AN EXTENSION OF OSTROWSKI'S INEQUALITY TO THE COMPLEX INTEGRAL

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we extend the Ostrowski inequality to the complex integral by providing upper bounds for the quantity

$$\left|f(v)(w-u)-\int_{\gamma} f(z)\,dz\right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t\in[a,b]$, $u=z(a)$, $v=z(x)$ with $x\in(a,b)$ and $w=z(b)$ while f is holomorphic in G, an open domain and $\gamma\subset G$. An application for circular paths is also given.

1. Introduction

In 1938, A. Ostrowski [7], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x\in[a,b]$.

Theorem 1 (Ostrowski, 1938 [7])**.** *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_{\infty}:=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_{\infty}(b-a), \tag{1.1}$$

for all $x\in[a,b]$ and the constant $\frac{1}{4}$ is the best possible.

In [5], S. S. Dragomir and S. Wang, by the use of the *Montgomery integral identity* [6, p. 565],

$$f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{1}{b-a}\int_a^b p(x,t)\,f'(t)\,dt,\quad x\in[a,b], \tag{1.2}$$

where $p:[a,b]^2\to\mathbb{R}$ is given by

$$p(x,t):=\begin{cases} t-a & \text{if} \quad t\in[a,x], \\ t-b & \text{if} \quad t\in(x,b], \end{cases}$$

gave a simple proof of Ostrowski's inequality and applied it for special means (identric mean, logarithmic mean, etc.) and to the problem of estimating the error bound in approximating the Riemann integral $\int_a^b f(t)\,dt$ by one arbitrary Riemann sum (see [5], Section 3).

The following result, which is an improvement on Ostrowski's inequality, holds.

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Theorem 2 (Dragomir, 2002 [1])**.** *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ whose derivative $f'\in L_{\infty}[a,b]$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{2}\left[\|f'\|_{[a,x],\infty}\left(\frac{x-a}{b-a}\right)^2+\|f'\|_{[x,b],\infty}\left(\frac{b-x}{b-a}\right)^2\right](b-a) \leq \|f'\|_{[a,b],\infty}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a); \tag{1.3}$$

for all $x\in[a,b]$, where $\|\cdot\|_{[m,n],\infty}$ denotes the usual norm on $L_{\infty}[m,n]$, i.e., we recall that

$$\|g\|_{[m,n],\infty}=\operatorname*{essup}_{t\in[m,n]}|g(t)|<\infty.$$

The corresponding version for the 1-norm is as follows:

Theorem 3 (Dragomir, 2002 [2])**.** *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{x-a}{b-a}\|f'\|_{[a,x],1}+\frac{b-x}{b-a}\|f'\|_{[x,b],1} \leq\left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\|f'\|_{[a,b],1}$$

for all $x\in[a,b]$, where $\|\cdot\|_{[m,n],1}$ denotes the usual norm on $L_1[m,n]$ with $m<n$, i.e., we recall that

$$\|g\|_{[m,n],1}:=\int_m^n|g(t)|\,dt<\infty.$$

The case of p-norm is as follows:

Theorem 4 (Dragomir, 2013 [3])**.** *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_p[a,b]$, then we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq\frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}\|f'\|_{[a,x],p}+\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}}\|f'\|_{[x,b],p}\right](b-a)^{1/q} \leq\frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\|f'\|_{[a,b],p}, \tag{1.4}$$

for all $x\in[a,b]$, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\|\cdot\|_{[a,b],p}$ is the p-Lebesgue norm on $L_p[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],p}:=\left(\int_a^b|g(t)|^p\,dt\right)^{1/p}.$$

For a recent survey on Ostrowski's inequality, see [4].

In order to extend this result for the complex integral, we need some preparations as follows.

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, and open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.5}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_{\gamma} f(z)\,dz\right| \le \int_{\gamma} |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.6}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_{\gamma} |f(z)|^p\,|dz|\right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_{\gamma} |f(z)|\,|dz|.$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}.$$

In this paper we extend the Ostrowski inequality to the complex integral by providing upper bounds for the quantity

$$\left| f(v)(w-u) - \int_\gamma f(z)\,dz \right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$, $u = z(a)$, $v = z(x)$ with $x \in (a,b)$ and $w = z(b)$ while f is holomorphic in G, an open domain and $\gamma \subset G$. An application for circular paths is also given.

2. Ostrowski Type Inequalities

We have the following result for functions of complex variable:

Theorem 5. *Let f be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(x)$ with $x \in (a,b)$, then $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$,*

$$\begin{aligned} &\left| f(v)(w-u) - \int_\gamma f(z)\,dz \right| \\ &\quad \le \|f'\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-u|\,|dz| + \|f'\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-w|\,|dz| \\ &\quad \le \left[\int_{\gamma_{u,v}} |z-u|\,|dz| + \int_{\gamma_{v,w}} |z-w|\,|dz| \right] \|f'\|_{\gamma_{u,w};\infty} \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &\left| f(v)(w-u) - \int_\gamma f(z)\,dz \right| \\ &\quad \le \max_{z\in\gamma_{u,v}} |z-u|\, \|f'\|_{\gamma_{u,v};1} + \max_{z\in\gamma_{v,w}} |z-w|\, \|f'\|_{\gamma_{v,w};1} \\ &\quad \le \max\left\{ \max_{z\in\gamma_{u,v}} |z-u|, \max_{z\in\gamma_{v,w}} |z-w| \right\} \|f'\|_{\gamma_{u,w};1}. \end{aligned} \tag{2.2}$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned} &\left| f(v)(w-u) - \int_\gamma f(z)\,dz \right| \\ &\le \left(\int_{\gamma_{u,v}} |z-u|^q\,|dz| \right)^{1/q} \|f'\|_{\gamma_{u,v};p} + \left(\int_{\gamma_{v,w}} |z-w|^q\,|dz| \right)^{1/q} \|f'\|_{\gamma_{v,w};p} \\ &\le \left(\int_{\gamma_{u,v}} |z-u|^q\,|dz| + \int_{\gamma_{v,w}} |z-w|^q\,|dz| \right)^{1/q} \|f'\|_{\gamma_{u,w};p}. \end{aligned} \tag{2.3}$$

Proof. Using the integration by parts formula (1.5) twice we have

$$\int_{\gamma_{u,v}} (z-u) f'(z)\,dz = (v-u) f(v) - \int_{\gamma_{u,v}} f(z)\,dz$$

and

$$\int_{\gamma_{v,w}} (z-w) f'(z)\,dz = (w-v) f(v) - \int_{\gamma_{v,w}} f(z)\,dz.$$

If we add these two equalities, we get

$$\begin{aligned}&\int_{\gamma_{u,v}}(z-u)\,f'(z)\,dz+\int_{\gamma_{v,w}}(z-w)\,f'(z)\,dz\\&=f(v)(w-u)-\int_{\gamma_{u,v}}f(z)\,dz-\int_{\gamma_{v,w}}f(z)\,dz,\end{aligned}$$

which gives the following equality of interest

$$f(v)(w-u)-\int_{\gamma}f(z)\,dz=\int_{\gamma_{u,v}}(z-u)\,f'(z)\,dz+\int_{\gamma_{v,w}}(z-w)\,f'(z)\,dz \tag{2.4}$$

that is a generalization of Montgomery identity for functions of real variables mentioned in (1.2).

Using the properties of modulus and the triangle inequality for the complex integral we have

$$\begin{aligned}&\left|f(v)(w-u)-\int_{\gamma}f(z)\,dz\right| \\&=\left|\int_{\gamma_{u,v}}(z-u)\,f'(z)\,dz+\int_{\gamma_{v,w}}(z-w)\,f'(z)\,dz\right|\\&\le\left|\int_{\gamma_{u,v}}(z-u)\,f'(z)\,dz\right|+\left|\int_{\gamma_{v,w}}(z-w)\,f'(z)\,dz\right|\\&\le\int_{\gamma_{u,v}}|z-u|\,|f'(z)|\,|dz|+\int_{\gamma_{v,w}}|z-w|\,|f'(z)|\,|dz|\\&\le\|f'\|_{\gamma_{u,v};\infty}\int_{\gamma_{u,v}}|z-u|\,|dz|+\|f'\|_{\gamma_{v,w};\infty}\int_{\gamma_{v,w}}|z-w|\,|dz|\\&\le\|f'\|_{\gamma_{u,w};\infty}\left[\int_{\gamma_{u,v}}|z-u|\,|dz|+\int_{\gamma_{v,w}}|z-w|\,|dz|\right],\end{aligned} \tag{2.5}$$

which proves the desired result (2.1).

We also have

$$\begin{aligned}&\int_{\gamma_{u,v}}|z-u|\,|f'(z)|\,|dz|+\int_{\gamma_{v,w}}|z-w|\,|f'(z)|\,|dz|\\&\le\max_{z\in\gamma_{u,v}}|z-u|\int_{\gamma_{u,v}}|f'(z)|\,|dz|+\max_{z\in\gamma_{v,w}}|z-w|\int_{\gamma_{v,w}}|f'(z)|\,|dz|\\&\le\max\left\{\max_{z\in\gamma_{u,v}}|z-u|,\max_{z\in\gamma_{v,w}}|z-w|\right\}\left[\int_{\gamma_{u,v}}|f'(z)|\,|dz|+\int_{\gamma_{v,w}}|f'(z)|\,|dz|\right]\\&=\max\left\{\max_{z\in\gamma_{u,v}}|z-u|,\max_{z\in\gamma_{v,w}}|z-w|\right\}\int_{\gamma_{u,w}}|f'(z)|\,|dz|\end{aligned}$$

and by (2.5) we get (2.5).

If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's weighted integral inequality we have

$$\begin{aligned}
&\int_{\gamma_{u,v}}|z-u|\,|f'(z)|\,|dz|+\int_{\gamma_{v,w}}|z-w|\,|f'(z)|\,|dz| \\
&\qquad\le\left(\int_{\gamma_{u,v}}|z-u|^{q}\,|dz|\right)^{1/q}\left(\int_{\gamma_{u,v}}|f'(z)|^{p}\,|dz|\right)^{1/p} \\
&\qquad\quad+\left(\int_{\gamma_{v,w}}|z-w|^{q}\,|dz|\right)^{1/q}\left(\int_{\gamma_{v,w}}|f'(z)|^{p}\,|dz|\right)^{1/p}=:B.
\end{aligned}$$

By the elementary inequality

$$ab+cd\le(a^{p}+c^{p})^{1/p}(b^{q}+d^{q})^{1/q},$$

where $a,\ b,\ c,\ d\ge0$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we also have

$$\begin{aligned}
B&\le\left(\int_{\gamma_{u,v}}|z-u|^{q}\,|dz|+\int_{\gamma_{v,w}}|z-w|^{q}\,|dz|\right)^{1/q} \\
&\qquad\times\left(\int_{\gamma_{u,v}}|f'(z)|^{p}\,|dz|+\int_{\gamma_{v,w}}|f'(z)|^{p}\,|dz|\right)^{1/p} \\
&=\left(\int_{\gamma_{u,v}}|z-u|^{q}\,|dz|+\int_{\gamma_{v,w}}|z-w|^{q}\,|dz|\right)^{1/q}\left(\int_{\gamma_{u,w}}|f'(z)|^{p}\,|dz|\right)^{1/p},
\end{aligned}$$

which together with (2.5) gives (2.3). □

If the path γ is a segment $[u,w]\subset G$ connecting two distinct points u and w in G then we write $\int_{\gamma}f(z)\,dz$ as $\int_{u}^{w}f(z)\,dz$.

Using the p-norms defined in the introduction for the segments, namely

$$\|h\|_{[u,w];\infty}=\sup_{z\in[u,w]}|h(z)|$$

and

$$\|h\|_{[u,w];p}=\left(\int_{u}^{w}|h(z)|^{p}\,|dz|\right)^{1/p}\ \text{ for } p\ge1,$$

we can state the following particular case as well:

Corollary 1. *Let f be holomorphic in G, an open domain and suppose $[u,w]\subset G$ is a segment connecting two distinct points u and w in G and $v\in[u,w]$. Then for $v=(1-s)\,u+sw$ with $s\in[0,1]$, we have*

$$\begin{aligned}
(2.6)\quad &\left|f(v)(w-u)-\int_{u}^{w}f(z)\,dz\right| \\
&\qquad\le\frac{1}{2}|w-u|^{2}\left[\|f'\|_{[u,v];\infty}\,s^{2}+\|f'\|_{[v,w];\infty}(1-s)^{2}\right] \\
&\qquad\qquad\le|w-u|^{2}\left[\frac{1}{4}+\left(s-\frac{1}{2}\right)^{2}\right]\|f'\|_{[u,w];\infty}
\end{aligned}$$

and

$$\left|f(v)(w-u)-\int_u^w f(z)\,dz\right| \le |w-u|\left\{s\|f'\|_{[u,v];1}+(1-s)\|f'\|_{[v,w];1}\right\} \le |w-u|\left(\frac{1}{2}+\left|s-\frac{1}{2}\right|\right)\|f'\|_{[u,w];1}. \tag{2.7}$$

If $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\left|f(v)(w-u)-\int_u^w f(z)\,dz\right| \le \frac{1}{(q+1)^{1/q}}|w-u|^{1+1/q}\left[s^{1+1/q}\|f'\|_{[u,v];p}+(1-s)^{1+1/q}\|f'\|_{[v,w];p}\right] \le \frac{1}{(q+1)^{1/q}}|w-u|^{1+1/q}\left[s^{q+1}+(1-s)^{q+1}\right]^{1/q}\|f'\|_{[u,w];p}. \tag{2.8}$$

Proof. Observe that if the segment $[u,w]$ is parametrized by $z(t)=(1-t)u+tw$, then

$$\int_u^v |z-u|\,|dz| = |w-u|\int_0^s |(1-t)u+tw-u|\,dt = |w-u|^2\int_0^s t\,dt = \frac{1}{2}s^2|w-u|^2$$

and

$$\int_v^w |z-w|\,|dz| = |w-u|\int_s^1 |(1-t)u+tw-w|\,dt = |w-u|^2\int_s^1 (1-t)\,dt = \frac{1}{2}(1-s)^2|w-u|^2.$$

Using (2.1) we get

$$\left|f(v)(w-u)-\int_u^w f(z)\,dz\right| \le \frac{1}{2}|w-u|^2\left[\|f'\|_{[u,v];\infty}s^2+\|f'\|_{[v,w];\infty}(1-s)^2\right] \le \frac{1}{2}|w-u|^2\left[s^2+(1-s)^2\right]\|f'\|_{[u,w];\infty} = |w-u|^2\left[\frac{1}{4}+\left(s-\frac{1}{2}\right)^2\right]\|f'\|_{[u,w];\infty}$$

and the inequality (2.6).

Also,

$$\max_{z\in\gamma_{u,v}}|z-u| = \max_{t\in[0,s]}\{|w-u|\,t\} = |w-u|\,s$$

and

$$\max_{z\in\gamma_{v,w}}|z-w| = \max_{t\in[s,1]}\{|w-u|(1-t)\} = |w-u|(1-s),$$

then by (2.2) we get

$$
\begin{aligned}
&\left|f(v)(w-u)-\int_u^w f(z)\,dz\right| \\
&\qquad\qquad \le |w-u|\left\{s\|f'\|_{[u,v];1}+(1-s)\|f'\|_{[v,w];1}\right\} \\
&\qquad \le |w-u|\max\{s,1-s\}\|f'\|_{[u,w];1}=|w-u|\left(\frac{1}{2}+\left|s-\frac{1}{2}\right|\right)\|f'\|_{[u,w];1}\,,
\end{aligned}
$$

which proves (2.7).

Finally, since

$$
\begin{aligned}
\int_u^v |z-u|^q\,|dz| &= |w-u|\int_0^s |(1-t)\,u+tw-u|^q\,dt=|w-u|^{q+1}\int_0^s t^q dt \\
&= \frac{1}{q+1}s^{q+1}\,|w-u|^{q+1}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_v^w |z-w|^q\,|dz| &= |w-u|\int_s^1 |(1-t)\,u+tw-w|^q\,dt=|w-u|^{q+1}\int_s^1 (1-t)^q\,dt \\
&= \frac{1}{q+1}(1-s)^{q+1}\,|w-u|^{q+1}\,,
\end{aligned}
$$

hence by (2.3) we get

$$
\begin{aligned}
&\left|f(v)(w-u)-\int_\gamma f(z)\,dz\right| \\
&\quad\le \frac{1}{(q+1)^{1/q}}\,|w-u|^{1+1/q}\left[s^{1+1/q}\|f'\|_{[u,v];p}+(1-s)^{1+1/q}\|f'\|_{[v,w];p}\right] \\
&\qquad\qquad \le \frac{1}{(q+1)^{1/q}}\,|w-u|^{1+1/q}\left[s^{q+1}+(1-s)^{q+1}\right]^{1/q}\|f'\|_{[u,w];p}\,.
\end{aligned}
$$

□

Remark 1. *If there exists $m\in(a,b)$ such that $z(m)=\frac{w+u}{2}$, with $w\neq u$ then from (2.1) we get*

$$
\begin{aligned}
&\left|f\left(\frac{w+u}{2}\right)(w-u)-\int_\gamma f(z)\,dz\right| \qquad (2.9)\\
&\quad\le \|f'\|_{\gamma_{u,\frac{w+u}{2}};\infty}\int_{\gamma_{u,\frac{w+u}{2}}}|z-u|\,|dz|+\|f'\|_{\gamma_{\frac{w+u}{2},w};\infty}\int_{\gamma_{\frac{w+u}{2},w}}|z-w|\,|dz| \\
&\qquad\qquad \le \left[\int_{\gamma_{u,\frac{w+u}{2}}}|z-u|\,|dz|+\int_{\gamma_{\frac{w+u}{2},w}}|z-w|\,|dz|\right]\|f'\|_{\gamma_{u,w};\infty}\,,
\end{aligned}
$$

from (2.2) that

$$\left|f\left(\frac{w+u}{2}\right)(w-u)-\int_{\gamma}f(z)\,dz\right| \tag{2.10}$$
$$\leq \max_{z\in\gamma_{u,\frac{w+u}{2}}}|z-u|\,\|f'\|_{\gamma_{u,\frac{w+u}{2}};1}+\max_{z\in\gamma_{\frac{w+u}{2},w}}|z-w|\,\|f'\|_{\gamma_{\frac{w+u}{2},w};1}$$
$$\leq \max\left\{\max_{z\in\gamma_{u,\frac{w+u}{2}}}|z-u|\,,\max_{z\in\gamma_{\frac{w+u}{2},w}}|z-w|\right\}\|f'\|_{\gamma_{u,w};1}$$

and from (2.3) that

$$\left|f\left(\frac{w+u}{2}\right)(w-u)-\int_{\gamma}f(z)\,dz\right| \tag{2.11}$$
$$\leq\left(\int_{\gamma_{u,\frac{w+u}{2}}}|z-u|^{q}\,|dz|\right)^{1/q}\|f'\|_{\gamma_{u,\frac{w+u}{2}};p}$$
$$+\left(\int_{\gamma_{\frac{w+u}{2},w}}|z-w|^{q}\,|dz|\right)^{1/q}\|f'\|_{\gamma_{\frac{w+u}{2},w};p}$$
$$\leq\left(\int_{\gamma_{u,\frac{w+u}{2}}}|z-u|^{q}\,|dz|+\int_{\gamma_{\frac{w+u}{2},w}}|z-w|^{q}\,|dz|\right)^{1/q}\|f'\|_{\gamma_{u,w};p}\,,$$

where $p,\,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

For $s=\frac{1}{2}$ in Corollary 1 we get

$$\left|f\left(\frac{w+u}{2}\right)(w-u)-\int_{u}^{w}f(z)\,dz\right| \tag{2.12}$$
$$\leq\frac{1}{8}|w-u|^{2}\left[\|f'\|_{[u,\frac{w+u}{2}];\infty}+\|f'\|_{[\frac{w+u}{2},w];\infty}\right]\leq\frac{1}{4}|w-u|^{2}\,\|f'\|_{[u,w];\infty}$$

and

$$\left|f\left(\frac{w+u}{2}\right)(w-u)-\int_{u}^{w}f(z)\,dz\right|\leq\frac{1}{2}|w-u|\,\|f'\|_{[u,w];1}\,. \tag{2.13}$$

If $p,\,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\left|f\left(\frac{w+u}{2}\right)(w-u)-\int_{u}^{w}f(z)\,dz\right| \tag{2.14}$$
$$\leq\frac{1}{2^{1+1/q}(q+1)^{1/q}}|w-u|^{1+1/q}\left[\|f'\|_{[u,\frac{w+u}{2}];p}+\|f'\|_{[\frac{w+u}{2},w];p}\right]$$
$$\leq\frac{1}{2(q+1)^{1/q}}|w-u|^{1+1/q}\,\|f'\|_{[u,w];p}\,.$$

Suppose that $\gamma\subset G$ is a smooth path from $z(a)=u$ to $z(b)=w$. If $v=z(x)$ with $x\in(a,b)$, then $\gamma_{u,w}=\gamma_{u,v}\cup\gamma_{v,w}$.

If we consider $f(z)=\exp(z)$ with $z\in\mathbb{C}$, then

$$\int_{\gamma_{u,w}}\exp(z)\,dz=\exp(w)-\exp(u),$$

$$|\exp(z)|=|\exp(\operatorname{Re}(z)+i\operatorname{Im}(z))|=\exp(\operatorname{Re}(z))$$

and by Theorem 5 we have

$$\begin{aligned}(2.15)\quad &|\exp(w)-\exp(u)-\exp(v)(w-u)|\\ &\le\|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{u,v};\infty}\int_{\gamma_{u,v}}|z-u|\,|dz|+\|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{v,w};\infty}\int_{\gamma_{v,w}}|z-w|\,|dz|\\ &\le\left[\int_{\gamma_{u,v}}|z-u|\,|dz|+\int_{\gamma_{v,w}}|z-w|\,|dz|\right]\|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{u,w};\infty}\end{aligned}$$

and

$$\begin{aligned}(2.16)\quad &|\exp(w)-\exp(u)-\exp(v)(w-u)|\\ &\le\max_{z\in\gamma_{u,v}}|z-u|\,\|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{u,v};1}+\max_{z\in\gamma_{v,w}}|z-w|\,\|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{v,w};1}\\ &\le\max\left\{\max_{z\in\gamma_{u,v}}|z-u|,\max_{z\in\gamma_{v,w}}|z-w|\right\}\|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{u,w};1}.\end{aligned}$$

If $p,\,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\begin{aligned}(2.17)\quad &|\exp(w)-\exp(u)-\exp(v)(w-u)|\\ &\le\left(\int_{\gamma_{u,v}}|z-u|^q\,|dz|\right)^{1/q}\|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{u,v};p}\\ &+\left(\int_{\gamma_{v,w}}|z-w|^q\,|dz|\right)^{1/q}\|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{v,w};p}\\ &\le\left(\int_{\gamma_{u,v}}|z-u|^q\,|dz|+\int_{\gamma_{v,w}}|z-w|^q\,|dz|\right)^{1/q}\|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{u,w};p}.\end{aligned}$$

With the same assumption on the path γ and if we consider $f(z)=z^n$ with $n\ge1$, then

$$\int_\gamma z^n dz=\frac{w^{n+1}-u^{n+1}}{n+1}$$

and by Theorem 5 we get, by denoting $\ell(z)=z$, $z\in\mathbb{C}$, that

$$\begin{aligned}(2.18)\quad &\left|\frac{w^{n+1}-u^{n+1}}{n+1}-v^n(w-u)\right|\\ &\le n\left[\left\|\ell^{n-1}\right\|_{\gamma_{u,v};\infty}\int_{\gamma_{u,v}}|z-u|\,|dz|+\left\|\ell^{n-1}\right\|_{\gamma_{v,w};\infty}\int_{\gamma_{v,w}}|z-w|\,|dz|\right]\\ &\le n\left[\int_{\gamma_{u,v}}|z-u|\,|dz|+\int_{\gamma_{v,w}}|z-w|\,|dz|\right]\left\|\ell^{n-1}\right\|_{\gamma_{u,w};\infty}\end{aligned}$$

and

$$\left|\frac{w^{n+1}-u^{n+1}}{n+1}-v^{n}(w-u)\right| \tag{2.19}$$
$$\leq n\left[\max_{z\in\gamma_{u,v}}|z-u|\left\|\ell^{n-1}\right\|_{\gamma_{u,v};1}+\max_{z\in\gamma_{v,w}}|z-w|\left\|\ell^{n-1}\right\|_{\gamma_{v,w};1}\right]$$
$$\leq n\max\left\{\max_{z\in\gamma_{u,v}}|z-u|,\max_{z\in\gamma_{v,w}}|z-w|\right\}\left\|\ell^{n-1}\right\|_{\gamma_{u,w};1}.$$

If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\left|\frac{w^{n+1}-u^{n+1}}{n+1}-v^{n}(w-u)\right| \tag{2.20}$$
$$\leq n\left[\left(\int_{\gamma_{u,v}}|z-u|^{q}|dz|\right)^{1/q}\left\|\ell^{n-1}\right\|_{\gamma_{u,v};p}+\left(\int_{\gamma_{v,w}}|z-w|^{q}|dz|\right)^{1/q}\left\|\ell^{n-1}\right\|_{\gamma_{v,w};p}\right]$$
$$\leq n\left(\int_{\gamma_{u,v}}|z-u|^{q}|dz|+\int_{\gamma_{v,w}}|z-w|^{q}|dz|\right)^{1/q}\left\|\ell^{n-1}\right\|_{\gamma_{u,w};p}.$$

3. Examples for Circular Paths

Let $[a,b]\subseteq[0,2\pi]$ and the circular path $\gamma_{[a,b],R}$ centered in 0 and with radius $R>0$

$$z(t)=R\exp(it)=R(\cos t+i\sin t),\ t\in[a,b].$$

If $[a,b]=[0,\pi]$ then we get a half circle while for $[a,b]=[0,2\pi]$ we get the full circle.

Since

$$\left|e^{is}-e^{it}\right|^{2}=\left|e^{is}\right|^{2}-2\operatorname{Re}\left(e^{i(s-t)}\right)+\left|e^{it}\right|^{2}$$
$$=2-2\cos(s-t)=4\sin^{2}\left(\frac{s-t}{2}\right)$$

for any $t,\ s\in\mathbb{R}$, then

$$\left|e^{is}-e^{it}\right|^{r}=2^{r}\left|\sin\left(\frac{s-t}{2}\right)\right|^{r} \tag{3.1}$$

for any $t,\ s\in\mathbb{R}$ and $r>0$. In particular,

$$\left|e^{is}-e^{it}\right|=2\left|\sin\left(\frac{s-t}{2}\right)\right|$$

for any $t,\ s\in\mathbb{R}$.

For $s=a$ and $s=b$ we have

$$\left|e^{ia}-e^{it}\right|=2\left|\sin\left(\frac{a-t}{2}\right)\right| \text{ and } \left|e^{ib}-e^{it}\right|=2\left|\sin\left(\frac{b-t}{2}\right)\right|.$$

If $u=R\exp(ia)$ and $w=R\exp(ib)$ then

$$w-u=R[\exp(ib)-\exp(ia)]=R[\cos b+i\sin b-\cos a-i\sin a]$$
$$=R[\cos b-\cos a+i(\sin b-\sin a)].$$

Since

$$\cos b-\cos a=-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

and

$$\sin b-\sin a=2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right),$$

hence

$$\begin{aligned}
w-u&=R\left[-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)+2i\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right)\right]\\
&=2R\sin\left(\frac{b-a}{2}\right)\left[-\sin\left(\frac{a+b}{2}\right)+i\cos\left(\frac{a+b}{2}\right)\right]\\
&=2Ri\sin\left(\frac{b-a}{2}\right)\left[\cos\left(\frac{a+b}{2}\right)+i\sin\left(\frac{a+b}{2}\right)\right]\\
&=2Ri\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right].
\end{aligned}$$

Moreover,

$$|z-u|=R\left|\exp(it)-\exp(ia)\right|=2R\left|\sin\left(\frac{t-a}{2}\right)\right|$$

and

$$|z-w|=R\left|\exp(it)-\exp(ib)\right|=2R\left|\sin\left(\frac{b-t}{2}\right)\right|$$

for $t\in[a,b]$.

If $[a,b]\subseteq[0,2\pi]$ then $0\leq\frac{t-a}{2},\ \frac{b-t}{2}\leq\pi$ for $t\in[a,b]$, therefore

$$|z-u|=2R\sin\left(\frac{t-a}{2}\right)\text{ and }|z-w|=2R\sin\left(\frac{b-t}{2}\right).$$

We also have

$$z'(t)=Ri\exp(it)\text{ and }|z'(t)|=R$$

for $t\in[a,b]$.

Proposition 1. *Let f be holomorphic in G, on open domain and suppose $\gamma_{[a,b],R}\subset G$ with $[a,b]\subseteq[0,2\pi]$ and $R>0$. If $x\in[a,b]$, then*

$$\begin{aligned}
&\left|2\sin\left(\frac{b-a}{2}\right)f(R\exp(ix))\exp\left(\left(\frac{a+b}{2}\right)i\right)\right.\\
&\qquad\qquad\left.-\int_a^b f(R\exp(it))\exp(it)\,dt\right|\\
&\leq 8R\left[\|f'(R\exp(i\cdot))\|_{[a,x],\infty}\sin^2\left(\frac{x-a}{4}\right)+\|f'(R\exp(i\cdot))\|_{[x,b],\infty}\sin^2\left(\frac{b-x}{4}\right)\right]\\
&\qquad\leq 8R\|f'(R\exp(i\cdot))\|_{[a,b],\infty}\left[\sin^2\left(\frac{x-a}{4}\right)+\sin^2\left(\frac{b-x}{4}\right)\right].
\end{aligned}\tag{3.2}$$

Proof. We write the inequality (2.1) for $\gamma_{[a,b],R}$ and $x \in [a,b]$ to get

$$\begin{aligned}
&\Bigg|2R\sin\left(\frac{b-a}{2}\right) if\left(R\exp\left(ix\right)\right)\exp\left[\left(\frac{a+b}{2}\right)i\right] \\
&\qquad -Ri\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\Bigg| \\
&\quad \le 2R^2\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,x],\infty}\int_a^x \sin\left(\frac{t-a}{2}\right)dt \\
&\qquad + 2R^2\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,b],\infty}\int_x^b \sin\left(\frac{b-t}{2}\right)dt \\
&\quad \le 2R^2\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,b],\infty}\left[\int_a^x \sin\left(\frac{t-a}{2}\right)dt + \int_x^b \sin\left(\frac{b-t}{2}\right)dt\right].
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&\Bigg|2\sin\left(\frac{b-a}{2}\right) f\left(R\exp\left(ix\right)\right)\exp\left(\left(\frac{a+b}{2}\right)i\right) \\
&\qquad -\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\Bigg| \\
&\quad \le 2R\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,x],\infty}\int_a^x \sin\left(\frac{t-a}{2}\right)dt \\
&\qquad + 2R\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,b],\infty}\int_x^b \sin\left(\frac{b-t}{2}\right)dt \\
&\quad \le 2R\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,b],\infty}\left[\int_a^x \sin\left(\frac{t-a}{2}\right)dt + \int_x^b \sin\left(\frac{b-t}{2}\right)dt\right].
\end{aligned} \tag{3.3}$$

Observe that

$$\int_a^x \sin\left(\frac{s-a}{2}\right)ds = 2-2\cos\left(\frac{x-a}{2}\right) = 4\sin^2\left(\frac{x-a}{4}\right)$$

and

$$\int_x^b \sin\left(\frac{b-s}{2}\right)ds = 2-2\cos\left(\frac{b-t}{2}\right) = 4\sin^2\left(\frac{b-x}{4}\right)$$

for $x \in [a,b]$.

By using (3.3) we then get the desired result (3.2). □

Corollary 2. *With the assumptions of Proposition 1 we have*

$$\begin{aligned}(3.4)\quad & \left|2\sin\left(\frac{b-a}{2}\right) f\left(R\exp\left(\frac{a+b}{2}i\right)\right)\exp\left(\frac{a+b}{2}i\right)\right. \\ & \qquad\left. -\int_a^b f\left(R\exp\left(i\cdot\right)\right)\exp\left(it\right)dt\right| \\ & \leq 8R\left[\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{\left[a,\frac{a+b}{2}\right],\infty}+\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{\left[\frac{a+b}{2},b\right],\infty}\right]\sin^2\left(\frac{b-a}{8}\right) \\ & \leq 16R\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,b],\infty}\sin^2\left(\frac{b-a}{8}\right).\end{aligned}$$

Remark 2. *The case of semi-circle, namely $a=0$ and $b=\pi$ in (3.2) gives the inequality*

$$\begin{aligned}(3.5)\quad & \left|2if\left(R\exp\left(ix\right)\right)-\int_0^\pi f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right| \\ & \leq 8R\left[\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[0,x],\infty}\sin^2\left(\frac{x}{4}\right)+\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,\pi],\infty}\sin^2\left(\frac{\pi-x}{4}\right)\right] \\ & \leq 8R\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,b],\infty}\left[\sin^2\left(\frac{x}{4}\right)+\sin^2\left(\frac{\pi-x}{4}\right)\right],\end{aligned}$$

for $x\in[0,\pi]$.

Since

$$\sin^2\left(\frac{\pi}{8}\right)=\frac{1-\cos\left(\frac{\pi}{4}\right)}{2}=\frac{1-\frac{\sqrt{2}}{2}}{2}=\frac{2-\sqrt{2}}{4},$$

then by taking $x=\frac{\pi}{2}$ in (3.5), we get

$$\begin{aligned}(3.6)\quad & \left|2if\left(Ri\right)-\int_0^\pi f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right| \\ & \leq 2\left(2-\sqrt{2}\right)\left[\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{\left[0,\frac{\pi}{2}\right],\infty}+\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{\left[\frac{\pi}{2},\pi\right],\infty}\right] \\ & \leq 4\left(2-\sqrt{2}\right)\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[0,\pi],\infty}.\end{aligned}$$

Proposition 2. *With the assumptions of Proposition 1 we have*

$$\begin{aligned}(3.7)\quad & \left|2\sin\left(\frac{b-a}{2}\right)f\left(R\exp\left(ix\right)\right)\exp\left(\left(\frac{a+b}{2}\right)i\right)\right. \\ & \qquad\left.-\int_a^b f\left(R\exp\left(i\cdot\right)\right)\exp\left(it\right)dt\right| \\ & \leq 2R\left[\max_{t\in[a,x]}\sin\left(\frac{t-a}{2}\right)\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,x],1}\right. \\ & \qquad\left.+\max_{t\in[x,b]}\sin\left(\frac{b-t}{2}\right)\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,b],1}\right] \\ & \leq 2R\max\left\{\max_{t\in[a,x]}\sin\left(\frac{t-a}{2}\right),\max_{t\in[x,b]}\sin\left(\frac{b-t}{2}\right)\right\}\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,b],1}\end{aligned}$$

for $x\in[a,b]\subseteq[0,2\pi]$.

Proof. We write the inequality (2.2) for $\gamma_{[a,b],R}$ and $x \in [a,b]$ to get

$$\begin{aligned}
&\left|2R\sin\left(\frac{b-a}{2}\right) if\left(R\exp\left(ix\right)\right)\exp\left[\left(\frac{a+b}{2}\right)i\right]\right. \\
&\qquad\qquad\left. -Ri\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right| \\
&\quad\le 2R^2\left[\max_{t\in[a,x]}\sin\left(\frac{t-a}{2}\right)\int_a^x\left|f'\left(R\exp\left(it\right)\right)\right|dt\right. \\
&\qquad\left.+\max_{t\in[x,b]}\sin\left(\frac{b-t}{2}\right)\int_x^b\left|f'\left(R\exp\left(it\right)\right)\right|dt\right] \\
&\quad\le 2R^2\max\left\{\max_{t\in[a,x]}\sin\left(\frac{t-a}{2}\right),\max_{t\in[x,b]}\sin\left(\frac{b-t}{2}\right)\right\}\int_a^b\left|f'\left(R\exp\left(it\right)\right)\right|dt,
\end{aligned}$$

which is equivalent to (3.7). □

Corollary 3. *With the assumptions of Proposition 1 we have*

$$\begin{aligned}
&\left|2\sin\left(\frac{b-a}{2}\right)f\left(R\exp\left(\left(\frac{a+b}{2}\right)i\right)\right)\exp\left(\left(\frac{a+b}{2}\right)i\right)\right. \\
&\qquad\qquad\left.-\int_a^b f\left(R\exp\left(i\cdot\right)\right)\exp\left(it\right)dt\right| \\
&\qquad\qquad\qquad\le 2R\sin\left(\frac{b-a}{4}\right)\int_a^b\left|f'\left(R\exp\left(it\right)\right)\right|dt.
\end{aligned} \tag{3.8}$$

Proof. If we take in (3.7) $x = \frac{a+b}{2}$, then we get

$$\begin{aligned}
&\left|2\sin\left(\frac{b-a}{2}\right)f\left(R\exp\left(\frac{a+b}{2}i\right)\right)\exp\left(\left(\frac{a+b}{2}\right)i\right)\right. \\
&\qquad\qquad\left.-\int_a^b f\left(R\exp\left(i\cdot\right)\right)\exp\left(it\right)dt\right| \\
&\quad\le 2R\left[\max_{t\in\left[a,\frac{a+b}{2}\right]}\sin\left(\frac{t-a}{2}\right)\int_a^{\frac{a+b}{2}}\left|f'\left(R\exp\left(it\right)\right)\right|dt\right. \\
&\qquad\left.+\max_{t\in\left[\frac{a+b}{2},b\right]}\sin\left(\frac{b-t}{2}\right)\int_{\frac{a+b}{2}}^b\left|f'\left(R\exp\left(it\right)\right)\right|dt\right] \\
&\le 2R\max\left\{\max_{t\in\left[a,\frac{a+b}{2}\right]}\sin\left(\frac{t-a}{2}\right),\max_{t\in\left[\frac{a+b}{2},b\right]}\sin\left(\frac{b-t}{2}\right)\right\}\int_a^b\left|f'\left(R\exp\left(it\right)\right)\right|dt.
\end{aligned} \tag{3.9}$$

Since the intervals $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$ have a length less than π, then

$$\max_{t\in\left[a,\frac{a+b}{2}\right]}\sin\left(\frac{t-a}{2}\right)=\sin\left(\frac{b-a}{4}\right),\quad \max_{t\in\left[\frac{a+b}{2},b\right]}\sin\left(\frac{b-t}{2}\right)=\sin\left(\frac{b-a}{4}\right)$$

and by (3.9) we get (3.8). □

The case of p-norms is as follows:

Proposition 3. *With the assumptions of Proposition 1 and* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *we have*

$$\begin{aligned}(3.10)\quad & \left| 2\sin\left(\frac{b-a}{2}\right) f\left(R\exp\left(ix\right)\right)\exp\left(\left(\frac{a+b}{2}\right)i\right) \right. \\ & \qquad \left. - \int_a^b f\left(R\exp\left(i\cdot\right)\right)\exp\left(it\right)dt \right| \\ & \leq 2R\left[\left(\int_a^x \sin^q\left(\frac{t-a}{2}\right)dt\right)^{1/q} \left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,x],p} \right. \\ & \qquad \left. + \left(\int_x^b \sin^q\left(\frac{b-t}{2}\right)dt\right)^{1/q} \left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,b],p}\right] \\ & \leq 2R\left[\int_a^x \sin^q\left(\frac{t-a}{2}\right)dt + \int_x^b \sin^q\left(\frac{b-t}{2}\right)dt\right]^{1/q} \left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,b],p}\end{aligned}$$

for $x \in [a,b] \subseteq [0, 2\pi]$ *.*

In particular, for $x = \frac{a+b}{2}$ *we get*

$$\begin{aligned}(3.11)\quad & \left| 2\sin\left(\frac{b-a}{2}\right) f\left(R\exp\left(\frac{a+b}{2}i\right)\right)\exp\left(\left(\frac{a+b}{2}\right)i\right) \right. \\ & \qquad \left. - \int_a^b f\left(R\exp\left(i\cdot\right)\right)\exp\left(it\right)dt \right| \\ & \leq 2R\left[\left(\int_a^{\frac{a+b}{2}} \sin^q\left(\frac{t-a}{2}\right)dt\right)^{1/q} \left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{\left[a,\frac{a+b}{2}\right],p} \right. \\ & \qquad \left. + \left(\int_{\frac{a+b}{2}}^b \sin^q\left(\frac{b-t}{2}\right)dt\right)^{1/q} \left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{\left[\frac{a+b}{2},b\right],p}\right] \\ & \leq 2R\left[\int_a^{\frac{a+b}{2}} \sin^q\left(\frac{t-a}{2}\right)dt + \int_{\frac{a+b}{2}}^b \sin^q\left(\frac{b-t}{2}\right)dt\right]^{1/q} \left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,b],p}.\end{aligned}$$

Proof. We write the inequality (2.3) for $\gamma_{[a,b],R}$ and $x \in [a,b]$ to get

$$\begin{aligned}& \left| 2R\sin\left(\frac{b-a}{2}\right) if\left(R\exp\left(ix\right)\right)\exp\left[\left(\frac{a+b}{2}\right)i\right] \right. \\ & \qquad \left. - Ri\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt \right|\end{aligned}$$

$$\leq 2R^2 \left(\int_a^x \sin^q \left(\frac{t-a}{2} \right) dt \right)^{1/q} \left\| f' \left(R \exp(i \cdot) \right) \right\|_{[a,x],p}$$

$$+ 2R^2 \left(\int_x^b \sin^q \left(\frac{b-t}{2} \right) dt \right)^{1/q} \left\| f' \left(R \exp(i \cdot) \right) \right\|_{[x,b],p}$$

$$\leq 2R^2 \left[\int_a^x \sin^q \left(\frac{t-a}{2} \right) dt + \int_x^b \sin^q \left(\frac{b-t}{2} \right) dt \right]^{1/q} \left\| f' \left(R \exp(i \cdot) \right) \right\|_{[a,b],p},$$

which proves the desired result (3.10). □

The interested reader may consider for examples some fundamental complex functions such as $f(z) = z^n$ with n a natural number, $f(z) = \exp(z)$ or f a trigonometric or a hyperbolic complex function. The details are omitted.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.org/dragomir`

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa