LOCAL EXTREME POINTS AND A YOUNG-TYPE INEQUALITY

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Abstract. In this paper is presented a Young-type inequality and then as an application is given a corresponding Holder-type inequality for isotonic linear functionals.

1. Introduction

The classical inequality of Young is

$$a^{\nu}b^{1-\nu} < \nu a + (1-\nu)b,$$

where a and b are distinct positive real numbers and $0 < \nu < 1$, see [14].

In [1] are given new results which extend many generalizations of Young's inequality given before. The following inequality is a refinement of the left-hand side of a refinement of the inequality of Young proved in 2010 and 2011 by Kittaneh and Manasrah in [12], [13]. Many generalizations and refinements of Young's inequality are presented also in [10], [8], [9] and references therein.

Theorem A([1]) *Let λ, ν and τ be real numbers with $\lambda \geq 1$ and $0 < \nu < \tau < 1$. Then*

$$\left(\frac{\nu}{\tau}\right)^{\lambda} < \frac{A_{\nu}(a,b)^{\lambda} - G_{\nu}(a,b)^{\lambda}}{A_{\tau}(a,b)^{\lambda} - G_{\tau}(a,b)^{\lambda}} < \left(\frac{1-\nu}{1-\tau}\right)^{\lambda},$$

for all positive and distinct real numbers a and b. Moreover, both bounds are sharp.

The following important definition is given in [3], [5] and we need to recall it here in order to help us to give new Young-type inequalities for isotonic linear functionals in Section 3.

Let E be a nonempty set and L be a class of real-valued functions $f : E \to \mathbf{R}$ having the following properties:

(L1) If $f, g \in L$ and $a, b \in \mathbf{R}$, then $(af + bg) \in L$.

(L2) If $f(t) = 1$ for all $t \in E$, then $f \in L$.

An *isotonic linear functional* is a functional $A : L \to \mathbf{R}$ having the following properties:

(A1) If $f, g \in L$ and $a, b \in \mathbf{R}$, then $A(af + bg) = aA(f) + bA(g)$.

(A2) If $f \in L$ and $f(t) \geq 0$ for all $t \in E$, then $A(f) \geq 0$.

The mapping A is said to be *normalised* if

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(A3) $A(\mathbf{1}) = 1$.

New inequalities concerning isotonic linear functionals can be also found in [7], [3], [5], [6] and refernces therein.

2. Local extreme points and a Young-type inequality for three numbers

In this section is given a new Young-type inequalitiy for three positive numbers which satisfies some conditions in Theorem 1 using the Lemma 1, where are stated several conditions for finding the local extreme point for a special function.

Lemma 1. *Let p_1, p_2, p_3, $p_1^{'}, p_2^{'}, p_3^{'}$ be strictly positive real numbers which satisfies the conditions,* $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, $\frac{1}{p_1^{'}} + \frac{1}{p_2^{'}} + \frac{1}{p_3^{'}} = 1$ *and*

$$p_1^{'}(1 - \frac{1}{p_2^{'}}) \neq p_1(1 - \frac{1}{p_2}).$$

(i) If $p_1^{'} < p_1$ *and*

$$\left(\frac{1}{p_1^{'}} - \frac{1}{p_1}\right)\left[-\frac{1}{p_2}(\frac{1}{p_2} - 1) + \frac{p_1^{'}}{p_1}\frac{1}{p_2^{'}}(\frac{1}{p_2^{'}} - 1)\right] > \frac{1}{p_1}\left(\frac{1}{p_2} - \frac{1}{p_2^{'}}\right)^2.$$

then $A(1,1)$ *is a local minimum point for the function*

$$f(x,y) = \frac{1}{p_1}x + \frac{1}{p_2}y + \frac{1}{p_3} - x^{\frac{1}{p_1}}y^{\frac{1}{p_2}} - \frac{p_1^{'}}{p_1}\left(\frac{1}{p_1^{'}}x + \frac{1}{p_2^{'}}y + \frac{1}{p_3^{'}} - x^{\frac{1}{p_1^{'}}}y^{\frac{1}{p_2^{'}}}\right),$$

defined on the interval $(0,\infty) \times (0,\infty)$.

(ii) If $p_1^{'} > p_1$ *and*

$$\left(\frac{1}{p_1^{'}} - \frac{1}{p_1}\right)\left[-\frac{1}{p_2}(\frac{1}{p_2} - 1) + \frac{p_1^{'}}{p_1}\frac{1}{p_2^{'}}(\frac{1}{p_2^{'}} - 1)\right] > \frac{1}{p_1}\left(\frac{1}{p_2} - \frac{1}{p_2^{'}}\right)^2.$$

then $A(1,1)$ *is a local maximum point for the function*

$$f(x,y) = \frac{1}{p_1}x + \frac{1}{p_2}y + \frac{1}{p_3} - x^{\frac{1}{p_1}}y^{\frac{1}{p_2}} - \frac{p_1^{'}}{p_1}\left(\frac{1}{p_1^{'}}x + \frac{1}{p_2^{'}}y + \frac{1}{p_3^{'}} - x^{\frac{1}{p_1^{'}}}y^{\frac{1}{p_2^{'}}}\right),$$

defined on the interval $(0,\infty) \times (0,\infty)$.

Proof. (i) We consider the function,

$$f(x,y) = \frac{1}{p_1}x + \frac{1}{p_2}y + \frac{1}{p_3} - x^{\frac{1}{p_1}}y^{\frac{1}{p_2}} - \frac{p_1^{'}}{p_1}\left(\frac{1}{p_1^{'}}x + \frac{1}{p_2^{'}}y + \frac{1}{p_3^{'}} - x^{\frac{1}{p_1^{'}}}y^{\frac{1}{p_2^{'}}}\right),$$

where the numbers $p_1, p_2, p_3, p_1^{'}, p_2^{'}, p_3^{'}$ satisfies the hypothesis and x, y are strictly positive real number with $x > 0$, and $y > 0$.

First, it is necessary to find the stationary points of f on $(0,\infty) \times (0,\infty)$ and for that we compute its first derivative, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. We have,

$$\frac{\partial f}{\partial x} = -\frac{1}{p_1}x^{\frac{1}{p_1}-1}y^{\frac{1}{p_2}} + \frac{1}{p_1}x^{\frac{1}{p_1^{'}}-1}y^{\frac{1}{p_2^{'}}}$$

and

$$\frac{\partial f}{\partial y} = \frac{1}{p_2} - \frac{1}{p_2}x^{\frac{1}{p_1}}y^{\frac{1}{p_2}-1} - \frac{p_1^{'}}{p_1}\frac{1}{p_2^{'}} + \frac{p_1^{'}}{p_1}\frac{1}{p_2^{'}}x^{\frac{1}{p_1^{'}}}y^{\frac{1}{p_2^{'}}-1}$$

and then we obtain the following system

$$(1)\qquad \begin{cases} x^{\frac{1}{p_1}}y^{\frac{1}{p_2}} = x^{\frac{1}{p_1'}}y^{\frac{1}{p_2'}} \\ \frac{1}{p_2}(1-x^{\frac{1}{p_1}}y^{\frac{1}{p_2}-1}) = \frac{p_1'}{p_1}\frac{1}{p_2'}(1-x^{\frac{1}{p_1'}}y^{\frac{1}{p_2'}-1}) \end{cases}$$

Using now the hypothesis, $\frac{p_2'}{p_2}-\frac{p_1'}{p_1}>0$ we get from the equation,

$$\left(1-x^{\frac{1}{p_1}}y^{\frac{1}{p_2}-1}\right)\left(\frac{1}{p_2}-\frac{p_1'}{p_1}\frac{1}{p_2'}\right)=0$$

that

$$x^{\frac{1}{p_1}}y^{\frac{1}{p_2}}=y,$$

where p_1, p_2 satisfy the hypothesis, being arbitrary numbers. Last equation becomes

$$x^{\frac{1}{p_1}}=y^{1-\frac{1}{p_2}},$$

when $x, y>0$.

Therefore, the last system will be

$$\begin{cases} y^{p_1'(1-\frac{1}{p_2'})} = y^{p_1(1-\frac{1}{p_2})} \\ y^{p_1(1-\frac{1}{p_2})} = y^{\frac{\frac{1}{p_2'}-\frac{1}{p_2}}{\frac{1}{p_1}-\frac{1}{p_1'}}} \end{cases}$$

Then we have

$$\begin{cases} p_1'(1-\frac{1}{p_2'}) = p_1(1-\frac{1}{p_2}) \\ p_1(1-\frac{1}{p_2}) = \frac{\frac{1}{p_2'}-\frac{1}{p_2}}{\frac{1}{p_1}-\frac{1}{p_1'}} \end{cases}$$

when $y\neq 1$, or the solution $x=y=1$. So we obtain in the second case, the stationary point $A(1,1)$.

First case, when $y\neq 1$, it is not interesting here because our hypothesis are not satisfied, i. e. from last system we have,

$$p_2'=\frac{1}{1-\frac{p_1}{p_1'}(1-\frac{1}{p_2})}$$

(which is already a restriction of p_2'), and in this way the second equation of last system in checked, but this is not our hypothesis.

We study now if $A(1,1)$ is an extreme point for the function f on the interval $(0,\infty)\times(0,\infty)$. For that we compute the second derivative of the function and then its hessian matrix in $A(1,1)$. We have,

$$\frac{\partial^2 f}{\partial x^2}=-\frac{1}{p_1}(\frac{1}{p_1}-1)x^{\frac{1}{p_1}-2}y^{\frac{1}{p_2}}+\frac{1}{p_1}(\frac{1}{p_1'}-1)x^{\frac{1}{p_1'}-2}y^{\frac{1}{p_2'}},$$

$$\frac{\partial^2 f}{\partial x^2}(1,1)=\frac{1}{p_1}\left(\frac{1}{p_1'}-\frac{1}{p_1}\right),$$

$$\frac{\partial^2 f}{\partial y^2}=-\frac{1}{p_2}(\frac{1}{p_2}-1)x^{\frac{1}{p_1}}y^{\frac{1}{p_2}-2}+\frac{p_1'}{p_1}\frac{1}{p_2'}(\frac{1}{p_2'}-1)x^{\frac{1}{p_1'}}y^{\frac{1}{p_2'}-2},$$

$$\frac{\partial^2 f}{\partial y^2}(1,1)=-\frac{1}{p_2}(\frac{1}{p_2}-1)+\frac{p_1'}{p_1}\frac{1}{p_2'}(\frac{1}{p_2'}-1),$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{1}{p_1 p_2} x^{\frac{1}{p_1}-1} y^{\frac{1}{p_2}-1} + \frac{1}{p_1 p_2^{'}} x^{\frac{1}{p_1^{'}}-1} y^{\frac{1}{p_2^{'}}-1},$$

$$\frac{\partial^2 f}{\partial x \partial y}(1,1) = \frac{1}{p_1}\left(\frac{1}{p_2^{'}} - \frac{1}{p_2}\right),$$

and also

$$\frac{\partial^2 f}{\partial y \partial x}(1,1) = \frac{1}{p_1}\left(\frac{1}{p_2^{'}} - \frac{1}{p_2}\right).$$

Now we can write the hessian matrix in $A(1,1)$,

$$H(1,1) = \begin{pmatrix} \frac{1}{p_1}\left(\frac{1}{p_1^{'}} - \frac{1}{p_1}\right) & \frac{1}{p_1}\left(\frac{1}{p_2^{'}} - \frac{1}{p_2}\right) \\ \frac{1}{p_1}\left(\frac{1}{p_2^{'}} - \frac{1}{p_2}\right) & -\frac{1}{p_2}(\frac{1}{p_2} - 1) + \frac{p_1^{'}}{p_1}\frac{1}{p_2^{'}}(\frac{1}{p_2^{'}} - 1) \end{pmatrix}$$

and if

$$\Delta_1 = \frac{1}{p_1}\left(\frac{1}{p_1^{'}} - \frac{1}{p_1}\right) > 0$$

and

$$\Delta_2 = \left(\frac{1}{p_1^{'}} - \frac{1}{p_1}\right)\left[-\frac{1}{p_2}(\frac{1}{p_2} - 1) + \frac{p_1^{'}}{p_1}\frac{1}{p_2^{'}}(\frac{1}{p_2^{'}} - 1)\right] - \frac{1}{p_1}\left(\frac{1}{p_2} - \frac{1}{p_2^{'}}\right)^2 > 0$$

then $A(1,1)$ is the local extreme point for the function f defined before.

For (ii) the proof is the same ∎

Example 1. *(i) We take into account the particular case for the function f when $p_1 = 5,\ p_2 = 6,\ p_3 = \frac{30}{19}$ and $p_1^{'} = 4,\ p_2^{'} = 5, p_3^{'} = \frac{20}{11}$, see also in Figures 1 and 2. We can easily notice that the conditions from hypothesis (i) are fulfilled for the function f, so that the point $A(1,1)$ is a local minimum point for f.*

(ii) Now, if we replace p_1 by 4 and $p_1^{'}$ by 7 in previous particular case, we can easily see that the conditions from hypothesis (ii) are satisfied for the function f, so the point $A(1,1)$ is a local maximum point for f.

Theorem 1. *Let $M > 1$ and $p_1, p_2, p_3,\ p_1^{'}, p_2^{'}, p_3^{'}$ be positive real numbers which satisfies the conditions, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1,\ \frac{1}{p_1^{'}} + \frac{1}{p_2^{'}} + \frac{1}{p_3^{'}} = 1,\ \frac{p_3^{'}}{p_3} > 1 > \frac{p_2^{'}}{p_2}$ and $p_2^{'}(1 - \frac{p_1^{'}}{p_1}) > \frac{p_2^{'}}{p_2} - \frac{p_1^{'}}{p_1} > 0$.*

(i) If x and y are two real numbers with $1 < x < M,\ 1 < y < M$ then the following inequality holds:

$$\frac{1}{p_1}x + \frac{1}{p_2}y + \frac{1}{p_3} - x^{\frac{1}{p_1}} y^{\frac{1}{p_2}} > \frac{p_1^{'}}{p_1}\left(\frac{1}{p_1^{'}}x + \frac{1}{p_2^{'}}y + \frac{1}{p_3^{'}} - x^{\frac{1}{p_1^{'}}} y^{\frac{1}{p_2^{'}}}\right).$$

(ii) Moreover, if a, b, c are three real numbers, $a > 0,\ b > 0,\ c > 0$ so that $c < a < Mc$ and $c < b < Mc$ then the following inequality takes place:

$$\frac{1}{p_1}a + \frac{1}{p_2}b + \frac{1}{p_3}c - a^{\frac{1}{p_1}} b^{\frac{1}{p_2}} c^{\frac{1}{p_3}} > \frac{p_1^{'}}{p_1}\left(\frac{1}{p_1^{'}}a + \frac{1}{p_2^{'}}b + \frac{1}{p_3^{'}}c - a^{\frac{1}{p_1^{'}}} b^{\frac{1}{p_2^{'}}} c^{\frac{1}{p_3^{'}}}\right).$$

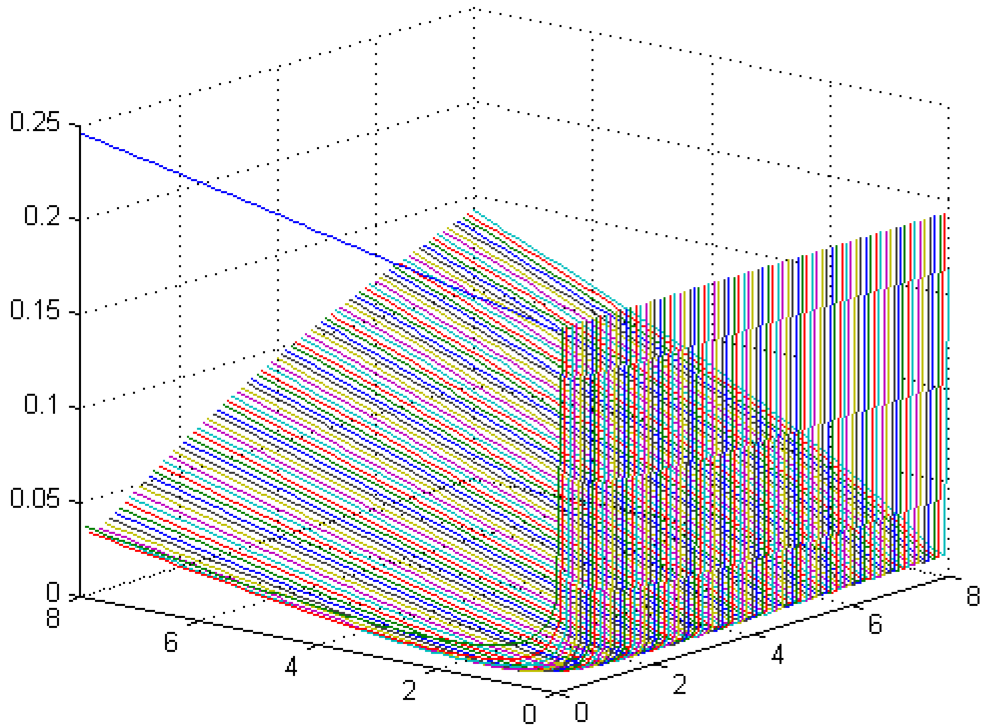

FIGURE 1. The function f(x,y) on $[0,8]\times[0,8]$ when $p_1=5, p_2=6, p_3=\frac{30}{19}$ and $p_1^{'}=4, p_2^{'}=5, p_3^{'}=\frac{20}{11}$.

Proof. Using Lemma 1, we know that $A(1,1)$ is a local minimum point for the function f on the interval $(1,M)\times(1,M)$, which it is the interior of the close interval $[1,M]\times[1,M]$. We study how will be the function on the frontier of the above interval. We see that the frontier of this interval from $\mathbb{R}^2$ is given by the sets, $\{x=1, y\in[1,M]\}$, $\{x=M, y\in[1,M]\}$, $\{x\in[1.M], y=1\}$ and $\{x\in[1,M], y=M\}$.

When $x=1,\ y\in[1,M]$ then

$$f(1,y)=y\frac{1}{p_2^{'}}\left(\frac{p_2^{'}}{p_2}-\frac{p_1^{'}}{p_1}\right)+\frac{1}{p_3^{'}}\left(\frac{p_3^{'}}{p_3}-\frac{p_1^{'}}{p_1}\right)+\frac{p_1^{'}}{p_1}y^{\frac{1}{p_2^{'}}}-y^{\frac{1}{p_2}}.$$

This function is increasing, as a function of variable y, from hypothesis of the above theorem, and then $f(1,1)<f(1,y)$, because $1<y$. Therefore, we find that $f(1,y)>f(1,1)=0$. Last function is increasing because its first derivative,

$$f^{'}(1,y)=\frac{1}{p_2}\left(1-y^{\frac{1}{p_2}-1}\right)-\frac{p_1^{'}}{p_1}\frac{1}{p_2^{'}}\left(1-y^{\frac{1}{p_2^{'}}-1}\right)>\frac{1}{p_2^{'}}\left(\frac{p_2^{'}}{p_2}-\frac{p_1^{'}}{p_1}\right)\left(1-y^{\frac{1}{p_2^{'}}-1}\right)>0.$$

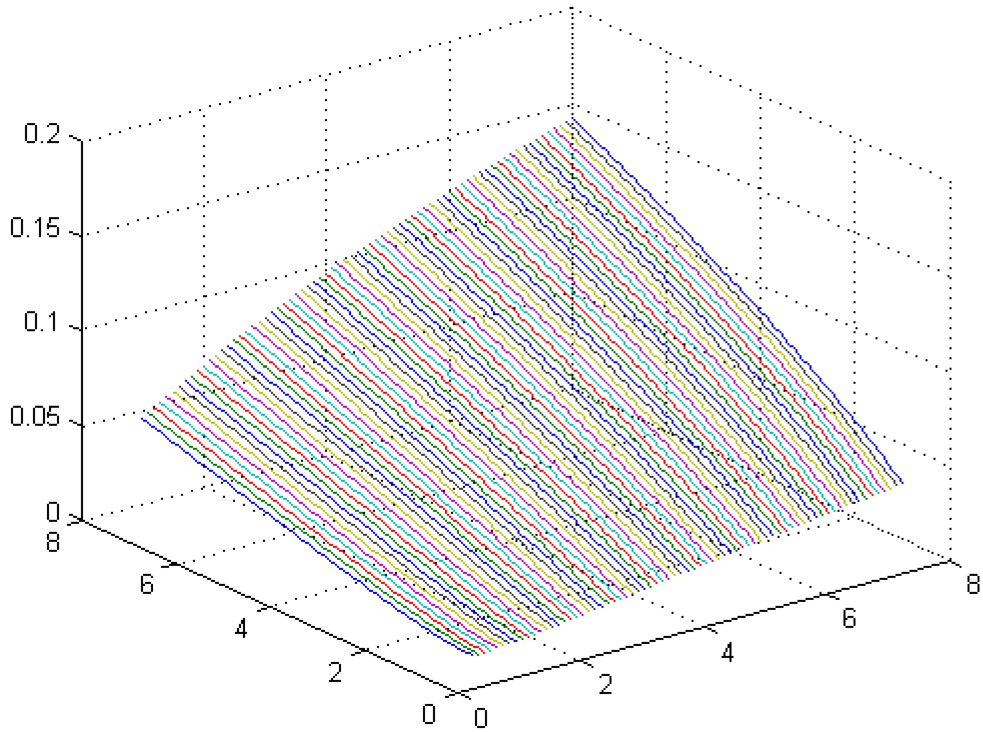

FIGURE 2. The function f(x,y) on $[1,8]\times[1,8]$ when $p_1=5, p_2=6, p_3=\frac{30}{19}$ and $p_1^{'}=4, p_2^{'}=5, p_3^{'}=\frac{20}{11}$.

Now, for $y=1,\ x\in[1,M]$, we have,

$$f(x,1)=1-\frac{p_1^{'}}{p_1}-x^{\frac{1}{p_1}}+\frac{p_1^{'}}{p_1}x^{\frac{1}{p_1^{'}}}.$$

This function is increasing because its first derivative,

$$f^{'}(x,1)=\frac{1}{p_1}\left(x^{\frac{1}{p_1^{'}}-1}-x^{\frac{1}{p_1}-1}\right)>0$$

, see hypothesis of our previous theorem. Thus we also have, $f(x,1)>f(1,1)=0$.

If $x\in[1,M],\ y=M$ then we obtain,

$$f(x,M)=M\frac{1}{p_2^{'}}\left(\frac{p_2^{'}}{p_2}-\frac{p_1^{'}}{p_1}\right)+\frac{1}{p_3^{'}}\left(\frac{p_3^{'}}{p_3}-\frac{p_1^{'}}{p_1}\right)+\frac{p_1^{'}}{p_1}x^{\frac{1}{p_1^{'}}}M^{\frac{1}{p_2^{'}}}-x^{\frac{1}{p_1}}M^{\frac{1}{p_2}},$$

and this function is increasing in x when $x\in[1,M]$, because

$$f^{'}(x,M)=\frac{1}{p_1}\left(x^{\frac{1}{p_1^{'}}-1}M^{\frac{1}{p_2^{'}}}-x^{\frac{1}{p_1}-1}M^{\frac{1}{p_2}}\right)>0.$$

From here, we get,

$$f(x,M) > f(1,M) > 0,$$

and we obtained this inequality before, see the case when $x = 1, y \in [1,M]$.

Last case, when $x = M,\ y \in [1,M]$ we have the function,

$$f(M,y) = y\frac{1}{p_2^{'}}\left(\frac{p_2^{'}}{p_2} - \frac{p_1^{'}}{p_1}\right) + \frac{1}{p_3^{'}}\left(\frac{p_3^{'}}{p_3} - \frac{p_1^{'}}{p_1}\right) + \frac{p_1^{'}}{p_1}M^{\frac{1}{p_1^{'}}}y^{\frac{1}{p_2^{'}}} - M^{\frac{1}{p_1}}y^{\frac{1}{p_2}},$$

which is increasing as a function of variable y, because its first derivative,

$$f^{'}(M,y) = \frac{1}{p_2^{'}}\left(\frac{p_2^{'}}{p_2} - \frac{p_1^{'}}{p_1}\right) + \frac{p_1^{'}}{p_1}\frac{1}{p_2^{'}}M^{\frac{1}{p_1^{'}}}y^{\frac{1}{p_2^{'}}-1} - \frac{1}{p_2}M^{\frac{1}{p_1}}y^{\frac{1}{p_2}-1} =$$

$$= \frac{1}{p_2^{'}}\left[\frac{p_2^{'}}{p_2}\left(1 - M^{\frac{1}{p_1}}y^{\frac{1}{p_2}-1}\right) - \frac{p_1^{'}}{p_1}\left(1 - M^{\frac{1}{p_1^{'}}}y^{\frac{1}{p_2^{'}}-1}\right)\right] > 0.$$

We used here that $\frac{p_2^{'}}{p_2} > \frac{p_1^{'}}{p_1}$ and $M^{\frac{1}{p_1^{'}}-\frac{1}{p_1}} > 1 > y^{\frac{1}{p_2}-\frac{1}{p_2^{'}}}$.

From the second case we get

$$f(M,1) = 1 - \frac{p_1^{'}}{p_1} + \frac{p_1^{'}}{p_1}M^{\frac{1}{p_1^{'}}} - M^{\frac{1}{p_1}} > 0$$

and then

$$f(M,y) > f(M,1) > 0.$$

Therefore the point$A(1,1)$ is the global minimum of the function f on the interval $[1,M] \times [1,M]$.

Taking into account hypothesis from Lemma 1, (i) and denoting by a, $\frac{p_1^{'}}{p_1}$, by b, $\frac{p_2^{'}}{p_2}$ and by c, $\frac{p_3^{'}}{p_3}$, we get $c > 1,\ a < b < 1$.

Condition $\Delta_2 > 0$ from the proof of Lemma 1 becomes,

$$\left(\frac{p_1}{p_1^{'}} - 1\right)\left[-\frac{p_2^{'}}{p_2}\left(\frac{p_2^{'}}{p_2} - p_2^{'}\right) + \frac{p_1^{'}}{p_1}\left(1 - p_2^{'}\right)\right] > \left(\frac{p_2^{'}}{p_2} - 1\right)^{,}2$$

or

$$\left(\frac{1}{a} - 1\right)[-b(b - p_2^{'}) + a(1 - p_2^{'})] > (b-1)^2$$

and by calculus, we have:

$$p_2^{'}(1-a) > b - a$$

, i. e. the condition

$$p_2^{'}\left(1 - \frac{p_1^{'}}{p_1}\right) > \frac{p_2^{'}}{p_2} - \frac{p_1^{'}}{p_1}$$

from our hypothesis.

(ii) We replace $x \in [1,M]$ by $\frac{a}{c}$ and $y \in [1,M]$ by $\frac{b}{c}$ and because $\frac{a}{c} \in [1,M]$ and $\frac{b}{c} \in [1,M]$ the inequality from (i) becomes:

$$\frac{1}{p_1}\frac{a}{c} + \frac{1}{p_2}\frac{b}{c} + \frac{1}{p_3} - \left(\frac{a}{c}\right)^{\frac{1}{p_1}}\left(\frac{b}{c}\right)^{\frac{1}{p_2}} > \frac{p_1^{'}}{p_1}\left[\frac{1}{p_1^{'}}\frac{a}{c} + \frac{1}{p_2^{'}}\frac{b}{c} + \frac{1}{p_3^{'}} - \left(\frac{a}{c}\right)^{\frac{1}{p_1^{'}}}\left(\frac{b}{c}\right)^{\frac{1}{p_2^{'}}}\right]$$

and multiplying by $c > 0$ we get the desired inequality.

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Example 2. *The particular case from Example 1 (i) satisfies the conditions of Theorem 1 (i), and then the point $A(1,1)$ is the global minimum for the function f and the inequality from Theorem 1 (i) takes place.*

3. Holder-type inequality for three functions

The following result is obtained as a consequence of Theorem 1 (ii) for isotonic linear functionals, being a Holder-type inequality in the case of three functions.

Theorem 2. *Let $M > 1$ and $p_1, p_2, p_3,\ p_1^{'}, p_2^{'}, p_3^{'}$ be positive real numbers which satisfies the conditions, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1,\ \frac{1}{p_1^{'}} + \frac{1}{p_2^{'}} + \frac{1}{p_3^{'}} = 1,\ \frac{p_3^{'}}{p_3} > 1 > \frac{p_2^{'}}{p_2}$ and $p_2^{'}(1 - \frac{p_1^{'}}{p_1}) > \frac{p_2^{'}}{p_2} - \frac{p_1^{'}}{p_1} > 0$, L satisfying conditions L1, L2 and A satisfying A1, A2 on the set E. Considering the nonnegative functions $f,\ g,\ h$ with $fgh,\ f^{\frac{p_1}{p_1^{'}}} g^{\frac{p_2}{p_2^{'}}} h^{\frac{p_3}{p_3^{'}}},\ f^{p_1},\ g^{p_2},\ h^{p_3} \in L$ and $A(f^{p_1}) > 0,\ A(g^{p_2}) > 0,\ A((h^{p_3}) > 0$, if in addition, $\frac{h^{p_3}}{A(h^{p_3})} < \frac{f^{p_1}}{A(f^{p_1})} < M \frac{h^{p_3}}{A(h^{p_3})}$ and $\frac{h^{p_3}}{A(h^{p_3})} < \frac{g^{p_2}}{A(g^{p_2})} < M \frac{h^{p_3}}{A(h^{p_3})}$ we will have,*

$$1 - \frac{A(fgh)}{A^{\frac{1}{p_1}}(f^{p_1}) A^{\frac{1}{p_2}}(g^{p_2}) A^{\frac{1}{p_3}}(h^{p_3})} > \frac{p_1^{'}}{p_1} \left(1 - \frac{A(f^{\frac{p_1}{p_1^{'}}} g^{\frac{p_2}{p_2^{'}}} h^{\frac{p_3}{p_3^{'}}})}{A^{\frac{1}{p_1^{'}}}(f^{p_1}) A^{\frac{1}{p_2^{'}}}(g^{p_2}) A^{\frac{1}{p_3^{'}}}(h^{p_3})} \right).$$

Proof. We use inequality from Theorem 1 (ii), for $a = \frac{f^{p_1}}{A(f^{p_1})}$, $b = \frac{g^{p_2}}{A(g^{p_2})}$ and $c = \frac{h^{p_3}}{A(h^{p_3})}$ and we have

$$\frac{1}{p_1} \frac{f^{p_1}}{A(f^{p_1})} + \frac{1}{p_2} \frac{g^{p_2}}{A(g^{p_2})} + \frac{1}{p_3} \frac{h^{p_3}}{A(h^{p_3})} - \frac{fgh}{A^{\frac{1}{p_1}}(f^{p_1}) A^{\frac{1}{p_2}}(g^{p_2}) A^{\frac{1}{p_3}}(h^{p_3})} >$$

$$> \frac{p_1^{'}}{p_1} \left(\frac{1}{p_1^{'}} \frac{f^{p_1}}{A(f^{p_1})} + \frac{1}{p_2^{'}} \frac{g^{p_2}}{A(g^{p_2})} + \frac{1}{p_3^{'}} \frac{h^{p_3}}{A(h^{p_3})} - \frac{f^{\frac{p_1}{p_1^{'}}} g^{\frac{p_2}{p_2^{'}}} h^{\frac{p_3}{p_3^{'}}}}{A^{\frac{1}{p_1^{'}}}(f^{p_1}) A^{\frac{1}{p_2^{'}}}(g^{p_2}) A^{\frac{1}{p_3^{'}}}(h^{p_3})} \right).$$

Now using hypothesis and condition A2, we get,

$$\frac{1}{p_1} \frac{A(f^{p_1})}{A(f^{p_1})} + \frac{1}{p_2} \frac{A(g^{p_2})}{A(g^{p_2})} + \frac{1}{p_3} \frac{A(h^{p_3})}{A(h^{p_3})} - \frac{A(fgh)}{A^{\frac{1}{p_1}}(f^{p_1}) A^{\frac{1}{p_2}}(g^{p_2}) A^{\frac{1}{p_3}}(h^{p_3})} >$$

$$> \frac{p_1^{'}}{p_1} \left(\frac{1}{p_1^{'}} \frac{A(f^{p_1})}{A(f^{p_1})} + \frac{1}{p_2^{'}} \frac{A(g^{p_2})}{A(g^{p_2})} + \frac{1}{p_3^{'}} \frac{A(h^{p_3})}{A(h^{p_3})} - \frac{A(f^{\frac{p_1}{p_1^{'}}} g^{\frac{p_2}{p_2^{'}}} h^{\frac{p_3}{p_3^{'}}})}{A^{\frac{1}{p_1^{'}}}(f^{p_1}) A^{\frac{1}{p_2^{'}}}(g^{p_2}) A^{\frac{1}{p_3^{'}}}(h^{p_3})} \right),$$

or by calculus we obtain the desired inequality.
∎

As a particular case, when instead of the isotonic linear functional, $A(f)$ we consider, as in [3], $\int_a^b f(x)dx$, Theorem 2 becomes:

Remark 1. *Let $M > 1$ and $p_1, p_2, p_3,\ p_1^{'}, p_2^{'}, p_3^{'}$ be positive real numbers which satisfies the conditions,* $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1,\ \frac{1}{p_1^{'}} + \frac{1}{p_2^{'}} + \frac{1}{p_3^{'}} = 1,\ \frac{p_3^{'}}{p_3} > 1 > \frac{p_2^{'}}{p_2}$ *and* $p_2^{'}(1 - \frac{p_1^{'}}{p_1}) > \frac{p_2^{'}}{p_2} - \frac{p_1^{'}}{p_1} > 0,$

Considering the continuous functions $f,\ g,\ h > 0$ on the interval $[a, b]$ with and $\frac{h^{p_3}(x)}{\int_a^b h^{p_3}(x)dx} < \frac{f^{p_1}(x)}{\int_a^b f^{p_1}(x)dx} < M\frac{h^{p_3}(x)}{\int_a^b h^{p_3}(x)}$ *and* $\frac{h^{p_3}(xdx)}{\int_a^b h^{p_3}(x)dx} < \frac{g^{p_2}(x)}{\int_a^b g^{p_2}(x)dx} < M\frac{h^{p_3}(x)}{\int_a^b h^{p_3}(x)dx}$ *we will have,*

$$1 - \frac{\int_a^b f(x)g(x)h(x)dx}{(\int_a^b f^{p_1}(x)dx)^{\frac{1}{p_1}}(\int_a^b g^{p_2}(x)dx)^{\frac{1}{p_2}}(\int_a^b h^{p_3}(x)dx)^{\frac{1}{p_3}}} >$$

$$> \frac{p_1^{'}}{p_1}\left(1 - \frac{\int_a^b f^{\frac{p_1}{p_1^{'}}}(x)g^{\frac{p_2}{p_2^{'}}}(x)h^{\frac{p_3}{p_3^{'}}}(x)dx}{(\int_a^b f^{p_1}(x)dx)^{\frac{1}{p_1^{'}}}(\int_a^b g^{p_2}(x)dx)^{\frac{1}{p_2^{'}}}(\int_a^b h^{p_3}(x)dx)^{\frac{1}{p_3^{'}}}}\right).$$

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