AN EXTENSION OF TRAPEZOID INEQUALITY TO THE COMPLEX INTEGRAL

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we extend the trapezoid inequality to the complex integral by providing upper bounds for the quantity

$$\left|(v-u)\,f(u)+(w-v)\,f(w)-\int_{\gamma}f(z)\,dz\right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t\in[a,b]$, $u=z(a)$, $v=z(x)$ with $x\in(a,b)$ and $w=z(b)$ while f is holomorphic in G, an open domain and $\gamma\subset G$. An application for circular paths is also given.

1. Introduction

Inequalities providing upper bounds for the quantity

$$\left|(t-a)\,f(a)+(b-t)\,f(b)-\int_a^b f(s)\,ds\right|,\qquad t\in[a,b] \tag{1.1}$$

are known in the literature as *generalized trapezoid inequalities* and it has been shown in [2] that

$$\begin{aligned}\left|(t-a)\,f(a)+(b-t)\,f(b)-\int_a^b f(s)\,ds\right| & \\ \leq\left[\frac{1}{2}+\left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right](b-a)\bigvee_a^b(f)&\end{aligned} \tag{1.2}$$

for any $t\in[a,b]$, provided that f is of bounded variation on $[a,b]$. The constant $\frac{1}{2}$ is the best possible.

If f is *absolutely continuous* on $[a,b]$, then (see [1, p. 93])

$$\left|(t-a)\,f(a)+(b-t)\,f(b)-\int_a^b f(s)\,ds\right| \tag{1.3}$$

$$\leq\begin{cases}\left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2\,\|f'\|_\infty & \text{if } f'\in L_\infty[a,b];\\[2ex] \frac{1}{(q+1)^{1/q}}\left[\left(\frac{t-a}{b-a}\right)^{q+1}+\left(\frac{b-t}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{1+1/q}\,\|f'\|_p & \text{if } f'\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \left[\frac{1}{2}+\left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right](b-a)\,\|f'\|_1 & \end{cases}$$

1991 *Mathematics Subject Classification.* 26D15, 26D10, 30A10, 30A86.

Key words and phrases. Complex integral, Continuous functions, Holomorphic functions, Trapezoid inequality.

RGMIA Res. Rep. Coll. 21 (2018), Art. 113, 16 pp. Received 27/09/18

for any $t \in [a,b]$. The constants $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{(q+1)^{1/q}}$ are the best possible.

Finally, for *convex functions* $f : [a,b] \to \mathbb{R}$, we have [4]

$$\frac{1}{2}\left[(b-t)^2 f'_+(t) - (t-a)^2 f'_-(t)\right] \leq (b-t) f(b) + (t-a) f(a) - \int_a^b f(s)\,ds \leq \frac{1}{2}\left[(b-t)^2 f'_-(b) - (t-a)^2 f'_-(a)\right] \tag{1.4}$$

for any $t \in (a,b)$, provided that $f'_-(b)$ and $f'_+(a)$ are finite. As above, the second inequality also holds for $t = a$ and $t = b$ and the constant $\frac{1}{2}$ is the best possible on both sides of (1.4).

For other recent results on the trapezoid inequality, see [3], [7], [8], [9] and [11].

In order to extend this result for the complex integral, we need some preparations as follows.

Suppose γ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are *piecewise smooth*. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.5}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right| \leq \int_\gamma |f(z)|\,|dz| \leq \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.6}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \geq 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p |dz| \right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)| |dz|.$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \leq [\ell(\gamma)]^{1/q} \|f\|_{\gamma,p}.$$

In this paper we extend the trapezoid inequality to the complex integral, by providing upper bounds for the quantity

$$\left| (v-u) f(u) + (w-v) f(w) - \int_\gamma f(z)\, dz \right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$, $u = z(a)$, $v = z(x)$ with $x \in (a,b)$ and $w = z(b)$ while f is holomorphic in G, an open domain and $\gamma \subset G$. An application for circular paths is also given.

2. Trapezoid Type Inequalities

We have the following result for functions of complex variable:

Theorem 1. *Let f be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(x)$ with $x \in (a,b)$, then $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$,*

$$\left| (v-u) f(u) + (w-v) f(w) - \int_\gamma f(z)\, dz \right| \leq \|f'\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-v| |dz| + \|f'\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-v| |dz| \leq \|f'\|_{\gamma_{u,w};\infty} \int_{\gamma_{u,w}} |z-v| |dz|, \tag{2.1}$$

and

$$\left| (v-u) f(u) + (w-v) f(w) - \int_\gamma f(z)\, dz \right| \leq \|f'\|_{\gamma_{u,v};1} \max_{z \in \gamma_{u,v}} |z-v| + \|f'\|_{\gamma_{v,w};1} \max_{z \in \gamma_{v,w}} |z-v| \leq \|f'\|_{\gamma_{u,w};1} \max_{z \in \gamma_{u,w}} |z-v|. \tag{2.2}$$

If $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1,$ *then*

$$\left|(v-u)\,f(u)+(w-v)\,f(w)-\int_{\gamma}f(z)\,dz\right| \tag{2.3}$$
$$\leq \|f'\|_{\gamma_{u,v};p}\left(\int_{\gamma_{u,v}}|z-v|^{q}\,|dz|\right)^{1/q}+\|f'\|_{\gamma_{v,w};p}\left(\int_{\gamma_{v,w}}|z-v|^{q}\,|dz|\right)^{1/q}$$
$$\leq \|f'\|_{\gamma_{u,w};p}\left(\int_{\gamma_{u,w}}|z-v|^{q}\,|dz|\right)^{1/q}.$$

Proof. Using the integration by parts formula (1.5) twice we have

$$\int_{\gamma_{u,v}}(z-v)\,f'(z)\,dz=(v-u)\,f(u)-\int_{\gamma_{u,v}}f(z)\,dz$$

and

$$\int_{\gamma_{v,w}}(z-v)\,f'(z)\,dz=(w-v)\,f(w)-\int_{\gamma_{v,w}}f(z)\,dz.$$

If we add these two equalities, we get the following equality of interest

$$(v-u)\,f(u)+(w-v)\,f(w)-\int_{\gamma}f(z)\,dz \tag{2.4}$$
$$=\int_{\gamma_{u,v}}(z-v)\,f'(z)\,dz+\int_{\gamma_{v,w}}(z-v)\,f'(z)\,dz=\int_{\gamma}(z-v)\,f'(z)\,dz$$

with the above assumptions for u, v and w on γ.

Using the properties of modulus and the triangle inequality for the complex integral we have

$$\left|(v-u)\,f(u)+(w-v)\,f(w)-\int_{\gamma}f(z)\,dz\right|$$
$$=\left|\int_{\gamma_{u,v}}(z-v)\,f'(z)\,dz+\int_{\gamma_{v,w}}(z-v)\,f'(z)\,dz\right|$$
$$\leq\left|\int_{\gamma_{u,v}}(z-v)\,f'(z)\,dz\right|+\left|\int_{\gamma_{v,w}}(z-v)\,f'(z)\,dz\right|$$
$$\leq\int_{\gamma_{u,v}}|z-v|\,|f'(z)|\,|dz|+\int_{\gamma_{v,w}}|z-v|\,|f'(z)|\,|dz|$$
$$\leq\|f'\|_{\gamma_{u,v};\infty}\int_{\gamma_{u,v}}|z-v|\,|dz|+\|f'\|_{\gamma_{v,w};\infty}\int_{\gamma_{v,w}}|z-v|\,|dz|\leq\|f'\|_{\gamma_{u,w};\infty}\int_{\gamma_{u,w}}|z-v|\,|dz|,$$

which proves the inequality (2.1).

We also have

$$\begin{aligned}
&\int_{\gamma_{u,v}} |z-v| \left|f'(z)\right| |dz| + \int_{\gamma_{v,w}} |z-v| \left|f'(z)\right| |dz| \\
&\leq \max_{z\in\gamma_{u,v}} |z-v| \int_{\gamma_{u,v}} \left|f'(z)\right| |dz| + \max_{z\in\gamma_{v,w}} |z-v| \int_{\gamma_{v,w}} \left|f'(z)\right| |dz| \\
&\leq \max\left\{ \max_{z\in\gamma_{u,v}} |z-v| \,, \max_{z\in\gamma_{v,w}} |z-v| \right\} \\
&\times \left(\int_{\gamma_{u,v}} \left|f'(z)\right| |dz| + \int_{\gamma_{v,w}} \left|f'(z)\right| |dz| \right) = \max_{z\in\gamma_{u,w}} |z-v| \int_{\gamma_{u,w}} \left|f'(z)\right| |dz| \,,
\end{aligned}$$

which proves the inequality (2.2).

If $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's weighted integral inequality we have

$$\begin{aligned}
&\int_{\gamma_{u,v}} |z-v| \left|f'(z)\right| |dz| + \int_{\gamma_{v,w}} |z-v| \left|f'(z)\right| |dz| \\
&\qquad \leq \left(\int_{\gamma_{u,v}} |z-v|^q |dz| \right)^{1/q} \left(\int_{\gamma_{u,v}} \left|f'(z)\right|^p |dz| \right)^{1/p} \\
&\qquad\quad + \left(\int_{\gamma_{v,w}} |z-v|^q |dz| \right)^{1/q} \left(\int_{\gamma_{v,w}} \left|f'(z)\right|^p |dz| \right)^{1/p} =: B.
\end{aligned}$$

By the elementary inequality

$$ab + cd \leq (a^p + c^p)^{1/p} (b^q + d^q)^{1/q} \,,$$

where $a,\, b,\, c,\, d \geq 0$ and $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we also have

$$\begin{aligned}
B &\leq \left(\int_{\gamma_{u,v}} |z-v|^q |dz| + \int_{\gamma_{v,w}} |z-v|^q |dz| \right)^{1/q} \\
&\times \left(\int_{\gamma_{u,v}} \left|f'(z)\right|^p |dz| + \int_{\gamma_{v,w}} \left|f'(z)\right|^p |dz| \right)^{1/p} \\
&= \left(\int_{\gamma_{u,w}} |z-v|^q |dz| \right)^{1/q} \left(\int_{\gamma_{u,w}} \left|f'(z)\right|^p |dz| \right)^{1/p} ,
\end{aligned}$$

which prove the desired result (2.3). □

If the path γ is a segment $[u, w] \subset G$ connecting two distinct points u and w in G then we write $\int_\gamma f(z)\, dz$ as $\int_u^w f(z)\, dz$.

Using the p-norms defined in the introduction for the segments, namely

$$\|h\|_{[u,w];\infty} = \sup_{z\in[u,w]} |h(z)|$$

and

$$\|h\|_{[u,w];p} = \left(\int_u^w |h(z)|^p |dz| \right)^{1/p} \quad \text{for } p \geq 1,$$

we can state the following particular case as well:

Corollary 1. *Let f be holomorphic in G, an open domain and suppose $[u,w]\subset G$ is a segment connecting two distinct points u and w in G and $v\in[u,w]$. Then for $v=(1-s)u+sw$ with $s\in[0,1]$, we have*

$$\begin{aligned}(2.5)\quad &\left|(v-u)f(u)+(w-v)f(w)-\int_u^w f(z)\,dz\right|\\ &\le\frac{1}{2}|w-u|^2\left[s^2\|f'\|_{\gamma_{u,v};\infty}+(1-s)^2\|f'\|_{\gamma_{v,w};\infty}\right]\\ &\le|w-u|^2\left[\frac{1}{4}+\left(s-\frac{1}{2}\right)^2\right]\|f'\|_{[u,w];\infty},\end{aligned}$$

and

$$\begin{aligned}(2.6)\quad &\left|(v-u)f(u)+(w-v)f(w)-\int_u^w f(z)\,dz\right|\\ &\le|w-u|\left\{s\|f'\|_{[u,v];1}+(1-s)\|f'\|_{[v,w];1}\right\}\\ &\le|w-u|\left(\frac{1}{2}+\left|s-\frac{1}{2}\right|\right)\|f'\|_{[u,w];1}.\end{aligned}$$

If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\begin{aligned}(2.7)\quad &\left|(v-u)f(u)+(w-v)f(w)-\int_\gamma f(z)\,dz\right|\\ &\le\frac{1}{(q+1)^{1/q}}|w-u|^{1+1/q}\left[s^{1+1/q}\|f'\|_{[u,v];p}+(1-s)^{1+1/q}\|f'\|_{[v,w];p}\right]\\ &\le\frac{1}{(q+1)^{1/q}}|w-u|^{1+1/q}\left[s^{q+1}+(1-s)^{q+1}\right]^{1/q}\|f'\|_{[u,w];p}.\end{aligned}$$

Proof. Observe that if the segment $[u,w]$ is parametrized by $z(t)=(1-t)u+tw$, then $z'(t)=w-u$

$$\begin{aligned}\int_u^v|z-v||dz|&=|w-u|\int_0^s|(1-t)u+tw-(1-s)u-sw|\,dt\\ &=|w-u|^2\int_0^s(s-t)\,dt=\frac{1}{2}|w-u|^2s^2\end{aligned}$$

and

$$\begin{aligned}\int_v^w|z-v||dz|&=|w-u|\int_s^1|(1-t)u+tw-(1-s)u-sw|\,dt\\ &=|w-u|^2\int_s^1(t-s)\,dt=\frac{1}{2}|w-u|^2(1-s)^2.\end{aligned}$$

Using the inequality (2.1) we get

$$\left|(v-u)f(u)+(w-v)f(w)-\int_\gamma f(z)\,dz\right|$$
$$\leq \frac{1}{2}|w-u|^2 s^2 \|f'\|_{\gamma_{u,v};\infty}+\frac{1}{2}|w-u|^2(1-s)^2\|f'\|_{\gamma_{v,w};\infty}$$
$$\leq \frac{1}{2}|w-u|^2\left[s^2+(1-s)^2\right]\|f'\|_{\gamma_{u,w};\infty}=|w-u|^2\left[\frac{1}{4}+\left(s-\frac{1}{2}\right)^2\right]\|f'\|_{[u,w];\infty},$$

which proves (2.5).

Also

$$\max_{z\in\gamma_{u,v}}|z-v|=\max_{t\in[0,s]}|(1-t)u+tw-(1-s)u-sw|=|w-u|\,s$$

and

$$\max_{z\in\gamma_{v,w}}|z-v|=\max_{t\in[s,1]}\{|w-u|(1-t)\}=|w-u|(1-s),$$

then by (2.2)

$$\left|(v-u)f(u)+(w-v)f(w)-\int_\gamma f(z)\,dz\right|$$
$$\leq |w-u|\left\{s\|f'\|_{[u,v];1}+(1-s)\|f'\|_{[v,w];1}\right\}$$
$$\leq |w-u|\max\{s,1-s\}\|f'\|_{[u,w];1}=|w-u|\left(\frac{1}{2}+\left|s-\frac{1}{2}\right|\right)\|f'\|_{[u,w];1},$$

which proves (2.6).

Finally, since

$$\int_u^v|z-v|^q|dz|=|w-u|\int_0^s|(1-t)u+tw-(1-s)u-sw|^q\,dt$$
$$=|w-u|^{q+1}\int_0^s(s-t)^q\,dt=\frac{1}{q+1}s^{q+1}|w-u|^{q+1}$$

and

$$\int_v^w|z-v|^q|dz|=|w-u|\int_s^1|(1-t)u+tw-(1-s)u-sw|^q\,dt$$
$$=|w-u|^{q+1}\int_s^1(t-s)^q\,dt=\frac{1}{q+1}(1-s)^{q+1}|w-u|^{q+1},$$

hence by (2.3) we get (2.7). □

Remark 1. *Let f be holomorphic in G, an open domain and suppose $[u,w]\subset G$ is a segment connecting two distinct points u and w in G. Then*

$$\left|\frac{f(u)+f(w)}{2}(w-u)-\int_u^w f(z)\,dz\right| \tag{2.8}$$
$$\leq \frac{1}{8}|w-u|^2\left[\|f'\|_{\gamma_{u,\frac{u+w}{2}};\infty}+\|f'\|_{\gamma_{\frac{u+w}{2},w};\infty}\right]\leq\frac{1}{4}|w-u|^2\|f'\|_{[u,w];\infty},$$

and

$$\left|\frac{f(u)+f(w)}{2}(w-u)-\int_u^w f(z)\,dz\right|\leq\frac{1}{2}|w-u|\,\|f'\|_{[u,w];1}. \tag{2.9}$$

If $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1,$ *then*

$$\begin{aligned}(2.10)\quad &\left|\frac{f(u)+f(w)}{2}(w-u)-\int_u^w f(z)\,dz\right| \\ &\le \frac{1}{2^{1+1/q}(q+1)^{1/q}}|w-u|^{1+1/q}\left[\|f'\|_{\left[u,\frac{u+w}{2}\right];p}+\|f'\|_{\left[\frac{u+w}{2},w\right];p}\right] \\ &\le \frac{1}{2(q+1)^{1/q}}|w-u|^{1+1/q}\|f'\|_{[u,w];p}.\end{aligned}$$

Suppose that $\gamma\subset G$ is a smooth path from $z(a)=u$ to $z(b)=w$. If $v=z(x)$ with $x\in(a,b)$, then $\gamma_{u,w}=\gamma_{u,v}\cup\gamma_{v,w}$.

If we consider $f(z)=\exp(z)$ with $z\in\mathbb{C}$, then

$$\int_{\gamma_{u,w}}\exp(z)\,dz=\exp(w)-\exp(u),$$

$$|\exp(z)|=|\exp(\operatorname{Re}(z)+i\operatorname{Im}(z))|=\exp(\operatorname{Re}(z))$$

and by Theorem 1 we have

$$\begin{aligned}(2.11)\quad &|(v-u)\exp u+(w-v)\exp w-\exp(w)+\exp(u)| \\ &\le \|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{u,v};\infty}\int_{\gamma_{u,v}}|z-v|\,|dz| \\ &+\|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{v,w};\infty}\int_{\gamma_{v,w}}|z-v|\,|dz| \\ &\le \|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{u,w};\infty}\int_{\gamma_{u,w}}|z-v|\,|dz|,\end{aligned}$$

and

$$\begin{aligned}(2.12)\quad &|(v-u)\exp u+(w-v)\exp w-\exp(w)+\exp(u)| \\ &\le \|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{u,v};1}\max_{z\in\gamma_{u,v}}|z-v|+\|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{v,w};1}\max_{z\in\gamma_{v,w}}|z-v| \\ &\le \|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{u,w};1}\max_{z\in\gamma_{u,w}}|z-v|.\end{aligned}$$

If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1,$ then

$$\begin{aligned}(2.13)\quad &|(v-u)\exp u+(w-v)\exp w-\exp(w)+\exp(u)| \\ &\le \|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{u,v};p}\left(\int_{\gamma_{u,v}}|z-v|^q\,|dz|\right)^{1/q} \\ &+\|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{v,w};p}\left(\int_{\gamma_{v,w}}|z-v|^q\,|dz|\right)^{1/q} \\ &\le \|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{u,w};p}\left(\int_{\gamma_{u,w}}|z-v|^q\,|dz|\right)^{1/q}.\end{aligned}$$

With the same assumption of the path γ and if we consider $f(z) = z^n$ with $n \geq 1$, then

$$\int_\gamma z^n dz = \frac{w^{n+1} - u^{n+1}}{n+1}$$

and by Theorem 1 we get, by denoting $\ell(z) = z$, $z \in \mathbb{C}$, that

$$\begin{aligned}(2.14)\quad & \left|(v-u)u^n + (w-v)w^n - \frac{w^{n+1} - u^{n+1}}{n+1}\right| \\ & \leq n\left[\left\|\ell^{n-1}\right\|_{\gamma_{u,v};\infty}\int_{\gamma_{u,v}}|z-v|\,|dz| + \left\|\ell^{n-1}\right\|_{\gamma_{v,w};\infty}\int_{\gamma_{v,w}}|z-v|\,|dz|\right] \\ & \leq n\left\|\ell^{n-1}\right\|_{\gamma_{u,w};\infty}\int_{\gamma_{u,w}}|z-v|\,|dz|,\end{aligned}$$

and

$$\begin{aligned}(2.15)\quad & \left|(v-u)u^n + (w-v)w^n - \frac{w^{n+1} - u^{n+1}}{n+1}\right| \\ & \leq n\left[\left\|\ell^{n-1}\right\|_{\gamma_{u,v};1}\max_{z\in\gamma_{u,v}}|z-v| + \left\|\ell^{n-1}\right\|_{\gamma_{v,w};1}\max_{z\in\gamma_{v,w}}|z-v|\right] \\ & \leq n\left\|\ell^{n-1}\right\|_{\gamma_{u,w};1}\max_{z\in\gamma_{u,w}}|z-v|.\end{aligned}$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned}(2.16)\quad & \left|(v-u)u^n + (w-v)w^n - \frac{w^{n+1} - u^{n+1}}{n+1}\right| \\ & \leq n\left[\left\|\ell^{n-1}\right\|_{\gamma_{u,v};p}\left(\int_{\gamma_{u,v}}|z-v|^q\,|dz|\right)^{1/q} + \left\|\ell^{n-1}\right\|_{\gamma_{v,w};p}\left(\int_{\gamma_{v,w}}|z-v|^q\,|dz|\right)^{1/q}\right] \\ & \leq n\left\|\ell^{n-1}\right\|_{\gamma_{u,w};p}\left(\int_{\gamma_{u,w}}|z-v|^q\,|dz|\right)^{1/q},\end{aligned}$$

where $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$ and $v = z(x)$ with $x \in (a,b)$.

3. Examples for Circular Paths

Let $[a,b] \subseteq [0, 2\pi]$ and the circular path $\gamma_{[a,b],R}$ centered in 0 and with radius $R > 0$

$$z(t) = R\exp(it) = R(\cos t + i\sin t),\ t \in [a,b].$$

If $[a,b] = [0,\pi]$ then we get a half circle while for $[a,b] = [0,2\pi]$ we get the full circle.

Since

$$\begin{aligned}\left|e^{is} - e^{it}\right|^2 &= \left|e^{is}\right|^2 - 2\operatorname{Re}\left(e^{i(s-t)}\right) + \left|e^{it}\right|^2 \\ &= 2 - 2\cos(s-t) = 4\sin^2\left(\frac{s-t}{2}\right)\end{aligned}$$

for any t, $s \in \mathbb{R}$, then

$$\left|e^{is}-e^{it}\right|^{r}=2^{r}\left|\sin\left(\frac{s-t}{2}\right)\right|^{r} \tag{3.1}$$

for any t, $s \in \mathbb{R}$ and $r > 0$. In particular,

$$\left|e^{is}-e^{it}\right|=2\left|\sin\left(\frac{s-t}{2}\right)\right|$$

for any t, $s \in \mathbb{R}$.

For t, $x \in [a,b] \subseteq [0,2\pi]$ we then have

$$\left|e^{ix}-e^{it}\right|=2\left|\sin\left(\frac{x-t}{2}\right)\right|.$$

If $u = R\exp(ia)$, $v = R\exp(ix)$ and $w = R\exp(ib)$ then

$$\begin{aligned} v-u &= R\left[\exp(ix)-\exp(ia)\right]=R\left[\cos x+i\sin x-\cos a-i\sin a\right] \\ &= R\left[\cos x-\cos a+i\left(\sin x-\sin a\right)\right]. \end{aligned}$$

Since

$$\cos x-\cos a=-2\sin\left(\frac{a+x}{2}\right)\sin\left(\frac{x-a}{2}\right)$$

and

$$\sin x-\sin a=2\sin\left(\frac{x-a}{2}\right)\cos\left(\frac{a+x}{2}\right),$$

hence

$$\begin{aligned} v-u &= R\left[-2\sin\left(\frac{a+x}{2}\right)\sin\left(\frac{x-a}{2}\right)+2i\sin\left(\frac{x-a}{2}\right)\cos\left(\frac{a+x}{2}\right)\right] \\ &= 2R\sin\left(\frac{x-a}{2}\right)\left[-\sin\left(\frac{a+x}{2}\right)+i\cos\left(\frac{a+x}{2}\right)\right] \\ &= 2Ri\sin\left(\frac{x-a}{2}\right)\left[\cos\left(\frac{a+x}{2}\right)+i\sin\left(\frac{a+x}{2}\right)\right] \\ &= 2Ri\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]. \end{aligned}$$

Similarly,

$$w-v=2Ri\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]$$

for $a \le x \le b$.

Moreover,

$$z-v=2Ri\sin\left(\frac{t-x}{2}\right)\exp\left[\left(\frac{t+b}{2}\right)i\right]$$

and

$$|z-v|=\left|2Ri\sin\left(\frac{t-x}{2}\right)\exp\left[\left(\frac{t+b}{2}\right)i\right]\right|=2R\left|\sin\left(\frac{t-x}{2}\right)\right|$$

for $a \le x$, $t \le b$.

We also have

$$z'(t)=Ri\exp(it) \text{ and } |z'(t)|=R$$

for $t \in [a,b]$.

Proposition 1. *Let f be holomorphic in G, on open domain and suppose $\gamma_{[a,b],R}\subset G$ with $[a,b]\subseteq[0,2\pi]$ and $R>0$. If $x\in[a,b]$, then*

$$\begin{aligned}(3.2)\quad &\left|\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]f\left(R\exp\left(ia\right)\right)\right.\\ &+\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]f\left(R\exp\left(ib\right)\right)\\ &\left.-\frac{1}{2}\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right|\\ &\leq 4R\left[\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,x],\infty}\sin^2\left(\frac{x-a}{4}\right)\right.\\ &\left.+\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,b],\infty}\sin^2\left(\frac{b-x}{4}\right)\right]\\ &\leq 4R\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,b],\infty}\left[\sin^2\left(\frac{x-a}{4}\right)+\sin^2\left(\frac{b-x}{4}\right)\right].\end{aligned}$$

Proof. We write the inequality (2.1) for $\gamma_{[a,b],R}$ and $x\in[a,b]$ to get

$$\begin{aligned}&\left|2Ri\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]f\left(R\exp\left(ia\right)\right)\right.\\ &+2Ri\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]f\left(R\exp\left(ib\right)\right)\\ &\left.-Ri\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right|\\ &\leq 2R^2\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,x],\infty}\int_a^b\left|\sin\left(\frac{t-x}{2}\right)\right|dt\\ &+2R^2\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,b],\infty}\int_x^x\left|\sin\left(\frac{t-x}{2}\right)\right|dt.\end{aligned}$$

This is equivalent to

$$\begin{aligned}(3.3)\quad &\left|\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]f\left(R\exp\left(ia\right)\right)\right.\\ &+\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]f\left(R\exp\left(ib\right)\right)\\ &\left.-\frac{1}{2}\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right|\\ &\leq R\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,x],\infty}\int_a^x\left|\sin\left(\frac{t-x}{2}\right)\right|dt\\ &+R\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,b],\infty}\int_x^b\left|\sin\left(\frac{t-x}{2}\right)\right|dt\end{aligned}$$

for $x\in[a,b]$.

Observe that

$$\int_a^x \left|\sin\left(\frac{t-x}{2}\right)\right| dt = \int_a^x \sin\left(\frac{x-t}{2}\right) dt = 2 - 2\cos\left(\frac{x-a}{2}\right)$$
$$= 4\sin^2\left(\frac{x-a}{4}\right)$$

and

$$\int_x^b \left|\sin\left(\frac{t-x}{2}\right)\right| dt = \int_x^b \sin\left(\frac{t-x}{2}\right) dt = 2 - 2\cos\left(\frac{b-t}{2}\right)$$
$$= 4\sin^2\left(\frac{b-x}{4}\right),$$

which by (3.3) produce the desired result (3.2). □

Corollary 2. *With the assumptions of Proposition 1 we have*

$$\begin{aligned}(3.4)\quad &\left|\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{3a+b}{4}\right)i\right] f(R\exp(ia))\right.\\ &+\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{a+3b}{4}\right)i\right] f(R\exp(ib))\\ &\left.-\frac{1}{2}\int_a^b f(R\exp(it))\exp(it)\,dt\right|\\ &\leq 4R\left[\|f'(R\exp(i\cdot))\|_{[a,x],\infty} + \|f'(R\exp(i\cdot))\|_{[x,b],\infty}\right]\sin^2\left(\frac{b-a}{8}\right)\\ &\leq 8R\|f'(R\exp(i\cdot))\|_{[a,b],\infty}\sin^2\left(\frac{b-a}{8}\right).\end{aligned}$$

Remark 2. *The case of semi-circle, namely $a = 0$ and $b = \pi$ in (3.2) gives the inequality*

$$\begin{aligned}(3.5)\quad &\left|\sin\left(\frac{x}{2}\right)\exp\left[\left(\frac{x}{2}\right)i\right] f(R) + i\cos\left(\frac{x}{2}\right)\exp\left[\left(\frac{x}{2}\right)i\right] f(-R)\right.\\ &\left.-\frac{1}{2}\int_0^\pi f(R\exp(it))\exp(it)\,dt\right|\\ &\leq 4R\left[\|f'(R\exp(i\cdot))\|_{[0,x],\infty}\sin^2\left(\frac{x}{4}\right)\right.\\ &\left.+\|f'(R\exp(i\cdot))\|_{[x,\pi],\infty}\sin^2\left(\frac{\pi-x}{4}\right)\right]\\ &\leq 4R\|f'(R\exp(i\cdot))\|_{[0,\pi],\infty}\left[\sin^2\left(\frac{x}{4}\right)+\sin^2\left(\frac{\pi-x}{4}\right)\right],\end{aligned}$$

for $x \in [0, \pi]$.

Since

$$\sin^2\left(\frac{\pi}{8}\right) = \frac{1-\cos\left(\frac{\pi}{4}\right)}{2} = \frac{1-\frac{\sqrt{2}}{2}}{2} = \frac{2-\sqrt{2}}{4},$$

then by taking $x = \frac{\pi}{2}$ in (3.5), we get

$$\begin{aligned}(3.6)\quad & \left|\frac{1+i}{2} f(R) + \frac{-1+i}{2} f(-R) - \frac{1}{2}\int_0^{\pi} f(R\exp(it))\exp(it)\,dt\right| \\ & \leq \left(2-\sqrt{2}\right)\left[\left\|f'(R\exp(i\cdot))\right\|_{\left[0,\frac{\pi}{2}\right],\infty} + \left\|f'(R\exp(i\cdot))\right\|_{\left[\frac{\pi}{2},\pi\right],\infty}\right] \\ & \leq 2\left(2-\sqrt{2}\right)\left\|f'(R\exp(i\cdot))\right\|_{[0,\pi],\infty}.\end{aligned}$$

Further, we have the following result as well:

Proposition 2. *With the assumptions of Proposition 1 we have*

$$\begin{aligned}(3.7)\quad & \left|\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right] f(R\exp(ia)) \right. \\ & + \sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right] f(R\exp(ib)) \\ & \left. -\frac{1}{2}\int_a^b f(R\exp(it))\exp(it)\,dt\right| \\ & \leq R\left[\max_{t\in[a,x]}\left|\sin\left(\frac{t-x}{2}\right)\right|\int_a^x \left|f'(R\exp(it))\right| dt \right. \\ & \left. + \max_{t\in[x,b]}\left|\sin\left(\frac{t-x}{2}\right)\right|\int_x^b \left|f'(R\exp(it))\right| dt\right] \\ & \leq R\max_{t\in[a,b]}\left|\sin\left(\frac{t-x}{2}\right)\right|\int_a^b \left|f'(R\exp(it))\right| dt.\end{aligned}$$

Proof. We write the inequality (2.2) for $\gamma_{[a,b],R}$ and $x \in [a,b]$ to get

$$\begin{aligned} & \left|2Ri\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right] f(R\exp(ia)) \right. \\ & + 2Ri\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right] f(R\exp(ib)) \\ & \left. -Ri\int_a^b f(R\exp(it))\exp(it)\,dt\right| \\ & \leq 2R^2\left[\max_{t\in[a,x]}\left|\sin\left(\frac{t-x}{2}\right)\right|\int_a^x \left|f'(R\exp(it))\right| dt \right. \\ & \left. + \max_{t\in[x,b]}\left|\sin\left(\frac{t-x}{2}\right)\right|\int_x^b \left|f'(R\exp(it))\right| dt\right] \\ & \leq 2R^2\max_{t\in[a,b]}\left|\sin\left(\frac{t-x}{2}\right)\right|\int_a^b \left|f'(R\exp(it))\right| dt,\end{aligned}$$

which is equivalent to (3.7). □

In particular, we have:

Corollary 3. *With the assumptions of Proposition 1 we have*

$$\begin{aligned}(3.8)\quad &\left|\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{3a+b}{4}\right)i\right]f\left(R\exp\left(ia\right)\right)\right.\\ &+\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{a+3b}{4}\right)i\right]f\left(R\exp\left(ib\right)\right)\\ &\left.-\frac{1}{2}\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right|\leq R\sin\left(\frac{b-a}{4}\right)\int_a^b\left|f'\left(R\exp\left(it\right)\right)\right|dt.\end{aligned}$$

Proof. If we take in (3.7) $x=\frac{a+b}{2}$, then we get

$$\begin{aligned}(3.9)\quad &\left|\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{3a+b}{4}\right)i\right]f\left(R\exp\left(ia\right)\right)\right.\\ &\left.+\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{a+3b}{4}\right)i\right]f\left(R\exp\left(ib\right)\right)-\frac{1}{2}\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right|\\ &\leq R\left[\max_{t\in\left[a,\frac{a+b}{2}\right]}\left|\sin\left(\frac{t-\frac{a+b}{2}}{2}\right)\right|\int_a^{\frac{a+b}{2}}\left|f'\left(R\exp\left(it\right)\right)\right|dt\right.\\ &\left.+\max_{t\in\left[\frac{a+b}{2},b\right]}\left|\sin\left(\frac{t-\frac{a+b}{2}}{2}\right)\right|\int_{\frac{a+b}{2}}^b\left|f'\left(R\exp\left(it\right)\right)\right|dt\right]\\ &\leq R\max_{t\in[a,b]}\left|\sin\left(\frac{t-\frac{a+b}{2}}{2}\right)\right|\int_a^b\left|f'\left(R\exp\left(it\right)\right)\right|dt.\end{aligned}$$

Since the intervals $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$ have a length less than π, then

$$\max_{t\in\left[a,\frac{a+b}{2}\right]}\left|\sin\left(\frac{t-\frac{a+b}{2}}{2}\right)\right|=\max_{t\in\left[\frac{a+b}{2},b\right]}\left|\sin\left(\frac{t-\frac{a+b}{2}}{2}\right)\right|=\sin\left(\frac{b-a}{4}\right)$$

and by (3.9) we get (3.8). □

The case of p-norms is as follows:

Proposition 3. *With the assumptions of Proposition 1 and* p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *we have*

$$\begin{aligned}(3.10)\quad &\left|\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]f\left(R\exp\left(ia\right)\right)\right.\\ &+\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]f\left(R\exp\left(ib\right)\right)\\ &\left.-\frac{1}{2}\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right|\end{aligned}$$

$$\leq R\left(\int_a^x \sin^q\left(\frac{x-t}{2}\right)dt\right)^{1/q}\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,x],p}$$
$$+R\left(\int_x^b \sin^q\left(\frac{t-x}{2}\right)dt\right)^{1/q}\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,b],p}$$
$$\leq R\left[\int_a^x \sin^q\left(\frac{x-t}{2}\right)dt+\int_x^b \sin^q\left(\frac{t-x}{2}\right)dt\right]^{1/q}\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,b],p}.$$

In particular, for $x=\frac{a+b}{2}$ we get

$$\left|\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{3a+b}{4}\right)i\right]f\left(R\exp\left(ia\right)\right)\right.$$
$$+\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{a+3b}{4}\right)i\right]f\left(R\exp\left(ib\right)\right)$$
$$\left.-\frac{1}{2}\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right| \tag{3.11}$$
$$\leq R\left(\int_a^{\frac{a+b}{2}} \sin^q\left(\frac{\frac{a+b}{2}-t}{2}\right)dt\right)^{1/q}\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{\left[a,\frac{a+b}{2}\right],p}$$
$$+R\left(\int_{\frac{a+b}{2}}^b \sin^q\left(\frac{t-\frac{a+b}{2}}{2}\right)dt\right)^{1/q}\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{\left[\frac{a+b}{2},b\right],p}$$
$$\leq R\left[\int_a^b \sin^q\left(\left|\frac{t-\frac{a+b}{2}}{2}\right|\right)dt\right]^{1/q}\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,b],p}.$$

Proof. By making use of the inequality (2.3) for $\gamma_{[a,b],R}$ and $x\in[a,b]$ we get

$$\left|2Ri\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]f\left(R\exp\left(ia\right)\right)\right.$$
$$\left.+2Ri\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]f\left(R\exp\left(ib\right)\right)-Ri\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right|$$
$$\leq 2R^2\left(\int_a^x \sin^q\left(\frac{x-t}{2}\right)dt\right)^{1/q}\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,x],p}$$
$$+2R^2\left(\int_x^b \sin^q\left(\frac{t-x}{2}\right)dt\right)^{1/q}\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,b],p}$$
$$\leq 2R^2\left[\int_a^x \sin^q\left(\frac{x-t}{2}\right)dt+\int_x^b \sin^q\left(\frac{t-x}{2}\right)dt\right]^{1/q}\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,b],p},$$

which proves the desired result (3.10). □

The interested reader may consider for examples some fundamental complex functions such as $f(z)=z^n$ with n a natural number, $f(z)=\exp(z)$ or f a trigonometric or a hyperbolic complex function. The details are omitted.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.org/dragomir`

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa