}}}dt\right]\right. \tag{119}$$

$$\left.+\left[\frac{\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(t)|\right)\|_{\alpha}^{x_0}T(f_i)\|_{p_3,[a,x_0]}(x_0-t)^{\beta+n-\frac{1}{p_3}}}{(p_1n+1)^{\frac{1}{p_1}}(p_2(\beta-1)+1)^{\frac{1}{p_2}}}dt\right]\right]=$$

$$\frac{1}{n!(p_1n+1)^{\frac{1}{p_1}}(p_2(\beta-1)+1)^{\frac{1}{p_2}}}$$

$$\sum_{i=1}^{r}\left[\left[\left(\int_{x_0}^{b}(t-x_0)^{n+\beta-\frac{1}{p_3}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(t)|\right)dt\right)\|T_{\alpha}^{x_0}(f_i)\|_{p_3,[x_0,b]}\right]\right.$$

$$\left.+\left[\left(\int_{a}^{x_0}(x_0-t)^{\beta+n-\frac{1}{p_3}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(t)|\right)dt\right)\|_{\alpha}^{x_0}T(f_i)\|_{p_3,[a,x_0]}\right]\right] \leq \tag{120}$$

$$\frac{1}{n!(p_1n+1)^{\frac{1}{p_1}}(p_2(\beta-1)+1)^{\frac{1}{p_2}}}$$

$$\sum_{i=1}^{r}\left[\left[\frac{(b-x_0)^{n+\beta+\frac{1}{p_1}+\frac{1}{p_2}}}{\left(n+\beta+\frac{1}{p_1}+\frac{1}{p_2}\right)}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j\|_{\infty,[x_0,b]}\right)\left\|T_{\alpha}^{x_0}(f_i)\right\|_{p_3,[x_0,b]}\right]\right. \tag{121}$$

$$\left.+\left[\frac{(x_0-a)^{n+\beta+\frac{1}{p_1}+\frac{1}{p_2}}}{\left(n+\beta+\frac{1}{p_1}+\frac{1}{p_2}\right)}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j\|_{\infty,[a,x_0]}\right)\left\|{}_{\alpha}^{x_0}T(f_i)\right\|_{p_3,[a,x_0]}\right]\right].$$

We have proved the L_p-Ostrowski type inequality:

$$|\theta(f_1,...,f_r)(x_0)|\leq$$

$$\frac{1}{n!\,(p_1n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(n+\beta+\frac{1}{p_1}+\frac{1}{p_2}\right)}$$

$$\sum_{i=1}^{r}\left[\left[(b-x_0)^{n+\beta+\frac{1}{p_1}+\frac{1}{p_2}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j\|_{\infty,[x_0,b]}\right)\left\|T_{\alpha}^{x_0}(f_i)\right\|_{p_3,[x_0,b]}\right]\right.$$

$$\left.+\left[(x_0-a)^{n+\beta+\frac{1}{p_1}+\frac{1}{p_2}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j\|_{\infty,[a,x_0]}\right)\left\|{}_{\alpha}^{x_0}T(f_i)\right\|_{p_3,[a,x_0]}\right]\right]. \tag{122}$$

From now on we assume $0<\alpha\leq 1$, i.e. $n=0$. So no initial conditions are needed.

Notice that

$$\Delta(f_1,...,f_r):=\int_a^b\theta(f_1,...,f_r)(x)\,dx=$$

$$r(b-a)\left(\int_a^b\left(\prod_{k=1}^{r}f_k(x)\,dx\right)\right)-$$

$$\sum_{i=1}^{r}\left[\left(\int_a^b f_i(x)\,dx\right)\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)dx\right)\right], \tag{123}$$

and it holds

$$|\Delta(f_1,...,f_r)|\leq\int_a^b|\theta(f_1,...,f_r)(x)|\,dx. \tag{124}$$

By (124) and (118) we get the ∞-Gruss type inequality (here $\alpha=\beta$):

$$|\Delta(f_1,...,f_r)|\leq\frac{\Gamma(\alpha)(b-a)^{a+2}}{\Gamma(\alpha+3)}$$

$$\sum_{i=1}^{r}\left[\left(\sup_{x_0\in[a,b]}\left\|T_{\alpha}^{x_0}(f_i)\right\|_{\infty,[x_0,b]}\right)\left(\prod_{\substack{j=1\\j\neq i}}^{r}\sup_{x_0\in[a,b]}\|f_j\|_{\infty,[x_0,b]}\right)\right. \tag{125}$$

$$\left.+\left(\sup_{x_0\in[a,b]}\left\|{}_{\alpha}^{x_0}T(f_i)\right\|_{\infty,[a,x_0]}\right)\left(\prod_{\substack{j=1\\j\neq i}}^{r}\sup_{x_0\in[a,b]}\|f_j\|_{\infty,[a,x_0]}\right)\right].$$

We have proved that

$$|\Delta(f_1,...,f_r)|\leq\frac{\Gamma(\alpha)(b-a)^{a+2}}{\Gamma(\alpha+3)}$$

$$\sum_{i=1}^{r}\left[\left(\sup_{x_0\in[a,b]}\left\|{}_{\alpha}^{x_0}T(f_i)\right\|_{\infty,[a,x_0]}+\sup_{x_0\in[a,b]}\left\|T_{\alpha}^{x_0}(f_i)\right\|_{\infty,[x_0,b]}\right)\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j\|_{\infty,[a,b]}\right)\right]. \tag{126}$$

Next by (122) we get the L_p-Gruss inequality:

$$|\Delta(f_1,...,f_r)|\leq$$

$$\frac{(b-a)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}+1}}{n!\,(p_1n+1)^{\frac{1}{p_1}}(p_2(\alpha-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}+1\right)}$$

$$\sum_{i=1}^{r}\left[\left(\sup_{x_0\in[a,b]}\left\|T_{\alpha}^{x_0}(f_i)\right\|_{p_3,[x_0,b]}+\sup_{x_0\in[a,b]}\left\|{}_{\alpha}^{x_0}T(f_i)\right\|_{p_3,[a,x_0]}\right)\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j\|_{\infty,[a,b]}\right)\right]. \tag{127}$$

We have proved the following results:

An ∞-Ostrowski type Conformable fractional inequality for several functions follows:

Theorem 29 *Let $\alpha\in(n,n+1]$, $n\in\mathbb{Z}_+$, $f_i\in C^{n+1}([a,b])$, $i=1,...,r\in\mathbb{N}$, $[a,b]\subset\mathbb{R}$, $\beta:=\alpha-n$, $x_0\in[a,b]$, and $f_i^{(k)}(x_0)=0$, $k=1,...,n$; $i=1,...,r$. Call*

$$\theta(f_1,...,f_r)(x_0):=r\int_a^b\left(\prod_{k=1}^{r}f_k(t)\right)dt-\sum_{i=1}^{r}\left[f_i(x_0)\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(t)\right)dt\right]. \tag{128}$$

Then

$$|\theta(f_1,...,f_r)(x_0)|\leq\frac{\Gamma(\beta)}{\Gamma(\alpha+2)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|T_{\alpha}^{x_0}\left(f_i\right)\right\|_{\infty,[x_0,b]}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j\right\|_{\infty,[x_0,b]}\right)\left(b-x_0\right)^{\alpha+1}\right]+\right.$$

$$\left.\left[\left\|{}_{\alpha}^{x_0}T\left(f_i\right)\right\|_{\infty,[a,x_0]}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j\right\|_{\infty,[a,x_0]}\right)\left(x_0-a\right)^{\alpha+1}\right]\right]. \quad (129)$$

Next follows the corresponding L_p-Ostrowski inequality for several functions.

Theorem 30 *All as in Theorem 29. Let* $p_1, p_2, p_3 > 1 : \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ *such that* $\beta > \frac{1}{p_1} + \frac{1}{p_3}$. *Then*

$$\left|\theta\left(f_1, ..., f_r\right)\left(x_0\right)\right| \leq \quad (130)$$

$$\frac{1}{n!\left(p_1 n+1\right)^{\frac{1}{p_1}}\left(p_2\left(\beta-1\right)+1\right)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}$$

$$\sum_{i=1}^{r}\left[\left[\left(b-x_0\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j\right\|_{\infty,[x_0,b]}\right)\left\|T_{\alpha}^{x_0}\left(f_i\right)\right\|_{p_3,[x_0,b]}\right]\right.$$

$$\left.+\left[\left(x_0-a\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j\right\|_{\infty,[a,x_0]}\right)\left\|{}_{\alpha}^{x_0}T\left(f_i\right)\right\|_{p_3,[a,x_0]}\right]\right].$$

The corresponding Gruss type inequalities follow:

Theorem 31 *Let all as in Theorem 29, with* $0 < \alpha \leq 1$. *We denote*

$$\Delta\left(f_1, ..., f_r\right) := \int_a^b \theta\left(f_1, ..., f_r\right)(x)\,dx =$$

$$r\left(b-a\right)\left(\int_a^b\left(\prod_{k=1}^{r} f_k(x)\,dx\right)\right)- \quad (131)$$

$$\sum_{i=1}^{r}\left[\left(\int_a^b f_i(x)\,dx\right)\left(\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)dx\right)\right].$$

Then

$$\left|\Delta\left(f_1, ..., f_r\right)\right| \leq \frac{\Gamma(\alpha)\left(b-a\right)^{a+2}}{\Gamma(\alpha+3)}$$

$$\sum_{i=1}^{r}\left[\left(\sup_{x_0\in[a,b]}\left\|{}_{\alpha}^{x_0}T\left(f_i\right)\right\|_{\infty,[a,x_0]}+\sup_{x_0\in[a,b]}\left\|T_{\alpha}^{x_0}\left(f_i\right)\right\|_{\infty,[x_0,b]}\right)\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j\right\|_{\infty,[a,b]}\right)\right]. \tag{132}$$

Theorem 32 *Here all as in Theorems 29 and 31. Let* $p_1,p_2,p_3>1:\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$, *with* $0<\alpha\leq 1$, *such that* $\beta>\frac{1}{p_1}+\frac{1}{p_3}$. *Then*

$$\left|\Delta\left(f_1,...,f_r\right)\right|\leq$$

$$\frac{(b-a)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}+1}}{n!\left(p_1n+1\right)^{\frac{1}{p_1}}\left(p_2\left(\alpha-1\right)+1\right)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}+1\right)}$$

$$\sum_{i=1}^{r}\left[\left(\sup_{x_0\in[a,b]}\left\|T_{\alpha}^{x_0}\left(f_i\right)\right\|_{p_3,[x_0,b]}+\sup_{x_0\in[a,b]}\left\|{}_{\alpha}^{x_0}T\left(f_i\right)\right\|_{p_3,[a,x_0]}\right)\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j\right\|_{\infty,[a,b]}\right)\right]. \tag{133}$$

We make

Remark 33 *Here we discuss about Conformable fractional left Opial inequality.*

Let $\alpha\in(n,n+1]$, $n\in\mathbb{Z}_+$, $f\in C^{n+1}([a,b])$ $(\beta:=\alpha-n,\ 0<\beta\leq 1)$. *Assume* $f^{(k)}(a)=0$, $k=0,1,...,n$, *then (by (69))*

$$f(t)=\frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\beta-1}\left(T_{\alpha}^{a}(f)\right)(x)\,dx, \tag{134}$$

$\forall\ t\in[a,b]$.

Let $a\leq w\leq t$, *then we have*

$$f(w)=\frac{1}{n!}\int_a^w (w-x)^n (x-a)^{\beta-1}\left(T_{\alpha}^{a}(f)\right)(x)\,dx. \tag{135}$$

Then

$$|f(w)|\leq\frac{(b-a)^n}{n!}\int_a^w (x-a)^{\beta-1}\left|T_{\alpha}^{a}(f)(x)\right|dx\leq$$

$$\frac{(b-a)^n}{n!}\left(\int_a^w (x-a)^{(\beta-1)p}\,dx\right)^{\frac{1}{p}}\left(\int_a^w \left|T_{\alpha}^{a}(f)(x)\right|^q dx\right)^{\frac{1}{q}}=$$

$$\frac{(b-a)^n}{n!}\left(\frac{(w-a)^{p(\beta-1)+1}}{p(\beta-1)+1}\right)^{\frac{1}{p}}(z(w))^{\frac{1}{q}}, \tag{136}$$

where

$$z(w):=\int_a^w \left|T_{\alpha}^{a}(f)(x)\right|^q dx,\quad \textit{all } a\leq w\leq t, \tag{137}$$

[we need $p(\beta-1)+1>0 \Leftrightarrow p(\beta-1)>-1 \Leftrightarrow \beta-1>-\frac{1}{p} \Leftrightarrow \beta>1-\frac{1}{p}=\frac{1}{q}$, *so we assume that* $\beta>\frac{1}{q}$*]*

and

$$z(a)=0. \tag{138}$$

Thus

$$z'(w)=\left|T_{\alpha}^{a}(f)(w)\right|^{q}, \text{ and } \left|T_{\alpha}^{a}f(w)\right|=(z'(w))^{\frac{1}{q}}. \tag{139}$$

Therefore we obtain

$$|f(w)|\left|T_{\alpha}^{a}f(w)\right| \leq \frac{(b-a)^{n}}{n!}\frac{(w-a)^{\frac{p(\beta-1)+1}{p}}}{(p(\beta-1)+1)^{\frac{1}{p}}}(z(w)z'(w))^{\frac{1}{q}}. \tag{140}$$

Integrating the last inequality we get

$$\int_{a}^{t}|f(w)|\left|T_{\alpha}^{a}f(w)\right|dw \leq$$

$$\frac{(b-a)^{n}}{n!(p(\beta-1)+1)^{\frac{1}{p}}}\int_{a}^{t}(w-a)^{\frac{p(\beta-1)+1}{p}}(z(w)z'(w))^{\frac{1}{q}}dw \leq$$

$$\frac{(b-a)^{n}}{n!(p(\beta-1)+1)^{\frac{1}{p}}}\left(\int_{a}^{t}(w-a)^{p(\beta-1)+1}dw\right)^{\frac{1}{p}}\left(\int_{a}^{t}z(w)z'(w)dw\right)^{\frac{1}{q}}= \tag{141}$$

$$\frac{(b-a)^{n}}{n!(p(\beta-1)+1)^{\frac{1}{p}}}\left(\frac{(t-a)^{p(\beta-1)+2}}{p(\beta-1)+2}\right)^{\frac{1}{p}}\left(\int_{a}^{t}z(w)dz(w)\right)^{\frac{1}{q}}=$$

$$\frac{(b-a)^{n}}{n!(p(\beta-1)+1)^{\frac{1}{p}}}\frac{(t-a)^{(\beta-1)+\frac{2}{p}}}{(p(\beta-1)+2)^{\frac{1}{p}}}\left(\frac{z^{2}(t)}{2}\right)^{\frac{1}{q}}=$$

$$\frac{(b-a)^{n}(t-a)^{\beta-1+\frac{2}{p}}}{n!2^{\frac{1}{q}}\left[(p(\beta-1)+1)(p(\beta-1)+2)\right]^{\frac{1}{p}}}\left(\int_{a}^{t}\left|T_{\alpha}^{a}(f)(x)\right|^{q}dx\right)^{\frac{2}{q}}. \tag{142}$$

We have proved the conformable left fractional Opial inequality:

Theorem 34 *Let* $\alpha\in(n,n+1]$, $n\in\mathbb{Z}_{+}$, $f\in C^{n+1}([a,b])$, $\beta:=\alpha-n$. *Assume* $f^{(k)}(a)=0$, $k=0,1,...,n$. *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\beta>\frac{1}{q}$. *Then*

$$\int_{a}^{t}|f(w)|\left|T_{\alpha}^{a}f(w)\right|dw \leq$$

$$\frac{(b-a)^{n}(t-a)^{\beta-1+\frac{2}{p}}}{n!2^{\frac{1}{q}}\left[(p(\beta-1)+1)(p(\beta-1)+2)\right]^{\frac{1}{p}}}\left(\int_{a}^{t}\left|T_{\alpha}^{a}(f)(x)\right|^{q}dx\right)^{\frac{2}{q}}, \tag{143}$$

$\forall\, t\in[a,b]$.

We make

Remark 35 *Here we discuss the Conformable right fractional Opial inequality.*

Let $\alpha \in (n, n+1]$, $n \in \mathbb{Z}_+$, $f \in C^{n+1}([a,b])$ $(\beta := \alpha - n,\ 0 < \beta \leq 1)$. *Assume that* $f^{(k)}(b) = 0$, $k = 0, 1, ..., n$, *then (by (70))*

$$f(t) = -\frac{1}{n!} \int_t^b (b-x)^{\beta-1} (x-t)^n \left({}^b_\alpha T(f) \right)(x)\, dx, \tag{144}$$

$\forall\ t \in [a,b]$.

Let $t \leq w \leq b$, *then we have*

$$f(w) = -\frac{1}{n!} \int_w^b (b-x)^{\beta-1} (x-w)^n \left({}^b_\alpha T(f) \right)(x)\, dx. \tag{145}$$

Then

$$|f(w)| \leq \frac{(b-a)^n}{n!} \int_w^b (b-x)^{\beta-1} \left| {}^b_\alpha T(f)(x) \right| dx \leq$$

$$\frac{(b-a)^n}{n!} \left(\int_w^b (b-x)^{p(\beta-1)} dx \right)^{\frac{1}{p}} \left(\int_w^b \left| {}^b_\alpha T(f)(x) \right|^q dx \right)^{\frac{1}{q}} =$$

$$\frac{(b-a)^n}{n!} \frac{(b-w)^{\frac{p(\beta-1)+1}{p}}}{(p(\beta-1)+1)^{\frac{1}{p}}} (z(w))^{\frac{1}{q}}, \tag{146}$$

where

$$z(w) := \int_w^b \left| {}^b_\alpha T(f)(x) \right|^q dx, \tag{147}$$

$t \leq w \leq b, z(b) = 0$. *Thus*

$$-z(w) := \int_b^w \left| {}^b_\alpha T(f)(x) \right|^q dx, \tag{148}$$

and

$$(-z(w))' = \left| {}^b_\alpha T(f)(x) \right|^q \geq 0, \tag{149}$$

and

$$\left| {}^b_\alpha T(f)(x) \right| = \left((-z(w))' \right)^{\frac{1}{q}} = (-z'(w))^{\frac{1}{q}}. \tag{150}$$

(want $p(\beta-1)+1 > 0 \Leftrightarrow p(\beta-1) > -1 \Leftrightarrow \beta - 1 > -\frac{1}{p} \Leftrightarrow \beta > 1 - \frac{1}{p} = \frac{1}{q}$, *so we assume* $\beta > \frac{1}{q}$*).*

Therefore we obtain

$$|f(w)| \left| {}^b_\alpha T(f)(w) \right| \leq \frac{(b-a)^n}{n!} \frac{(b-w)^{\frac{p(\beta-1)+1}{p}}}{(p(\beta-1)+1)^{\frac{1}{p}}} \left(z(w)(-z'(w)) \right)^{\frac{1}{q}}, \tag{151}$$

all $t \leq w \leq b$.

Hence it holds

$$\int_t^b |f(w)| \left| {}_\alpha^b T(f)(w) \right| dw \leq$$

$$\frac{(b-a)^n}{n!\,(p(\beta-1)+1)^{\frac{1}{p}}} \int_t^b (b-w)^{\frac{p(\beta-1)+1}{p}} \left(z(w)\left(-z'(w)\right)\right)^{\frac{1}{q}} dw \leq$$

$$\frac{(b-a)^n}{n!\,(p(\beta-1)+1)^{\frac{1}{p}}} \left(\int_t^b (b-w)^{p(\beta-1)+1} dw\right)^{\frac{1}{p}} \left(\int_t^b z(w)\left(-z'(w)\right) dw\right)^{\frac{1}{q}} = \tag{152}$$

$$\frac{(b-a)^n}{n!\,(p(\beta-1)+1)^{\frac{1}{p}}} \frac{(b-t)^{(\beta-1)+\frac{2}{p}}}{(p(\beta-1)+2)^{\frac{1}{p}}} \frac{(z(t))^{\frac{2}{q}}}{2^{\frac{1}{q}}} =$$

$$\frac{(b-a)^n (b-t)^{\beta-1+\frac{2}{p}}}{n!2^{\frac{1}{q}} \left[(p(\beta-1)+1)(p(\beta-1)+2)\right]^{\frac{1}{p}}} \left(\int_t^b \left| {}_\alpha^b T(f)(x) \right|^q dx\right)^{\frac{2}{q}}. \tag{153}$$

We have proved the Conformable right fractional Opial type inequality:

Theorem 36 *Let* $\alpha \in (n, n+1]$, $n \in \mathbb{Z}_+$, $\beta := \alpha - n$, $f \in C^{n+1}([a,b])$. *Assume* $f^{(k)}(b) = 0$, $k = 0, 1, ..., n$. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ *such that* $\beta > \frac{1}{q}$. *Then*

$$\int_t^b |f(w)| \left| {}_\alpha^b T(f)(w) \right| dw \leq$$

$$\frac{(b-a)^n (b-t)^{\beta-1+\frac{2}{p}}}{2^{\frac{1}{q}} n! \left[(p(\beta-1)+1)(p(\beta-1)+2)\right]^{\frac{1}{p}}} \left(\int_t^b \left| {}_\alpha^b T(f)(x) \right|^q dx\right)^{\frac{2}{q}}, \tag{154}$$

$\forall\ t \in [a,b]$.

Next we give a left conformable fractional Poincare type inequality:

Theorem 37 *Let* $\alpha \in (n, n+1]$, $n \in \mathbb{Z}_+$, $f \in C^{n+1}([a,b])$, $\beta := \alpha - n$. *Assume* $f^{(k)}(a) = 0$, $k = 0, 1, ..., n$. *Let* $p_1, p_2, p_3 > 1 : \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, *such that* $\beta > \frac{1}{p_1} + \frac{1}{p_3}$. *Then*

$$\|f\|_{p_3,[a,b]} \leq \frac{(b-a)^\alpha \|T_\alpha^a f\|_{p_3,[a,b]}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}} (\alpha p_3)^{\frac{1}{p_3}}}. \tag{155}$$

Proof. Since $f^{(k)}(a) = 0$, $k = 0, 1, ..., n$, then (by (69))

$$f(t) = \frac{1}{n!} \int_a^t (t-x)^n (x-a)^{\beta-1} \left(T_\alpha^a(f)\right)(x)\, dx, \tag{156}$$

$\forall\ t \in [a,b]$.

Let $p_1, p_2, p_3 > 1 : \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$. Then

$$|f(t)| \leq \frac{1}{n!} \int_a^t (t-x)^n (x-a)^{\beta-1} |T_\alpha^a(f)(x)| \, dx \leq$$

$$\frac{1}{n!} \left(\int_a^t (t-x)^{p_1 n} \, dx \right)^{\frac{1}{p_1}} \left(\int_a^t (x-a)^{p_2(\beta-1)} \, dx \right)^{\frac{1}{p_2}} \|T_\alpha^a f\|_{p_3,[a,b]} = \tag{157}$$

$$\frac{1}{n!} \frac{(t-a)^{\frac{p_1 n+1}{p_1}}}{(p_1 n+1)^{\frac{1}{p_1}}} \frac{(t-a)^{\frac{p_2(\beta-1)+1}{p_2}}}{(p_2(\beta-1)+1)^{\frac{1}{p_2}}} \|T_\alpha^a f\|_{p_3,[a,b]} =$$

$$\frac{(t-a)^{n+\frac{1}{p_1}+\beta-1+\frac{1}{p_2}}}{n! (p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}}} \|T_\alpha^a f\|_{p_3,[a,b]} = \tag{158}$$

$$\frac{(t-a)^{n+\beta-\frac{1}{p_3}}}{n! (p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}}} \|T_\alpha^a f\|_{p_3,[a,b]} .$$

We have proved

$$|f(t)| \leq \frac{(t-a)^{n+\beta-\frac{1}{p_3}}}{n! (p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}}} \|T_\alpha^a f\|_{p_3,[a,b]} , \tag{159}$$

$\forall \, t \in [a,b]$.

Then

$$|f(t)|^{p_3} \leq \frac{(t-a)^{p_3(n+\beta)-1}}{(n!)^{p_3} (p_1 n+1)^{\frac{p_3}{p_1}} (p_2(\beta-1)+1)^{\frac{p_3}{p_2}}} \|T_\alpha^a f\|_{p_3,[a,b]}^{p_3} . \tag{160}$$

Therefore, it holds

$$\int_a^b |f(t)|^{p_3} \, dt \leq \frac{\int_a^b (t-a)^{p_3(n+\beta)-1} \, dt}{(n!)^{p_3} (p_1 n+1)^{\frac{p_3}{p_1}} (p_2(\beta-1)+1)^{\frac{p_3}{p_2}}} \|T_\alpha^a f\|_{p_3,[a,b]}^{p_3} =$$

$$\frac{(b-a)^{p_3(n+\beta)} \|T_\alpha^a f\|_{p_3,[a,b]}^{p_3}}{(n!)^{p_3} (p_1 n+1)^{\frac{p_3}{p_1}} (p_2(\beta-1)+1)^{\frac{p_3}{p_2}} p_3(n+\beta)} . \tag{161}$$

Consequently, we get

$$\|f\|_{p_3,[a,b]} \leq \frac{(b-a)^{(n+\beta)} \|T_\alpha^a f\|_{p_3,[a,b]}}{n! (p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}} (p_3(n+\beta))^{\frac{1}{p_3}}} . \tag{162}$$

(we want $p_2(\beta-1)+1 > 0 \Leftrightarrow p_2(\beta-1) > -1 \Leftrightarrow \beta - 1 > -\frac{1}{p_2} \Leftrightarrow \beta > 1 - \frac{1}{p_2} \Leftrightarrow \beta > \frac{1}{p_1} + \frac{1}{p_3}$, by assumption). ■

It follows the right conformable fractional Poincare type inequality:

Theorem 38 *Let* $\alpha \in (n, n+1]$, $n \in \mathbb{Z}_+$, $f \in C^{n+1}([a,b])$, $\beta := \alpha - n$, $f^{(k)}(b) = 0$, $k = 0, 1, ..., n$. *Let* $p_1, p_2, p_3 > 1 : \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, *such that* $\beta > \frac{1}{p_1} + \frac{1}{p_3}$. *Then*

$$\|f\|_{p_3,[a,b]} \leq \frac{(b-a)^{\alpha} \left\| {}_{\alpha}^{b}T(f) \right\|_{p_3,[a,b]}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}} (\alpha p_3)^{\frac{1}{p_3}}}. \tag{163}$$

Proof. By (70) we get ($\forall\ t \in [a,b]$)

$$f(t) = -\frac{1}{n!} \int_t^b (b-x)^{\beta-1} (x-t)^n \left({}_{\alpha}^{b}T(f) \right)(x)\,dx. \tag{164}$$

Hence

$$|f(t)| \leq \frac{1}{n!} \int_t^b (b-x)^{\beta-1} (x-t)^n \left| {}_{\alpha}^{b}T(f)(x) \right| dx \leq$$

$$\frac{1}{n!} \left(\int_t^b (x-t)^{p_1 n} dx \right)^{\frac{1}{p_1}} \left(\int_t^b (b-x)^{p_2(\beta-1)} dx \right)^{\frac{1}{p_2}} \left\| {}_{\alpha}^{b}T(f) \right\|_{p_3,[a,b]} = \tag{165}$$

$$\frac{1}{n!} \frac{(b-t)^{\frac{p_1 n+1}{p_1}}}{(p_1 n+1)^{\frac{1}{p_1}}} \frac{(b-t)^{\frac{p_2(\beta-1)+1}{p_2}}}{(p_2(\beta-1)+1)^{\frac{1}{p_2}}} \left\| {}_{\alpha}^{b}T(f) \right\|_{p_3,[a,b]} =$$

$$\frac{(b-t)^{n+\frac{1}{p_1}+\beta-1+\frac{1}{p_2}}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}}} \left\| {}_{\alpha}^{b}T(f) \right\|_{p_3,[a,b]} =$$

$$\frac{(b-t)^{n+\beta-\frac{1}{p_3}}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}}} \left\| {}_{\alpha}^{b}T(f) \right\|_{p_3,[a,b]}.$$

We have proved

$$|f(t)| \leq \frac{(b-t)^{n+\beta-\frac{1}{p_3}}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}}} \left\| {}_{\alpha}^{b}T(f) \right\|_{p_3,[a,b]}, \tag{166}$$

$\forall\ t \in [a,b]$.

Then, it holds

$$|f(t)|^{p_3} \leq \frac{(b-t)^{p_3(n+\beta)-1} \left\| {}_{\alpha}^{b}T(f) \right\|_{p_3,[a,b]}^{p_3}}{(n!)^{p_3} (p_1 n+1)^{\frac{p_3}{p_1}} (p_2(\beta-1)+1)^{\frac{p_3}{p_2}}}, \tag{167}$$

$\forall\ t \in [a,b]$. Hence, we derive

$$\int_a^b |f(t)|^{p_3} dt \leq \frac{(b-a)^{p_3(n+\beta)} \left\| {}_{\alpha}^{b}T(f) \right\|_{p_3,[a,b]}^{p_3}}{(n!)^{p_3} (p_1 n+1)^{\frac{p_3}{p_1}} (p_2(\beta-1)+1)^{\frac{p_3}{p_2}} p_3(n+\beta)}. \tag{168}$$

Then, raise (168) to the power $\frac{1}{p_3}$, and we are done. ■

Next we give a left conformable fractional Sobolev type inequality:

Theorem 39 *All assumptions as in Theorem 37 and $r > 0$. Then*

$$\|f\|_{r,[a,b]} \leq \frac{(b-a)^{\left(\alpha - \frac{1}{p_3} + \frac{1}{r}\right)} \left\| T_{\alpha}^{a} f \right\|_{p_3,[a,b]}}{n! \left(p_1 n + 1\right)^{\frac{1}{p_1}} \left(p_2 \left(\beta - 1\right) + 1\right)^{\frac{1}{p_2}} \left[r \left(\alpha - \frac{1}{p_3} \right) + 1 \right]^{\frac{1}{r}}}. \tag{169}$$

Proof. We use (159). Hence it holds

$$|f(t)|^r \leq \frac{(t-a)^{r\left(n+\beta-\frac{1}{p_3}\right)} \left\| T_{\alpha}^{a} f \right\|_{p_3,[a,b]}^r}{(n!)^r \left(p_1 n + 1\right)^{\frac{r}{p_1}} \left(p_2 \left(\beta - 1\right) + 1\right)^{\frac{r}{p_2}}}, \tag{170}$$

$\forall\ t \in [a,b]$.

Consequently we obtain

$$\int_a^b |f(t)|^r dt \leq \frac{(b-a)^{r\left(n+\beta-\frac{1}{p_3}\right)+1} \left\| T_{\alpha}^{a} f \right\|_{p_3,[a,b]}^r}{(n!)^r \left(p_1 n + 1\right)^{\frac{r}{p_1}} \left(p_2 \left(\beta - 1\right) + 1\right)^{\frac{r}{p_2}} \left[r \left(n + \beta - \frac{1}{p_3} \right) + 1 \right]}. \tag{171}$$

We have proved that

$$\|f\|_{r,[a,b]} \leq \frac{(b-a)^{\left(n+\beta - \frac{1}{p_3} + \frac{1}{r}\right)} \left\| T_{\alpha}^{a} f \right\|_{p_3,[a,b]}}{n! \left(p_1 n + 1\right)^{\frac{1}{p_1}} \left(p_2 \left(\beta - 1\right) + 1\right)^{\frac{1}{p_2}} \left[r \left(n + \beta - \frac{1}{p_3} \right) + 1 \right]^{\frac{1}{r}}}. \tag{172}$$

We have established (169). ■

It follows the right conformable fractional Sobolev type inequality:

Theorem 40 *All assumptions as in Theorem 38, and $r > 0$. Then*

$$\|f\|_{r,[a,b]} \leq \frac{(b-a)^{\left(\alpha - \frac{1}{p_3} + \frac{1}{r}\right)} \left\| {}_{\alpha}^{b}T(f) \right\|_{p_3,[a,b]}}{n! \left(p_1 n + 1\right)^{\frac{1}{p_1}} \left(p_2 \left(\beta - 1\right) + 1\right)^{\frac{1}{p_2}} \left[r \left(\alpha - \frac{1}{p_3} \right) + 1 \right]^{\frac{1}{r}}}. \tag{173}$$

Proof. We use (166). We get that

$$|f(t)|^r \leq \frac{(b-t)^{r\left(n+\beta-\frac{1}{p_3}\right)} \left\| {}_{\alpha}^{b}T(f) \right\|_{p_3,[a,b]}^r}{(n!)^r \left(p_1 n + 1\right)^{\frac{r}{p_1}} \left(p_2 \left(\beta - 1\right) + 1\right)^{\frac{r}{p_2}}}, \tag{174}$$

and

$$\int_a^b |f(t)|^r dt \leq \frac{(b-a)^{r\left(n+\beta-\frac{1}{p_3}\right)+1} \left\| {}_{\alpha}^{b}T(f) \right\|_{p_3,[a,b]}^r}{(n!)^r \left(p_1 n + 1\right)^{\frac{r}{p_1}} \left(p_2 \left(\beta - 1\right) + 1\right)^{\frac{r}{p_2}} \left[r \left(n + \beta - \frac{1}{p_3} \right) + 1 \right]}. \tag{175}$$

Finally, we derive

$$\|f\|_{r,[a,b]} \leq \frac{(b-a)^{\left(n+\beta-\frac{1}{p_3}+\frac{1}{r}\right)} \left\| {}_{\alpha}^{b}T(f) \right\|_{p_3,[a,b]}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}} \left[r\left(n+\beta-\frac{1}{p_3}\right)+1\right]^{\frac{1}{r}}}, \quad (176)$$

proving the claim. ■

We need

Corollary 41 *(of Theorem 26) Let* $\alpha \in (0,1]$, $f \in C^1([a,b])$, $[a,b] \subset \mathbb{R}$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(a) \right| \leq \frac{(b-a)^{\alpha}}{\alpha(\alpha+1)} \|T_{\alpha}^{a}(f)\|_{\infty,[a,b]}, \quad (177)$$

and

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(b) \right| \leq \frac{(b-a)^{\alpha}}{\alpha(\alpha+1)} \left\| {}_{\alpha}^{b}T(f) \right\|_{\infty,[a,b]}. \quad (178)$$

We need

Corollary 42 *Let* $\alpha \in (0,1]$, *any* $[a,b] \subset \mathbb{R}_+$, $f \in C^1(\mathbb{R}_+)$ *with* $\left\|T_{\alpha}^{0}(f)\right\|_{\infty,\mathbb{R}_+} < +\infty$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(a) \right| \leq \frac{(b-a)^{\alpha}}{\alpha(\alpha+1)} \left\|T_{\alpha}^{0}(f)\right\|_{\infty,\mathbb{R}_+}. \quad (179)$$

Proof. It comes from (177), and the following:

Here

$$T_{\alpha}^{a}(f)(x) = (x-a)^{1-\alpha} f'(x),$$

all $x \in [a,b]$, $0 \leq a < b$.

Then

$$|T_{\alpha}^{a}(f)(x)| = (x-a)^{1-\alpha} |f'(x)| \leq x^{1-\alpha} |f'(x)| = \left|T_{\alpha}^{0}(f)(x)\right|,$$

$\forall\, x \in [a,b]$.

Therefore it holds

$$\|T_{\alpha}^{a}(f)\|_{\infty,[a,b]} \leq \left\|T_{\alpha}^{0}(f)\right\|_{\infty,\mathbb{R}_+}. \quad (180)$$

■

Corollary 43 *Let* $\alpha \in (0,1]$, *any* $[a,b] \subset \mathbb{R}_-$, $f \in C^1(\mathbb{R}_-)$ *with* $\left\| {}_{\alpha}^{0}T(f) \right\|_{\infty,\mathbb{R}_-} < +\infty$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(b) \right| \leq \frac{(b-a)^{\alpha}}{\alpha(\alpha+1)} \left\| {}_{\alpha}^{0}T(f) \right\|_{\infty,\mathbb{R}_-}. \quad (181)$$

Proof. It comes from (178), and the following:

Here

$$-\left({}_{\alpha}^{b}T\left(f\right)\right)\left(x\right)=\left(b-x\right)^{1-\alpha}f'\left(x\right),$$

all $x\in[a,b]$, $a<b\leq 0$.

Then

$$\left|{}_{\alpha}^{b}T\left(f\right)\left(x\right)\right|=\left(b-x\right)^{1-\alpha}\left|f'\left(x\right)\right|\leq\left(-x\right)^{1-\alpha}\left|f'\left(x\right)\right|=\left|{}_{\alpha}^{0}T\left(f\right)\left(x\right)\right|,$$

$\forall\ x\in[a,b]$.

Therefore it holds

$$\left\|{}_{\alpha}^{b}T\left(f\right)\right\|_{\infty,[a,b]}\leq\left\|{}_{\alpha}^{0}T\left(f\right)\right\|_{\infty,\mathbb{R}_{-}}. \tag{182}$$

■

We need

Corollary 44 *(to Theorem 27) Let* $\alpha\in(0,1]$, $f\in C^{1}([a,b])$, $[a,b]\subset\mathbb{R}$. *Let* $p_1,p_2,p_3>1:\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$, *with* $\alpha>\frac{1}{p_1}+\frac{1}{p_3}$. *Then*

$$\left|\frac{1}{b-a}\int_a^b f\left(t\right)dt-f\left(a\right)\right|\leq$$

$$\frac{1}{\left(p_1+1\right)^{\frac{1}{p_1}}\left(p_2\left(\alpha-1\right)+1\right)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_2}+\frac{1}{p_3}\right)}\left(b-a\right)^{\alpha-\frac{1}{p_1}}\left\|T_{\alpha}^{a}\left(f\right)\right\|_{p_3,[a,b]}, \tag{183}$$

and

$$\left|\frac{1}{b-a}\int_a^b f\left(t\right)dt-f\left(b\right)\right|\leq$$

$$\frac{1}{\left(p_1+1\right)^{\frac{1}{p_1}}\left(p_2\left(\alpha-1\right)+1\right)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_2}+\frac{1}{p_3}\right)}\left(b-a\right)^{\alpha-\frac{1}{p_1}}\left\|{}_{\alpha}^{b}T\left(f\right)\right\|_{p_3,[a,b]}. \tag{184}$$

We need

Corollary 45 *Let* $\alpha\in(0,1]$, $f\in C^{1}\left(\mathbb{R}_{+}\right)$, *any* $[a,b]\subset\mathbb{R}_{+}$. *Let* $p_1,p_2,p_3>1:$ $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$, *with* $\alpha>\frac{1}{p_1}+\frac{1}{p_3}$. *We assume that* $\left\|T_{\alpha}^{0}\left(f\right)\right\|_{p_3,\mathbb{R}_{+}}<+\infty$. *Then*

$$\left|\frac{1}{b-a}\int_a^b f\left(t\right)dt-f\left(a\right)\right|\leq$$

$$\frac{1}{\left(p_1+1\right)^{\frac{1}{p_1}}\left(p_2\left(\alpha-1\right)+1\right)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_2}+\frac{1}{p_3}\right)}\left(b-a\right)^{\alpha-\frac{1}{p_1}}\left\|T_{\alpha}^{0}\left(f\right)\right\|_{p_3,\mathbb{R}_{+}}. \tag{185}$$

Proof. As in the proof of Corollary 42 we have that

$$\left|T_{\alpha}^{a}\left(f\right)\left(x\right)\right| \leq \left|T_{\alpha}^{0}\left(f\right)\left(x\right)\right|,$$

$\forall\, x \in [a,b]$.

Clearly then

$$\left\|T_{\alpha}^{a}\left(f\right)\right\|_{p_3,[a,b]} \leq \left\|T_{\alpha}^{0}\left(f\right)\right\|_{p_3,[a,b]} \leq \left\|T_{\alpha}^{0}\left(f\right)\right\|_{p_3,\mathbb{R}_+}. \tag{186}$$

■

Corollary 46 *Let $\alpha \in (0,1]$, $f \in C^1(\mathbb{R}_-)$, any $[a,b] \subset \mathbb{R}_-$. Let $p_1, p_2, p_3 > 1$: $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, with $\alpha > \frac{1}{p_1} + \frac{1}{p_3}$. We assume that $\left\|{}_{\alpha}^{0}T\left(f\right)\right\|_{p_3,\mathbb{R}_-} < +\infty$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(b)\right| \leq$$

$$\frac{1}{(p_1+1)^{\frac{1}{p_1}}\left(p_2(\alpha-1)+1\right)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_2}+\frac{1}{p_3}\right)}(b-a)^{\alpha-\frac{1}{p_1}}\left\|{}_{\alpha}^{0}T\left(f\right)\right\|_{p_3,\mathbb{R}_-}. \tag{187}$$

We make

Proof. As in the proof of Corollary 43 we have that

$$\left|{}_{\alpha}^{b}T\left(f\right)\left(x\right)\right| \leq \left|{}_{\alpha}^{0}T\left(f\right)\left(x\right)\right|,$$

$\forall\, x \in [a,b]$.

Clearly then

$$\left\|{}_{\alpha}^{b}T\left(f\right)\right\|_{p_3,[a,b]} \leq \left\|{}_{\alpha}^{0}T\left(f\right)\right\|_{p_3,[a,b]} \leq \left\|{}_{\alpha}^{0}T\left(f\right)\right\|_{p_3,\mathbb{R}_-}. \tag{188}$$

■

We make

Remark 47 *Let $\alpha \in (0,1]$, any $[a,b] \subset \mathbb{R}_+$, $f \in C^2(\mathbb{R}_+)$, with $\left\|T_{\alpha}^{0}\left(f'\right)\right\|_{\infty,\mathbb{R}_+} < +\infty$. Then (by (179))*

$$\left|\frac{1}{b-a}\int_a^b f'(t)\,dt - f'(a)\right| \leq \frac{(b-a)^{\alpha}}{\alpha(\alpha+1)}\left\|T_{\alpha}^{0}\left(f'\right)\right\|_{\infty,\mathbb{R}_+}. \tag{189}$$

That is

$$\left|\frac{1}{b-a}\left(f(b)-f(a)\right) - f'(a)\right| \leq \frac{(b-a)^{\alpha}}{\alpha(\alpha+1)}\left\|T_{\alpha}^{0}\left(f'\right)\right\|_{\infty,\mathbb{R}_+}. \tag{190}$$

Hence it holds

$$\left|f'(a)\right| - \frac{1}{b-a}\left|f(b)-f(a)\right| \leq \frac{(b-a)^{\alpha}}{\alpha(\alpha+1)}\left\|T_{\alpha}^{0}\left(f'\right)\right\|_{\infty,\mathbb{R}_+}. \tag{191}$$

Equivalently, we can write

$$|f'(a)| \leq \frac{1}{b-a}|f(b)-f(a)| + \frac{(b-a)^{\alpha}}{\alpha(\alpha+1)}\left\|T_{\alpha}^{0}(f')\right\|_{\infty,\mathbb{R}_+} \leq \tag{192}$$

$$\frac{2\|f\|_{\infty,\mathbb{R}_+}}{b-a} + \frac{(b-a)^{\alpha}}{\alpha(\alpha+1)}\left\|T_{\alpha}^{0}(f')\right\|_{\infty,\mathbb{R}_+},$$

$\forall\ a,b \in \mathbb{R}_+ : a < b.$

The last right hand side of (192) depends only on $(b-a)$.

Therefore it holds

$$\|f'\|_{\infty,\mathbb{R}_+} \leq \frac{2\|f\|_{\infty,\mathbb{R}_+}}{b-a} + \frac{(b-a)^{\alpha}}{\alpha(\alpha+1)}\left\|T_{\alpha}^{0}(f')\right\|_{\infty,\mathbb{R}_+}. \tag{193}$$

Set $t := b-a > 0$. *Thus*

$$\|f'\|_{\infty,\mathbb{R}_+} \leq \frac{2\|f\|_{\infty,\mathbb{R}_+}}{t} + \frac{t^{\alpha}}{\alpha(\alpha+1)}\left\|T_{\alpha}^{0}(f')\right\|_{\infty,\mathbb{R}_+}, \tag{194}$$

$\forall\ t > 0.$

Call

$$\mu := 2\|f\|_{\infty,\mathbb{R}_+},$$

$$\text{and} \tag{195}$$

$$\theta := \frac{\left\|T_{\alpha}^{0}(f')\right\|_{\infty,\mathbb{R}_+}}{\alpha(\alpha+1)},\ \alpha \in (0,1],$$

both μ, θ *are greater than zero.*

That is we have

$$\|f'\|_{\infty,\mathbb{R}_+} \leq \frac{\mu}{t} + \theta t^{\alpha},\ \forall\ t > 0. \tag{196}$$

Consider the function

$$y(t) := \frac{\mu}{t} + \theta t^{\alpha},\ t > 0,\ \alpha \in (0,1]. \tag{197}$$

Next we act as in [8], pp. 80-82. The only critical number here is

$$t_0 = \left(\frac{\mu}{\alpha\theta}\right)^{\frac{1}{\alpha+1}}, \tag{198}$$

and y *has a global minimum at* t_0, *which is*

$$y(t_0) = (\theta\mu^{\alpha})^{\frac{1}{\alpha+1}}(\alpha+1)\,\alpha^{-\left(\frac{\alpha}{\alpha+1}\right)}. \tag{199}$$

Thus, we have proved:

$$\|f'\|_{\infty,\mathbb{R}_+} \leq \left[\frac{\left\|T_{\alpha}^{0}(f')\right\|_{\infty,\mathbb{R}_+}}{\alpha(\alpha+1)}\left(2\|f\|_{\infty,\mathbb{R}_+}\right)^{\alpha}\right]^{\frac{1}{\alpha+1}}(\alpha+1)\,\alpha^{-\left(\frac{\alpha}{\alpha+1}\right)}, \tag{200}$$

under the assumption $\|f\|_{\infty,\mathbb{R}_+} < +\infty.$

We have established the following ∞-conformable left fractional alternative Landau type inequality:

Theorem 48 *Let* $\alpha \in (0,1]$, $f \in C^2(\mathbb{R}_+)$; $\|f\|_{\infty,\mathbb{R}_+}, \left\|T_\alpha^0(f')\right\|_{\infty,\mathbb{R}_+} < +\infty$. *Then*

$$\|f'\|_{\infty,\mathbb{R}_+} \leq \left(\frac{\alpha+1}{\alpha}\right)\left(\frac{2^\alpha}{\alpha+1}\right)^{\frac{1}{\alpha+1}} \|f\|_{\infty,\mathbb{R}_+}^{\frac{\alpha}{\alpha+1}} \left\|T_\alpha^0(f')\right\|_{\infty,\mathbb{R}_+}^{\frac{1}{\alpha+1}}. \tag{201}$$

That is $\|f'\|_{\infty,\mathbb{R}_+} < +\infty$.

We make

Remark 49 *Let* $\alpha \in (0,1]$, *any* $[a,b] \subset \mathbb{R}_-$, $f \in C^2(\mathbb{R}_-)$, *with* $\left\|{}_\alpha^0 T(f')\right\|_{\infty,\mathbb{R}_-} < +\infty$. *Assume also* $\|f\|_{\infty,\mathbb{R}_-} < +\infty$. *Then (by (181)) we get*

$$\left|\frac{1}{b-a}\int_a^b f'(t)\,dt - f'(b)\right| \leq \frac{(b-a)^\alpha}{\alpha(\alpha+1)} \left\|{}_\alpha^0 T(f')\right\|_{\infty,\mathbb{R}_-}. \tag{202}$$

That is

$$\left|\frac{1}{b-a}(f(b)-f(a)) - f'(b)\right| \leq \frac{(b-a)^\alpha}{\alpha(\alpha+1)} \left\|{}_\alpha^0 T(f')\right\|_{\infty,\mathbb{R}_-}. \tag{203}$$

Hence it holds

$$|f'(b)| - \frac{1}{b-a}|f(b)-f(a)| \leq \frac{(b-a)^\alpha}{\alpha(\alpha+1)} \left\|T_\alpha^0(f')\right\|_{\infty,\mathbb{R}_-}. \tag{204}$$

Equivalently, we can write

$$|f'(b)| \leq \frac{1}{b-a}|f(b)-f(a)| + \frac{(b-a)^\alpha}{\alpha(\alpha+1)} \left\|{}_\alpha^0 T(f')\right\|_{\infty,\mathbb{R}_-} \leq$$

$$\frac{2\|f\|_{\infty,\mathbb{R}_-}}{b-a} + \frac{(b-a)^\alpha}{\alpha(\alpha+1)} \left\|{}_\alpha^0 T(f')\right\|_{\infty,\mathbb{R}_-}, \tag{205}$$

$\forall\ a,b \in \mathbb{R}_- : a < b$.

The last right hand side of (205) depends only on $(b-a)$.

Therefore it holds

$$\|f'\|_{\infty,\mathbb{R}_-} \leq \frac{2\|f\|_{\infty,\mathbb{R}_-}}{b-a} + \frac{(b-a)^\alpha}{\alpha(\alpha+1)} \left\|{}_\alpha^0 T(f')\right\|_{\infty,\mathbb{R}_-}. \tag{206}$$

Set $t := b-a > 0$. *Thus*

$$\|f'\|_{\infty,\mathbb{R}_-} \leq \frac{2\|f\|_{\infty,\mathbb{R}_-}}{t} + \frac{t^\alpha}{\alpha(\alpha+1)} \left\|{}_\alpha^0 T(f')\right\|_{\infty,\mathbb{R}_-}, \tag{207}$$

$\forall\ t>0.$

Call

$$\overline{\mu} := 2\left\|f\right\|_{\infty,\mathbb{R}_-},$$
$$and \tag{208}$$
$$\overline{\theta} := \frac{\left\|{}_{\alpha}^{0}T\left(f'\right)\right\|_{\infty,\mathbb{R}_-}}{\alpha\left(\alpha+1\right)},\ \ \alpha\in(0,1],$$

both $\overline{\mu},\overline{\theta}>0$.

That is we have

$$\left\|f'\right\|_{\infty,\mathbb{R}_-}\leq\frac{\overline{\mu}}{t}+\overline{\theta}t^{\alpha},\ \ \forall\ t>0. \tag{209}$$

Consider the function

$$\overline{y}\left(t\right) := \frac{\overline{\mu}}{t}+\overline{\theta}t^{\alpha},\ \ t>0,\ \ \alpha\in(0,1]. \tag{210}$$

Next we act as in [8], pp. 80-82. The only critical number here is

$$\overline{t_0}=\left(\frac{\overline{\mu}}{\alpha\overline{\theta}}\right)^{\frac{1}{\alpha+1}}, \tag{211}$$

and $\overline{y}$ *has a global minimum at* $\overline{t_0}$, *which is*

$$\overline{y}\left(\overline{t_0}\right)=\left(\overline{\theta}\overline{\mu}^{\alpha}\right)^{\frac{1}{\alpha+1}}\left(\alpha+1\right)\alpha^{-\left(\frac{\alpha}{\alpha+1}\right)}. \tag{212}$$

Thus, we have proved:

$$\left\|f'\right\|_{\infty,\mathbb{R}_-}\leq\left[\frac{\left\|{}_{\alpha}^{0}T\left(f'\right)\right\|_{\infty,\mathbb{R}_-}}{\alpha\left(\alpha+1\right)}\left(2\left\|f\right\|_{\infty,\mathbb{R}_-}\right)^{\alpha}\right]^{\frac{1}{\alpha+1}}\left(\alpha+1\right)\alpha^{-\left(\frac{\alpha}{\alpha+1}\right)}. \tag{213}$$

We have established the following ∞-conformable right fractional alternative Landau type inequality:

Theorem 50 *Let* $\alpha\in(0,1],\ f\in C^{2}\left(\mathbb{R}_-\right);\ \left\|f\right\|_{\infty,\mathbb{R}_-},\left\|{}_{\alpha}^{0}T\left(f'\right)\right\|_{\infty,\mathbb{R}_-}<+\infty.$ *Then*

$$\left\|f'\right\|_{\infty,\mathbb{R}_-}\leq\left(\frac{\alpha+1}{\alpha}\right)\left(\frac{2^{\alpha}}{\alpha+1}\right)^{\frac{1}{\alpha+1}}\left\|f\right\|_{\infty,\mathbb{R}_-}^{\frac{\alpha}{\alpha+1}}\left\|{}_{\alpha}^{0}T\left(f'\right)\right\|_{\infty,\mathbb{R}_-}^{\frac{1}{\alpha+1}}. \tag{214}$$

That is $\left\|f'\right\|_{\infty,\mathbb{R}_-}<+\infty$.

We make

Remark 51 *Let* $\alpha \in (0,1]$*, any* $[a,b] \subset \mathbb{R}_+$*,* $f \in C^2(\mathbb{R}_+)$*,* $\|f\|_{\infty,\mathbb{R}_+} < +\infty$*. Let* $p_1,p_2,p_3 > 1 : \frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} = 1$*, with* $\alpha > \frac{1}{p_1}+\frac{1}{p_3}$*. We assume that* $\left\|T_\alpha^0(f')\right\|_{p_3,\mathbb{R}_+} < +\infty$*. Then (by (185))*

$$\left|\frac{1}{b-a}\int_a^b f'(t)\,dt - f'(a)\right| \leq \gamma(b-a)^\delta \left\|T_\alpha^0(f')\right\|_{p_3,\mathbb{R}_+}, \tag{215}$$

where

$$\gamma := \frac{1}{(p_1+1)^{\frac{1}{p_1}}(p_2(\alpha-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_2}+\frac{1}{p_3}\right)}, \tag{216}$$

and

$$\delta := \alpha - \frac{1}{p_1}. \tag{217}$$

Since $\alpha < 1+\frac{1}{p_1}$*, then* $\alpha - \frac{1}{p_1} < 1$*. Since* $\alpha > \frac{1}{p_1}+\frac{1}{p_3} > \frac{1}{p_1}$*, then* $\alpha - \frac{1}{p_1} > 0$*. Hence* $0 < \delta < 1$*. That is*

$$\left|\frac{1}{b-a}(f(b)-f(a)) - f'(a)\right| \leq \gamma(b-a)^\delta \left\|T_\alpha^0(f')\right\|_{p_3,\mathbb{R}_+}. \tag{218}$$

Hence it holds

$$|f'(a)| - \frac{1}{b-a}|f(b)-f(a)| \leq \gamma(b-a)^\delta \left\|T_\alpha^0(f')\right\|_{p_3,\mathbb{R}_+}. \tag{219}$$

Equivalently, we can write

$$|f'(a)| \leq \frac{1}{b-a}|f(b)-f(a)| + \gamma(b-a)^\delta \left\|T_\alpha^0(f')\right\|_{p_3,\mathbb{R}_+} \leq \tag{220}$$

$$\frac{2\|f\|_{\infty,\mathbb{R}_+}}{b-a} + (b-a)^\delta \gamma \left\|T_\alpha^0(f')\right\|_{p_3,\mathbb{R}_+},$$

$\forall\ a,b \in \mathbb{R}_+ : a < b$.

The last right hand side of (220) depends only on $(b-a)$.

Therefore it holds

$$\|f'\|_{\infty,\mathbb{R}_+} \leq \frac{2\|f\|_{\infty,\mathbb{R}_+}}{b-a} + (b-a)^\delta \gamma \left\|T_\alpha^0(f')\right\|_{p_3,\mathbb{R}_+}. \tag{221}$$

Set $t := b-a > 0$*. Thus*

$$\|f'\|_{\infty,\mathbb{R}_+} \leq \frac{2\|f\|_{\infty,\mathbb{R}_+}}{t} + t^\delta \gamma \left\|T_\alpha^0(f')\right\|_{p_3,\mathbb{R}_+}, \tag{222}$$

$\forall\ t > 0$.

Call

$$\mu := 2\|f\|_{\infty,\mathbb{R}_+},$$

and (223)

$$\theta := \gamma \left\| T_{\alpha}^{0}\left(f'\right)\right\|_{p_3,\mathbb{R}_+},\ \ \alpha \in (0,1],$$

both $\mu, \theta > 0$.

That is we have

$$\|f'\|_{\infty,\mathbb{R}_+} \leq \frac{\mu}{t} + \theta t^{\delta},\ \ \forall\ t > 0. \tag{224}$$

Consider the function

$$y(t) := \frac{\mu}{t} + \theta t^{\delta},\ \ \forall\ t > 0,\ \ 0 < \delta < 1. \tag{225}$$

Next we act as in [8], pp. 80-82. The only critical number here is

$$t_0 = \left(\frac{\mu}{\delta\theta}\right)^{\frac{1}{\delta+1}}, \tag{226}$$

and y *has a global minimum at* t_0*, which is*

$$y(t_0) = \left(\theta\mu^{\delta}\right)^{\frac{1}{\delta+1}} (\delta+1)\, \delta^{-\left(\frac{\delta}{\delta+1}\right)}. \tag{227}$$

Thus, we have proved:

$$\|f'\|_{\infty,\mathbb{R}_+} \leq \left[\left(\gamma \left\|T_{\alpha}^{0}(f')\right\|_{p_3,\mathbb{R}_+}\right)\left(2\|f\|_{\infty,\mathbb{R}_+}\right)^{\delta}\right]^{\frac{1}{\delta+1}} (\delta+1)\, \delta^{-\left(\frac{\delta}{\delta+1}\right)}. \tag{228}$$

We have established the following L_p-conformable left fractional alternative Landau type inequality:

Theorem 52 *Let* $\alpha \in (0,1]$*,* $f \in C^2(\mathbb{R}_+)$*,* $\|f\|_{\infty,\mathbb{R}_+} < +\infty$*. Let* $p_1, p_2, p_3 > 1$: $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$*, with* $\alpha > \frac{1}{p_1} + \frac{1}{p_3}$*. We assume that* $\left\|T_{\alpha}^{0}(f')\right\|_{p_3,\mathbb{R}_+} < +\infty$*. Set*

$$\gamma := \frac{1}{(p_1+1)^{\frac{1}{p_1}} (p_2(\alpha-1)+1)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_2} + \frac{1}{p_3}\right)},$$

$$\text{and} \tag{229}$$

$$\delta := \alpha - \frac{1}{p_1}.$$

Then

$$\|f'\|_{\infty,\mathbb{R}_+} \leq \|f\|_{\infty,\mathbb{R}_+}^{\frac{\delta}{\delta+1}} \left\|T_{\alpha}^{0}(f')\right\|_{p_3,\mathbb{R}_+}^{\frac{1}{\delta+1}} \gamma^{\frac{1}{\delta+1}} 2^{\frac{\delta}{\delta+1}} (\delta+1)\, \delta^{-\left(\frac{\delta}{\delta+1}\right)}. \tag{230}$$

That is $\|f'\|_{\infty,\mathbb{R}_+} < +\infty$.

We make

Remark 53 *Let* $\alpha \in (0,1]$*, any* $[a,b] \subset \mathbb{R}_-$*,* $f \in C^2(\mathbb{R}_-)$*,* $\|f\|_{\infty,\mathbb{R}_-} < +\infty$*. Let* $p_1, p_2, p_3 > 1 : \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$*, with* $\alpha > \frac{1}{p_1} + \frac{1}{p_3}$*. We assume that* $\left\| {}^{0}_{\alpha}T(f') \right\|_{p_3,\mathbb{R}_-} < +\infty$*. Then (by (187))*

$$\left| \frac{1}{b-a} \int_a^b f'(t)\,dt - f'(b) \right| \leq \gamma (b-a)^{\delta} \left\| {}^{0}_{\alpha}T(f') \right\|_{p_3,\mathbb{R}_-}, \tag{231}$$

where

$$\gamma := \frac{1}{(p_1+1)^{\frac{1}{p_1}} (p_2(\alpha-1)+1)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_2} + \frac{1}{p_3} \right)}, \tag{232}$$

and

$$\delta := \alpha - \frac{1}{p_1}. \tag{233}$$

It holds $0 < \delta < 1$*. That is*

$$\left| \frac{1}{b-a} (f(b) - f(a)) - f'(b) \right| \leq \gamma (b-a)^{\delta} \left\| {}^{0}_{\alpha}T(f') \right\|_{p_3,\mathbb{R}_-}. \tag{234}$$

Hence it holds

$$|f'(b)| - \frac{1}{b-a} |f(b) - f(a)| \leq \gamma (b-a)^{\delta} \left\| {}^{0}_{\alpha}T(f') \right\|_{p_3,\mathbb{R}_-}. \tag{235}$$

Equivalently, we can write

$$|f'(b)| \leq \frac{1}{b-a} |f(b) - f(a)| + \gamma (b-a)^{\delta} \left\| {}^{0}_{\alpha}T(f') \right\|_{p_3,\mathbb{R}_-} \leq \tag{236}$$

$$\frac{2\|f\|_{\infty,\mathbb{R}_-}}{b-a} + (b-a)^{\delta} \gamma \left\| {}^{0}_{\alpha}T(f') \right\|_{p_3,\mathbb{R}_-},$$

$\forall\ a, b \in \mathbb{R}_- : a < b$.

The last right hand side of (236) depends only on $(b-a)$.

Therefore it holds

$$\|f'\|_{\infty,\mathbb{R}_-} \leq \frac{2\|f\|_{\infty,\mathbb{R}_-}}{b-a} + (b-a)^{\delta} \gamma \left\| {}^{0}_{\alpha}T(f') \right\|_{p_3,\mathbb{R}_-}. \tag{237}$$

Set $t := b - a > 0$*. Thus*

$$\|f'\|_{\infty,\mathbb{R}_-} \leq \frac{2\|f\|_{\infty,\mathbb{R}_-}}{t} + t^{\delta} \gamma \left\| {}^{0}_{\alpha}T(f') \right\|_{p_3,\mathbb{R}_-}, \tag{238}$$

$\forall\ t > 0$.

Call

$$\overline{\mu} := 2\|f\|_{\infty,\mathbb{R}_-},$$

and (239)

$$\overline{\theta} := \gamma \left\| {}_{\alpha}^{0}T\left(f'\right)\right\|_{p_3,\mathbb{R}_-},\ \alpha \in (0,1],$$

both $\overline{\mu}, \overline{\theta} > 0$.

That is we have

$$\|f'\|_{\infty,\mathbb{R}-} \leq \frac{\overline{\mu}}{t} + \overline{\theta} t^{\delta},\ \forall\ t > 0. \tag{240}$$

Consider the function

$$\overline{y}(t) := \frac{\overline{\mu}}{t} + \overline{\theta} t^{\delta},\ \forall\ t > 0,\ 0 < \delta < 1. \tag{241}$$

Next we act as in [8], pp. 80-82. The only critical number here is

$$\overline{t_0} = \left(\frac{\overline{\mu}}{\delta\overline{\theta}}\right)^{\frac{1}{\delta+1}}, \tag{242}$$

and $\overline{y}$ *has a global minimum at* $\overline{t_0}$*, which is*

$$\overline{y}\left(\overline{t_0}\right) = \left(\overline{\theta}\overline{\mu}^{\delta}\right)^{\frac{1}{\delta+1}} (\delta+1)\, \delta^{-\left(\frac{\delta}{\delta+1}\right)}. \tag{243}$$

Thus, we have proved:

$$\|f'\|_{\infty,\mathbb{R}_-} \leq \left[\left(\gamma \left\| {}_{\alpha}^{0}T\left(f'\right)\right\|_{p_3,\mathbb{R}_-}\right)\left(2\|f\|_{\infty,\mathbb{R}_-}\right)^{\delta}\right]^{\frac{1}{\delta+1}} (\delta+1)\, \delta^{-\left(\frac{\delta}{\delta+1}\right)}. \tag{244}$$

We have established the following L_p-conformable right fractional alternative Landau type inequality:

Theorem 54 *Let* $\alpha \in (0,1]$, $f \in C^2(\mathbb{R}_-)$, $\|f\|_{\infty,\mathbb{R}_-} < +\infty$. *Let* $p_1, p_2, p_3 > 1$: $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, *with* $\alpha > \frac{1}{p_1} + \frac{1}{p_3}$. *We assume that* $\left\|T_{\alpha}^{0}\left(f'\right)\right\|_{p_3,\mathbb{R}_-} < +\infty$. *Set*

$$\gamma := \frac{1}{(p_1+1)^{\frac{1}{p_1}} \left(p_2(\alpha-1)+1\right)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_2} + \frac{1}{p_3}\right)},$$

$$\text{and} \tag{245}$$

$$\delta := \alpha - \frac{1}{p_1}.$$

Then

$$\|f'\|_{\infty,\mathbb{R}_-} \leq \|f\|_{\infty,\mathbb{R}_-}^{\frac{\delta}{\delta+1}} \left\| {}_{\alpha}^{0}T\left(f'\right)\right\|_{p_3,\mathbb{R}_-}^{\frac{1}{\delta+1}} \gamma^{\frac{1}{\delta+1}} 2^{\frac{\delta}{\delta+1}} (\delta+1)\, \delta^{-\left(\frac{\delta}{\delta+1}\right)}. \tag{246}$$

That is $\|f'\|_{\infty,\mathbb{R}_-} < +\infty$.

Comment 55 *Let* $f, g \geq 0$ *be functions. Then it is well-known that*

$$\sup(fg) \leq (\sup f)(\sup g). \tag{247}$$

Property (247) strongly supports our inverstigations throughout Section 3.

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