OSTROWSKI'S TYPE INEQUALITIES FOR THE COMPLEX INTEGRAL ON PATHS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we extend the Ostrowski inequality to the integral with respect to arc-length by providing upper bounds for the quantity

$$\left| f(v)\,\ell(\gamma) - \int_\gamma f(z)\,|dz| \right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with the length $\ell(\gamma)$, $u = z(a)$, $v = z(x)$ with $x \in (a,b)$ and $w = z(b)$ while f is holomorphic in G, an open domain and $\gamma \subset G$. An application for circular paths is also given.

1. Introduction

In 1938, A. Ostrowski [8], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x \in [a,b]$.

Theorem 1 (Ostrowski, 1938 [8])**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a), \tag{1.1}$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

In [6], S. S. Dragomir and S. Wang, by the use of the *Montgomery integral identity* [7, p. 565],

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b p(x,t)\,f'(t)\,dt, \quad x \in [a,b], \tag{1.2}$$

where $p : [a,b]^2 \to \mathbb{R}$ is given by

$$p(x,t) := \begin{cases} t-a & \text{if} \quad t \in [a,x], \\ t-b & \text{if} \quad t \in (x,b], \end{cases}$$

gave a simple proof of Ostrowski's inequality and applied it for special means (identric mean, logarithmic mean, etc.) and to the problem of estimating the error bound in approximating the Riemann integral $\int_a^b f(t)\,dt$ by one arbitrary Riemann sum (see [6], Section 3).

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For extensions of Ostrowski's inequality in terms of the p-norms of the derivative, see [1], [2] and [3]. For a recent survey on Ostrowski's inequality, see [4].

In order to extend this result for the complex integral, we need some preparations as follows.

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, and open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.3}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.4}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\,|dz|\right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|.$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}.$$

In the recent paper [5] we obtained the following result for functions of complex variable:

Theorem 2. *Let f be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(x)$ with $x \in (a,b)$, then* $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$,

$$\left|f(v)(w-u) - \int_\gamma f(z)\,dz\right| \tag{1.5}$$
$$\leq \|f'\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-u|\,|dz| + \|f'\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-w|\,|dz|$$
$$\leq \left[\int_{\gamma_{u,v}} |z-u|\,|dz| + \int_{\gamma_{v,w}} |z-w|\,|dz|\right] \|f'\|_{\gamma_{u,w};\infty}$$

and

$$\left|f(v)(w-u) - \int_\gamma f(z)\,dz\right| \tag{1.6}$$
$$\leq \max_{z\in\gamma_{u,v}} |z-u|\, \|f'\|_{\gamma_{u,v};1} + \max_{z\in\gamma_{v,w}} |z-w|\, \|f'\|_{\gamma_{v,w};1}$$
$$\leq \max\left\{\max_{z\in\gamma_{u,v}} |z-u|, \max_{z\in\gamma_{v,w}} |z-w|\right\} \|f'\|_{\gamma_{u,w};1}.$$

If $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left|f(v)(w-u) - \int_\gamma f(z)\,dz\right| \tag{1.7}$$
$$\leq \left(\int_{\gamma_{u,v}} |z-u|^q\,|dz|\right)^{1/q} \|f'\|_{\gamma_{u,v};p} + \left(\int_{\gamma_{v,w}} |z-w|^q\,|dz|\right)^{1/q} \|f'\|_{\gamma_{v,w};p}$$
$$\leq \left(\int_{\gamma_{u,v}} |z-u|^q\,|dz| + \int_{\gamma_{v,w}} |z-w|^q\,|dz|\right)^{1/q} \|f'\|_{\gamma_{u,w};p}.$$

Motivated by the above results, in this paper we extend the Ostrowski inequality to the complex integral, by providing upper bounds for the quantity

$$\left|f(v)\,\ell(\gamma) - \int_\gamma f(z)\,|dz|\right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$, with the length $\ell(\gamma)$, $u = z(a)$, $v = z(x)$ with $x \in (a,b)$ and $w = z(b)$ while f is holomorphic in G, an open domain and $\gamma \subset G$. An application for circular paths is also given.

2. Ostrowski Type Results

We have the following result for functions of complex variable:

Theorem 3. *Let f be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(x)$ with $x \in (a,b)$, then*

$\gamma_{u,w}=\gamma_{u,v}\cup\gamma_{v,w}$ *and*

$$\left|f(v)\,\ell(\gamma)-\int_{\gamma}f(z)\,|dz|\right|\le\ell(\gamma_{u,v})\,\|f'\|_{\gamma_{u,v};1}+\ell(\gamma_{v,w})\,\|f'\|_{\gamma_{v,w};1} \tag{2.1}$$

$$\le\begin{cases}\frac{1}{2}\left[\ell(\gamma_{u,w})+\left|\ell(\gamma_{u,v})-\ell(\gamma_{v,w})\right|\right]\|f'\|_{\gamma_{u,w};1}\,,\\[2ex] \left[\ell^{p}(\gamma_{u,v})+\ell^{p}(\gamma_{v,w})\right]^{1/p}\left(\|f'\|_{\gamma_{u,v};1}^{q}+\|f'\|_{\gamma_{v,w};1}^{q}\right)^{1/q}\\ p,\ q>1\ \text{and}\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \frac{1}{2}\ell(\gamma_{u,w})\left[\|f'\|_{\gamma_{u,w};1}+\left|\|f'\|_{\gamma_{u,v};1}-\|f'\|_{\gamma_{v,w};1}\right|\right].\end{cases}$$

Proof. Using the integration by parts formula we have

$$\begin{aligned}\int_{\gamma_{u,v}}f(z)\,|dz|&=\int_a^x f(z(t))\,|z'(t)|\,dt=\int_a^x f(z(t))\,d\left(\int_a^t|z'(s)|\,ds\right)\\&=f(z(t))\int_a^t|z'(s)|\,ds\Big|_a^x-\int_a^x\frac{df(z(t))}{dt}\left(\int_a^t|z'(s)|\,ds\right)dt\\&=f(z(x))\int_a^x|z'(s)|\,ds-\int_a^x f'(z(t))\left(\int_a^t|z'(s)|\,ds\right)z'(t)\,dt\\&=f(v)\,\ell(\gamma_{u,v})-\int_a^x f'(z(t))\left(\int_a^t|z'(s)|\,ds\right)z'(t)\,dt\end{aligned}$$

and

$$\begin{aligned}\int_{\gamma_{v,w}}f(z)\,|dz|&=\int_x^b f(z(t))\,|z'(t)|\,dt=-\int_x^b f(z(t))\,d\left(\int_t^b|z'(s)|\,ds\right)\\&=-\,f(z(t))\int_t^b|z'(s)|\,ds\Big|_x^b+\int_x^b\frac{df(z(t))}{dt}\left(\int_t^b|z'(s)|\,ds\right)dt\\&=f(z(x))\int_x^b|z'(s)|\,ds+\int_x^b f'(z(t))\left(\int_t^b|z'(s)|\,ds\right)z'(t)\,dt\\&=f(v)\,\ell(\gamma_{v,w})+\int_x^b f'(z(t))\left(\int_t^b|z'(s)|\,ds\right)z'(t)\,dt.\end{aligned}$$

If we add these two equalities we get

$$\begin{aligned}&\int_{\gamma_{u,v}}f(z)\,|dz|+\int_{\gamma_{v,w}}f(z)\,|dz|=f(v)\,\ell(\gamma_{u,v})+f(v)\,\ell(\gamma_{v,w})\\&-\int_a^x f'(z(t))\left(\int_a^t|z'(s)|\,ds\right)z'(t)\,dt+\int_x^b f'(z(t))\left(\int_t^b|z'(s)|\,ds\right)z'(t)\,dt,\end{aligned}$$

which gives the following equality of interest

$$f(v)\,\ell(\gamma_{u,w}) - \int_{\gamma_{u,w}} f(z)\,|dz| \tag{2.2}$$
$$= \int_a^x f'(z(t)) \left(\int_a^t |z'(s)|\,ds \right) z'(t)\,dt - \int_x^b f'(z(t)) \left(\int_t^b |z'(s)|\,ds \right) z'(t)\,dt.$$

By taking the modulus in (2.2) and using the properties of modulus, we get

$$\left| f(v)\,\ell(\gamma_{u,w}) - \int_{\gamma_{u,w}} f(z)\,|dz| \right| \tag{2.3}$$
$$\leq \left| \int_a^x f'(z(t)) \left(\int_a^t |z'(s)|\,ds \right) z'(t)\,dt \right|$$
$$+ \left| \int_x^b f'(z(t)) \left(\int_t^b |z'(s)|\,ds \right) z'(t)\,dt \right|$$
$$\leq \int_a^x |f'(z(t))|\,|z'(t)| \left(\int_a^t |z'(s)|\,ds \right) dt$$
$$+ \int_x^b |f'(z(t))|\,|z'(t)| \left(\int_t^b |z'(s)|\,ds \right) dt =: B(x)$$

for $x \in [a,b]$.

We have

$$\int_a^t |z'(s)|\,ds \leq \int_a^x |z'(s)|\,ds \text{ for } t \in [a,x]$$

and

$$\int_t^b |z'(s)|\,ds \leq \int_x^b |z'(s)|\,ds \text{ for } t \in [x,b],$$

then

$$B(x) \leq \int_a^x |z'(s)|\,ds \int_a^x |f'(z(t))|\,|z'(t)|\,dt$$
$$+ \int_x^b |z'(s)|\,ds \int_x^b |f'(z(t))|\,|z'(t)|\,dt$$
$$= \ell(\gamma_{u,v}) \int_{\gamma_{u,v}} |f'(z)|\,|dz| + \ell(\gamma_{v,w}) \int_{\gamma_{v,w}} |f'(z)|\,|dz|$$

and by (2.3) we get the first inequality in (2.1).

The second part follows by Hölder's inequalities

$$mn + cd \leq \begin{cases} \max\{m,c\}(n+d) \\ (m^p + c^p)^{1/p} (n^q + d^q)^{1/q}, \text{ for } p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

where $m,\ n,\ c,\ d \geq 0$. □

Corollary 1. *With the assumption of Theorem 3 and if there exists $m \in \gamma$ such that $\ell(\gamma_{u,m}) = \ell(\gamma_{m,w})$, then*

$$\left| f(m)\,\ell(\gamma) - \int_\gamma f(z)\,|dz| \right| \le \frac{1}{2}\ell(\gamma)\,\|f'\|_{\gamma_{u,w};1} \tag{2.4}$$

and if $s \in \gamma$ such that $\int_{\gamma_{u,s}} |f'(z)|\,|dz| = \int_{\gamma_{s,w}} |f'(z)|\,|dz|$, then

$$\left| f(s)\,\ell(\gamma) - \int_\gamma f(z)\,|dz| \right| \le \frac{1}{2}\ell(\gamma)\,\|f'\|_{\gamma_{u,w};1}\,. \tag{2.5}$$

We have also the following result for p-norms:

Theorem 4. *With the assumption of Theorem 3 we have for $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that*

$$\left| f(v)\,\ell(\gamma) - \int_\gamma f(z)\,|dz| \right| \tag{2.6}$$

$$\le \frac{1}{q+1}\left[\ell^{1+1/q}(\gamma_{u,v})\,\|f'\|_{\gamma_{u,v};p} + \ell^{1+1/q}(\gamma_{v,w})\,\|f'\|_{\gamma_{v,w};p} \right]$$

$$\le \frac{1}{q+1}\begin{cases} \frac{1}{2^{1+1/q}}\left[\ell(\gamma_{u,w}) + \left|\ell(\gamma_{u,v}) - \ell(\gamma_{v,w})\right|\right]^{1+1/q} \\ \times\left[\|f'\|_{\gamma_{u,v};p} + \|f'\|_{\gamma_{v,w};p}\right], \\ \\ \left[\ell^{q+1}(\gamma_{u,v}) + \ell^{q+1}(\gamma_{v,w})\right]^{1/q}\|f'\|_{\gamma_{u,w};p}\,, \\ \\ \frac{1}{2^{1/p}}\left[\|f'\|^p_{\gamma_{u,w};p} + \left|\|f'\|^p_{\gamma_{u,v};p} - \|f'\|^p_{\gamma_{v,w};p}\right|\right]^{1/p} \\ \times\left[\ell^{1+1/q}(\gamma_{u,v}) + \ell^{1+1/q}(\gamma_{v,w})\right]. \end{cases}$$

Proof. Using the weighted Hölder integral inequality for $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\int_a^x |f'(z(t))|\,|z'(t)|\left(\int_a^t |z'(s)|\,ds\right)dt$$
$$\le \left(\int_a^x |f'(z(t))|^p\,|z'(t)|\,dt\right)^{1/p}\left(\int_a^x \left(\int_a^t |z'(s)|\,ds\right)^q |z'(t)|\,dt\right)^{1/q}$$
$$= \left(\int_a^x |f'(z(t))|^p\,|z'(t)|\,dt\right)^{1/p}\left(\frac{\left(\int_a^x |z'(s)|\,ds\right)^{q+1}}{q+1}\right)^{1/q}$$
$$= \frac{\left(\int_a^x |z'(s)|\,ds\right)^{1+1/q}}{(q+1)^{1/q}}\left(\int_a^x |f'(z(t))|^p\,|z'(t)|\,dt\right)^{1/p}$$

and

$$\begin{aligned}
&\int_x^b |f'(z(t))| |z'(t)| \left(\int_t^b |z'(s)| ds\right) dt \\
&\quad\le \left(\int_x^b |f'(z(t))|^p |z'(t)| dt\right)^{1/p} \left(\int_x^b \left(\int_t^b |z'(s)| ds\right)^q |z'(t)| dt\right)^{1/q} \\
&\quad= \left(\int_x^b |f'(z(t))|^p |z'(t)| dt\right)^{1/p} \left(\frac{\left(\int_x^b |z'(s)| ds\right)^{q+1}}{q+1}\right)^{1/q} \\
&\quad= \frac{\left(\int_x^b |z'(s)| ds\right)^{1+1/q}}{(q+1)^{1/q}} \left(\int_x^b |f'(z(t))|^p |z'(t)| dt\right)^{1/p}
\end{aligned}$$

for $x \in (a,b)$.

If we add these two inequalities we get

$$\begin{aligned}
B(x) &\le \frac{\left(\int_a^x |z'(s)| ds\right)^{1+1/q}}{(q+1)^{1/q}} \left(\int_a^x |f'(z(t))|^p |z'(t)| dt\right)^{1/p} \\
&\quad + \frac{\left(\int_x^b |z'(s)| ds\right)^{1+1/q}}{(q+1)^{1/q}} \left(\int_x^b |f'(z(t))|^p |z'(t)| dt\right)^{1/p} \\
&= \frac{1}{q+1}\left[\ell^{1+1/q}(\gamma_{u,v}) \left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \ell^{1+1/q}(\gamma_{v,w}) \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p}\right],
\end{aligned}$$

which proves the first inequality in (2.6).

We also have

$$\begin{aligned}
&\ell^{1+1/q}(\gamma_{u,v}) \left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \ell^{1+1/q}(\gamma_{v,w}) \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p} \\
&\qquad \le \max\left\{\ell^{1+1/q}(\gamma_{u,v}), \ell^{1+1/q}(\gamma_{v,w})\right\} \\
&\qquad\quad \times \left[\left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p}\right] \\
&= \left[\max\left\{\ell(\gamma_{u,v}), \ell(\gamma_{v,w})\right\}\right]^{1+1/q} \left[\left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p}\right] \\
&\qquad = \frac{1}{2^{1+1/q}}\left[\ell(\gamma_{u,w}) + |\ell(\gamma_{u,v}) - \ell(\gamma_{v,w})|\right]^{1+1/q} \\
&\qquad\quad \times \left[\left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p}\right]
\end{aligned}$$

and

$$\ell^{1+1/q}\left(\gamma_{u,v}\right)\left(\int_{\gamma_{u,v}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right)^{1/p}+\ell^{1+1/q}\left(\gamma_{v,w}\right)\left(\int_{\gamma_{v,w}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right)^{1/p}$$
$$\leq\max\left\{\left(\int_{\gamma_{u,v}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right)^{1/p},\left(\int_{\gamma_{v,w}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right)^{1/p}\right\}$$
$$\times\left[\ell^{1+1/q}\left(\gamma_{u,v}\right)+\ell^{1+1/q}\left(\gamma_{v,w}\right)\right]$$
$$=\left[\max\left\{\int_{\gamma_{u,v}}\left|f'\left(z\right)\right|^{p}\left|dz\right|,\int_{\gamma_{v,w}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right\}\right]^{1/p}$$
$$\times\left[\ell^{1+1/q}\left(\gamma_{u,v}\right)+\ell^{1+1/q}\left(\gamma_{v,w}\right)\right]$$
$$=\frac{1}{2^{1/p}}\left[\int_{\gamma_{u,w}}\left|f'\left(z\right)\right|^{p}\left|dz\right|+\left|\int_{\gamma_{u,v}}\left|f'\left(z\right)\right|^{p}\left|dz\right|-\int_{\gamma_{v,w}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right|\right]^{1/p}$$
$$\times\left[\ell^{1+1/q}\left(\gamma_{u,v}\right)+\ell^{1+1/q}\left(\gamma_{v,w}\right)\right].$$

By Hölder's discrete inequality we have

$$\ell^{1+1/q}\left(\gamma_{u,v}\right)\left(\int_{\gamma_{u,v}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right)^{1/p}+\ell^{1+1/q}\left(\gamma_{v,w}\right)\left(\int_{\gamma_{v,w}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right)^{1/p}$$
$$\leq\left[\ell^{q+1}\left(\gamma_{u,v}\right)+\ell^{q+1}\left(\gamma_{v,w}\right)\right]^{1/q}\left[\int_{\gamma_{u,v}}\left|f'\left(z\right)\right|^{p}\left|dz\right|+\int_{\gamma_{v,w}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right]^{1/p}$$
$$=\left[\ell^{q+1}\left(\gamma_{u,v}\right)+\ell^{q+1}\left(\gamma_{v,w}\right)\right]^{1/q}\left(\int_{\gamma_{u,w}}\left|f'\left(z\right)\right|^{p}\left|dz\right|\right)^{1/p},$$

which prove the last part of (2.6). □

We have:

Corollary 2. *With the assumption of Theorem 4 and if there exists $m\in\gamma$ such that $\ell\left(\gamma_{u,m}\right)=\ell\left(\gamma_{m,w}\right)$, then*

$$\left|f\left(m\right)\ell\left(\gamma\right)-\int_{\gamma}f\left(z\right)\left|dz\right|\right|\leq\frac{1}{2^{1+1/q}\left(q+1\right)}\ell^{1+1/q}\left(\gamma_{u,w}\right)\left[\left\|f'\right\|_{\gamma_{u,m};p}+\left\|f'\right\|_{\gamma_{m,w};p}\right]\tag{2.7}$$

and if $s\in\gamma$ such that $\int_{\gamma_{u,s}}\left|f'\left(z\right)\right|^{p}\left|dz\right|=\int_{\gamma_{s,w}}\left|f'\left(z\right)\right|^{p}\left|dz\right|$, then

$$\left|f\left(s\right)\ell\left(\gamma\right)-\int_{\gamma}f\left(z\right)\left|dz\right|\right|\leq\frac{1}{2^{1/p}\left(q+1\right)}\left[\ell^{1+1/q}\left(\gamma_{u,s}\right)+\ell^{1+1/q}\left(\gamma_{s,w}\right)\right]\left\|f'\right\|_{\gamma_{u,w};p}.\tag{2.8}$$

Finally we have:

Theorem 5. *With the assumption of Theorem 3 we have*

$$\left|f(v)\ell(\gamma)-\int_{\gamma}f(z)|dz|\right| \tag{2.9}$$

$$\leq\frac{1}{2}\left[\ell^{2}\left(\gamma_{u,v}\right)\|f'\|_{\gamma_{u,v};\infty}+\ell^{2}\left(\gamma_{v,w}\right)\|f'\|_{\gamma_{v,w};\infty}\right]$$

$$\leq\frac{1}{2}\begin{cases}\frac{1}{4}\left[\ell\left(\gamma_{u,w}\right)+\left|\ell\left(\gamma_{u,v}\right)-\ell\left(\gamma_{v,w}\right)\right|\right]^{2}\\ \times\left[\|f'\|_{\gamma_{u,v};\infty}+\|f'\|_{\gamma_{v,w};\infty}\right],\\ \\ \left[\ell^{2q}\left(\gamma_{u,v}\right)+\ell^{2q}\left(\gamma_{v,w}\right)\right]^{1/q}\left(\|f'\|_{\gamma_{u,v};\infty}^{p}+\|f'\|_{\gamma_{v,w};\infty}^{p}\right)^{1/p},\\ p,\ q>1\ \text{and}\ \frac{1}{p}+\frac{1}{q}=1,\\ \\ \left[\ell^{2}\left(\gamma_{u,v}\right)+\ell^{2}\left(\gamma_{v,w}\right)\right]\|f'\|_{\gamma_{u,w};\infty}.\end{cases}$$

Proof. We have

$$\begin{aligned}&\int_{a}^{x}\left|f'(z(t))\right|\left|z'(t)\right|\left(\int_{a}^{t}\left|z'(s)\right|ds\right)dt\\ &\qquad\leq\max_{t\in[a,x]}\left|f'(z(t))\right|\int_{a}^{x}\left(\int_{a}^{t}\left|z'(s)\right|ds\right)\left|z'(t)\right|dt\\ &\qquad=\frac{1}{2}\max_{t\in[a,x]}\left|f'(z(t))\right|\left(\int_{a}^{x}\left|z'(s)\right|ds\right)^{2}=\frac{1}{2}\ell^{2}\left(\gamma_{u,v}\right)\|f'\|_{\gamma_{u,v};\infty}\end{aligned}$$

and

$$\begin{aligned}&\int_{x}^{b}\left|f'(z(t))\right|\left|z'(t)\right|\left(\int_{t}^{b}\left|z'(s)\right|ds\right)dt\\ &\qquad\leq\max_{t\in[x,b]}\left|f'(z(t))\right|\int_{x}^{b}\left(\int_{t}^{b}\left|z'(s)\right|ds\right)\left|z'(t)\right|dt\\ &\qquad=\frac{1}{2}\max_{t\in[x,b]}\left|f'(z(t))\right|\left(\int_{x}^{b}\left|z'(s)\right|ds\right)^{2}=\frac{1}{2}\ell^{2}\left(\gamma_{v,w}\right)\|f'\|_{\gamma_{v,w};\infty},\end{aligned}$$

which by addition produce

$$B(x)\leq\frac{1}{2}\left[\ell^{2}\left(\gamma_{u,v}\right)\|f'\|_{\gamma_{u,v};\infty}+\ell^{2}\left(\gamma_{v,w}\right)\|f'\|_{\gamma_{v,w};\infty}\right]$$

and by utilising the inequality (2.3) we get the first part of (2.9).

The second part is obvious and we omit the details. □

Corollary 3. *With the assumption of Theorem 3 and if there exists* $m \in \gamma$ *such that* $\ell\left(\gamma_{u,m}\right) = \ell\left(\gamma_{m,w}\right)$, *then*

$$\left|f(m)\,\ell(\gamma) - \int_{\gamma} f(z)\,|dz|\right| \leq \frac{1}{8}\left[\|f'\|_{\gamma_{u,m};\infty} + \|f'\|_{\gamma_{m,w};\infty}\right]\ell^{2}\left(\gamma_{u,w}\right) \leq \frac{1}{4}\|f'\|_{\gamma_{u,w};\infty}\,\ell^{2}\left(\gamma_{u,w}\right). \tag{2.10}$$

3. Examples for Circular Paths

Let $[a,b] \subseteq [0,2\pi]$ and the circular path $\gamma_{[a,b],R}$ centered in 0 and with radius $R > 0$

$$z(t) = R\exp(it) = R(\cos t + i\sin t),\ t \in [a,b].$$

If $[a,b] = [0,\pi]$ then we get a half circle while for $[a,b] = [0,2\pi]$ we get the full circle.

We have

$$z'(t) = Ri\exp(it),\ t \in [a,b]$$

and $|z'(t)| = R$ for $t \in [a,b]$ giving that

$$\ell\left(\gamma_{[a,b],R}\right) = \int_a^b |z'(t)|\,dt = R(b-a).$$

If $x \in [a,b]$ and $v = R\exp(ix)$, then by (2.1) we have

$$\left|R(b-a)\,f(R\exp(ix)) - R\int_a^b f(R\exp(it))\,dt\right| \leq R(x-a)\,R\int_a^x |f'(R\exp(it))|\,dt + R(b-x)\,R\int_x^b |f'(R\exp(it))|\,dt$$

that is equivalent to

$$\left|(b-a)\,f(R\exp(ix)) - \int_a^b f(R\exp(it))\,dt\right| \leq R(x-a)\int_a^x |f'(R\exp(it))|\,dt + R(b-x)\int_x^b |f'(R\exp(it))|\,dt \tag{3.1}$$

for $x \in [a,b]$.

In particular, if we take $x = \frac{a+b}{2}$ in (3.1), then we get

$$\left|(b-a)\,f\left(R\exp\left(\frac{a+b}{2}i\right)\right) - \int_a^b f(R\exp(it))\,dt\right| \leq \frac{1}{2}R(b-a)\int_a^b |f'(R\exp(it))|\,dt. \tag{3.2}$$

If $m \in [a,b]$ is such that

$$\int_a^m |f'(R\exp(it))|\,dt = \int_m^b |f'(R\exp(it))|\,dt,$$

then from (3.1) we get

$$\left|(b-a) f(R\exp(mi)) - \int_a^b f(R\exp(it))\,dt\right| \leq \frac{1}{2} R(b-a) \int_a^b |f'(R\exp(it))|\,dt. \tag{3.3}$$

By making use of (2.6) we get

$$\begin{aligned}&\left|R(b-a) f(R\exp(ix)) - R\int_a^b f(R\exp(it))\,dt\right| \\ &\leq \frac{1}{q+1}\left[R^{1+1/q}(x-a)^{1+1/q} R^{1/p}\left(\int_a^x |f'(R\exp(it))|^p\,dt\right)^{1/p}\right. \\ &\qquad \left.+R^{1+1/q}(b-x)^{1+1/q} R^{1/p}\left(\int_x^b |f'(R\exp(it))|^p\,dt\right)^{1/p}\right]\end{aligned}$$

that is equivalent to

$$\begin{aligned}&\left|(b-a) f(R\exp(ix)) - \int_a^b f(R\exp(it))\,dt\right| \\ &\leq \frac{1}{q+1} R\left[(x-a)^{1+1/q}\left(\int_a^x |f'(R\exp(it))|^p\,dt\right)^{1/p}\right. \\ &\qquad \left.+(b-x)^{1+1/q}\left(\int_x^b |f'(R\exp(it))|^p\,dt\right)^{1/p}\right]\end{aligned} \tag{3.4}$$

for $x \in [a,b]$.

If we take $x = \frac{a+b}{2}$ in (3.4), then we get

$$\begin{aligned}&\left|(b-a) f\left(R\exp\left(\frac{a+b}{2} i\right)\right) - \int_a^b f(R\exp(it))\,dt\right| \\ &\leq \frac{1}{(q+1)\,2^{1+1/q}} R(b-a)^{1+1/q}\left[\left(\int_a^{\frac{a+b}{2}} |f'(R\exp(it))|^p\,dt\right)^{1/p}\right. \\ &\qquad \left.+\left(\int_{\frac{a+b}{2}}^b |f'(R\exp(it))|^p\,dt\right)^{1/p}\right].\end{aligned} \tag{3.5}$$

If $c \in [a,b]$ is such that

$$\int_a^c |f'(R\exp(it))|^p\,dt = \int_c^b |f'(R\exp(it))|^p\,dt,$$

then by (3.4) we get

$$\left|(b-a)f(R\exp(ic))-\int_a^b f(R\exp(it))\,dt\right| \le \frac{1}{(q+1)2^{1/p}}R\left[(c-a)^{1+1/q}+(b-c)^{1+1/q}\right] \times\left(\int_a^b |f'(R\exp(it))|^p\,dt\right)^{1/p}. \tag{3.6}$$

Further, if we use (2.9), then we have

$$\begin{aligned}&\left|R(b-a)f(R\exp(ix))-R\int_a^b f(R\exp(it))\,dt\right|\\ &\le \frac{1}{2}R^2\left[(x-a)^2\sup_{t\in[a,x]}|f'(R\exp(it))|+(b-x)^2\sup_{t\in[x,b]}|f'(R\exp(it))|\right]\\ &\le \frac{1}{2}R^2\left[(x-a)^2+(b-x)^2\right]\sup_{t\in[a,b]}|f'(R\exp(it))|\end{aligned}$$

that is equivalent to

$$\begin{aligned}&\left|(b-a)f(R\exp(xi))-\int_a^b f(R\exp(it))\,dt\right|\\ &\le \frac{1}{2}R\left[(x-a)^2\sup_{t\in[a,x]}|f'(R\exp(it))|+(b-x)^2\sup_{t\in[x,b]}|f'(R\exp(it))|\right]\\ &\le R\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\sup_{t\in[a,b]}|f'(R\exp(it))|\end{aligned} \tag{3.7}$$

for $x\in[a,b]$.

In particular, we have

$$\begin{aligned}&\left|(b-a)f\left(R\exp\left(\frac{a+b}{2}i\right)\right)-\int_a^b f(R\exp(it))\,dt\right|\\ &\le \frac{1}{8}R(b-a)^2\left[\sup_{t\in\left[a,\frac{a+b}{2}\right]}|f'(R\exp(it))|+\sup_{t\in\left[\frac{a+b}{2},b\right]}|f'(R\exp(it))|\right]\\ &\le \frac{1}{4}R(b-a)^2\sup_{t\in[a,b]}|f'(R\exp(it))|.\end{aligned} \tag{3.8}$$

We give now examples for some fundamental complex functions.

Consider the function $f(z)=z^n$, $z\in\mathbb{C}$ with $n\ge 1$. Then $f'(z)=nz^{n-1}$,

$$f(R\exp(ix))=R^n\exp(nxi),$$
$$|f'(R\exp(it))|=nR^{n-1}|\exp((n-1)it)|=nR^{n-1},\ t\in[a,b]$$

and

$$\int_a^b f(R\exp(it))\,dt=R^n\int_a^b\exp(nti)\,dt=R^n\frac{\exp(nbi)-\exp(nai)}{ni}.$$

Making use of the inequality (3.7) we get

$$\left|(b-a)R^n\exp(nxi)-R^n\frac{\exp(nbi)-\exp(nai)}{ni}\right| \le R\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]nR^{n-1},$$

which is equivalent to

$$\left|(b-a)\exp(nxi)-\frac{\exp(nbi)-\exp(nai)}{ni}\right| \le n\left[\frac{1}{2}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right], \tag{3.9}$$

for $x\in[a,b]$.

If we take in (3.9) $x=\frac{a+b}{2}$, then we get

$$\left|(b-a)\exp\left(n\left(\frac{a+b}{2}\right)i\right)-\frac{\exp(nbi)-\exp(nai)}{ni}\right|\le\frac{1}{4}n(b-a)^2, \tag{3.10}$$

for $n\ge 1$.

Consider the exponential function $f(z)=\exp(z)$, $z\in\mathbb{C}$. Then $f'(z)=\exp(z)$,

$$|f'(R\exp(it))|=|\exp(R(\cos t+i\sin t))|=\exp(R\cos t),\ t\in[a,b]$$

and by the inequality (3.1) we get

$$\left|(b-a)\exp(R\exp(ix))-\int_a^b\exp(R\exp(it))\,dt\right| \le R\left[(x-a)\int_a^x\exp(R\cos t)\,dt+(b-x)\int_x^b\exp(R\cos t)\,dt\right], \tag{3.11}$$

while from the inequality (3.7) we get

$$\begin{aligned}\left|(b-a)\exp(R\exp(ix))-\int_a^b\exp(R\exp(it))\,dt\right| &\le\frac{1}{2}R\left[(x-a)^2\sup_{t\in[a,x]}\exp(R\cos t)+(b-x)^2\sup_{t\in[x,b]}\exp(R\cos t)\right]\\ &\le R\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\sup_{t\in[a,b]}\exp(R\cos t)\end{aligned} \tag{3.12}$$

for $x\in[a,b]$.

From the inequality (3.4) we get

$$\left|(b-a)\exp(R\exp(ix))-\int_a^b \exp(R\exp(it))\,dt\right| \le \frac{1}{q+1}R\left[(x-a)^{1+1/q}\left(\int_a^x \exp(pR\cos t)\,dt\right)^{1/p} + (b-x)^{1+1/q}\left(\int_x^b \exp(pR\cos t)\,dt\right)^{1/p}\right] \tag{3.13}$$

for $x\in[a,b]$, where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

If in the inequality (3.11) we take $x=\frac{a+b}{2}$, then we get

$$\left|(b-a)\exp\left(R\exp\left(\frac{a+b}{2}i\right)\right)-\int_a^b \exp(R\exp(it))\,dt\right| \le \frac{1}{2}R(b-a)\int_a^b \exp(R\cos t)\,dt, \tag{3.14}$$

while from the inequality (2.8) we get

$$\left|(b-a)\exp\left(R\exp\left(\frac{a+b}{2}i\right)\right)-\int_a^b \exp(R\exp(it))\,dt\right| \le \frac{1}{8}(b-a)^2 R\left[\sup_{t\in\left[a,\frac{a+b}{2}\right]}\exp(R\cos t)+\sup_{t\in\left[\frac{a+b}{2},b\right]}\exp(R\cos t)\right] \le \frac{1}{4}R(b-a)^2\sup_{t\in[a,b]}\exp(R\cos t). \tag{3.15}$$

From (3.13) we have

$$\left|(b-a)\exp\left(R\exp\left(\frac{a+b}{2}i\right)\right)-\int_a^b \exp(R\exp(it))\,dt\right| \le \frac{1}{(q+1)2^{1+1/q}}R(b-a)^{1+1/q} \times\left[\left(\int_a^{\frac{a+b}{2}} \exp(pR\cos t)\,dt\right)^{1/p}+\left(\int_{\frac{a+b}{2}}^b \exp(pR\cos t)\,dt\right)^{1/p}\right]. \tag{3.16}$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa