TRAPEZOID TYPE INEQUALITIES FOR THE COMPLEX INTEGRAL ON PATHS

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ABSTRACT. In this paper we extend the trapezoid inequality to the integral with respect to arc-length by providing upper bounds for the quantity

$$\left|(v-u) f(u)+(w-v) f(w)-\int_{\gamma} f(z) d z\right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with the length $\ell(\gamma)$, $u=z(a)$, $v=z(x)$ with $x \in (a,b)$ and $w=z(b)$ while f is holomorphic in G, an open domain and $\gamma \subset G$. An application for circular paths is also given.

1. Introduction

Inequalities providing upper bounds for the quantity

$$\left|(t-a) f(a)+(b-t) f(b)-\int_{a}^{b} f(s) d s\right|, \qquad t \in [a,b] \tag{1.1}$$

are known in the literature as *generalized trapezoid inequalities* and it has been shown in [2] that

$$\begin{aligned} &\left|(t-a) f(a)+(b-t) f(b)-\int_{a}^{b} f(s) d s\right| \\ &\leq\left[\frac{1}{2}+\left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right](b-a) \bigvee_{a}^{b}(f) \end{aligned} \tag{1.2}$$

for any $t \in [a,b]$, provided that f is of bounded variation on $[a,b]$. The constant $\frac{1}{2}$ is the best possible.

If f is *absolutely continuous* on $[a,b]$, then (see [1, p. 93])

$$\left|(t-a) f(a)+(b-t) f(b)-\int_{a}^{b} f(s) d s\right| \tag{1.3}$$

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$$\leq \begin{cases} \left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^{2}\right](b-a)^{2}\left\|f'\right\|_{\infty} & \text{if } f' \in L_{\infty}[a,b]; \\ \frac{1}{(q+1)^{1/q}}\left[\left(\frac{t-a}{b-a}\right)^{q+1}+\left(\frac{b-t}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{1+1/q}\left\|f'\right\|_{p} & \text{if } f' \in L_{p}[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}+\left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right](b-a)\left\|f'\right\|_{1} & \end{cases}$$

for any $t \in [a,b]$. The constants $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{(q+1)^{1/q}}$ are the best possible.

For other recent results on the trapezoid inequality, see [3], [4], [8], [9], [10] and [12].

In order to extend this result for the complex integral, we need some preparations as follows.

Suppose γ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a)=u$ and $z(b)=w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w}=\gamma$ as

$$\int_{\gamma} f(z)\,dz=\int_{\gamma_{u,w}} f(z)\,dz:=\int_{a}^{b} f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are *piecewise smooth.* Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz:=\int_{\gamma_{u,v}} f(z)\,dz+\int_{\gamma_{v,w}} f(z)\,dz$$

where $v:=z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz|:=\int_{a}^{b} f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma)=\int_{\gamma_{u,w}} |dz|=\int_{a}^{b} |z'(t)|\,dt.$$

Let f and g be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz=f(w)\,g(w)-f(u)\,g(u)-\int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.4}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_{\gamma} f(z)\,dz\right| \leq \int_{\gamma} |f(z)|\,|dz| \leq \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.5}$$

where $\|f\|_{\gamma,\infty}:=\sup_{z\in\gamma}|f(z)|$.

We also define the p-norm with $p \geq 1$ by

$$\|f\|_{\gamma,p}:=\left(\int_{\gamma} |f(z)|^{p}\,|dz|\right)^{1/p}.$$

For $p=1$ we have

$$\|f\|_{\gamma,1}:=\int_{\gamma}|f(z)|\,|dz|\,.$$

If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1}\leq[\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}\,.$$

We have the following recent result for functions of complex variable [6]:

Theorem 1. *Let f be holomorphic in G, an open domain and suppose $\gamma\subset G$ is a smooth path from $z(a)=u$ to $z(b)=w$. If $v=z(x)$ with $x\in(a,b)$, then $\gamma_{u,w}=\gamma_{u,v}\cup\gamma_{v,w}$,*

$$\left|(v-u)f(u)+(w-v)f(w)-\int_{\gamma}f(z)\,dz\right| \leq\|f'\|_{\gamma_{u,v};\infty}\int_{\gamma_{u,v}}|z-v|\,|dz|+\|f'\|_{\gamma_{v,w};\infty}\int_{\gamma_{v,w}}|z-v|\,|dz| \leq\|f'\|_{\gamma_{u,w};\infty}\int_{\gamma_{u,w}}|z-v|\,|dz|\,, \tag{1.6}$$

and

$$\left|(v-u)f(u)+(w-v)f(w)-\int_{\gamma}f(z)\,dz\right| \leq\|f'\|_{\gamma_{u,v};1}\max_{z\in\gamma_{u,v}}|z-v|+\|f'\|_{\gamma_{v,w};1}\max_{z\in\gamma_{v,w}}|z-v| \leq\|f'\|_{\gamma_{u,w};1}\max_{z\in\gamma_{u,w}}|z-v|\,. \tag{1.7}$$

If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\left|(v-u)f(u)+(w-v)f(w)-\int_{\gamma}f(z)\,dz\right| \leq\|f'\|_{\gamma_{u,v};p}\left(\int_{\gamma_{u,v}}|z-v|^{q}\,|dz|\right)^{1/q}+\|f'\|_{\gamma_{v,w};p}\left(\int_{\gamma_{v,w}}|z-v|^{q}\,|dz|\right)^{1/q} \leq\|f'\|_{\gamma_{u,w};p}\left(\int_{\gamma_{u,w}}|z-v|^{q}\,|dz|\right)^{1/q}. \tag{1.8}$$

In this paper we extend the trapezoid inequality to the complex integral, by providing upper bounds for the quantity

$$\left|f(u)\,\ell\left(\gamma_{u,v}\right)+f(w)\,\ell\left(\gamma_{v,w}\right)-\int_{\gamma}f(z)\,|dz|\right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t\in[a,b]$, $u=z(a)$, $v=z(x)$ with $x\in(a,b)$ and $w=z(b)$ while f is holomorphic in G, an open domain and $\gamma\subset G$. An application for circular paths is also given.

2. Trapezoid Inequalities

We have:

Theorem 2. *Let f be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(x)$ with $x \in (a,b)$, then $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$ and*

$$\left| f(u)\,\ell\left(\gamma_{u,v}\right) + f(w)\,\ell\left(\gamma_{v,w}\right) - \int_{\gamma} f(z)\,|dz| \right| \tag{2.1}$$

$$\leq \ell\left(\gamma_{u,v}\right) \|f'\|_{\gamma_{u,v};1} + \ell\left(\gamma_{v,w}\right) \|f'\|_{\gamma_{v,w};1}$$

$$\leq \begin{cases} \frac{1}{2}\left[\ell\left(\gamma_{u,w}\right) + \left|\ell\left(\gamma_{u,v}\right) - \ell\left(\gamma_{v,w}\right)\right|\right] \|f'\|_{\gamma_{u,w};1}\,, \\ \\ \left[\ell^p\left(\gamma_{u,v}\right) + \ell^p\left(\gamma_{v,w}\right)\right]^{1/p} \left(\|f'\|^q_{\gamma_{u,v};1} + \|f'\|^q_{\gamma_{v,w};1}\right)^{1/q} \\ p,\ q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \frac{1}{2}\ell\left(\gamma_{u,w}\right)\left[\|f'\|_{\gamma_{u,w};1} + \left|\|f'\|_{\gamma_{u,v};1} - \|f'\|_{\gamma_{v,w};1}\right|\right]. \end{cases}$$

Proof. Observe that for $x \in (a,b)$ we have

$$\int_a^b f(z(t))\,d\left(\int_x^t |z'(s)|\,ds\right) = \int_a^b f(z(t))\,|z'(t)|\,dt = \int_{\gamma_{u,w}} f(z)\,|dz| \tag{2.2}$$

and, integrating by parts

$$\begin{aligned} &\int_a^b f(z(t))\,d\left(\int_x^t |z'(s)|\,ds\right) \\ &= f(z(t)) \int_x^t |z'(s)|\,ds\Big|_a^b - \int_a^b \left(\int_x^t |z'(s)|\,ds\right) f'(z(t))\,z'(t)\,dt \\ &= f(z(b)) \int_x^b |z'(s)|\,ds - f(z(a)) \int_x^a |z'(s)|\,ds \\ &\quad - \int_a^b \left(\int_x^t |z'(s)|\,ds\right) f'(z(t))\,z'(t)\,dt \\ &= f(w)\,\ell\left(\gamma_{v,w}\right) + f(u)\,\ell\left(\gamma_{v,w}\right) \\ &\quad - \int_a^x \left(\int_x^t |z'(s)|\,ds\right) f'(z(t))\,z'(t)\,dt - \int_x^b \left(\int_x^t |z'(s)|\,ds\right) f'(z(t))\,z'(t)\,dt \\ &= f(w)\,\ell\left(\gamma_{v,w}\right) + f(u)\,\ell\left(\gamma_{v,w}\right) \\ &\quad + \int_a^x \left(\int_t^x |z'(s)|\,ds\right) f'(z(t))\,z'(t)\,dt - \int_x^b \left(\int_x^t |z'(s)|\,ds\right) f'(z(t))\,z'(t)\,dt. \end{aligned} \tag{2.3}$$

By using (2.2) and (2.3) we get the following equality of interest

$$\begin{aligned}(2.4)\quad & f(w)\,\ell(\gamma_{v,w}) + f(u)\,\ell(\gamma_{v,w}) - \int_{\gamma_{u,w}} f(z)\,|dz| \\ &= \int_x^b \left(\int_x^t |z'(s)|\,ds\right) f'(z(t))\,z'(t)\,dt - \int_a^x \left(\int_t^x |z'(s)|\,ds\right) f'(z(t))\,z'(t)\,dt,\end{aligned}$$

for $x \in (a,b)$, where u, v, w are as in the statement of the theorem.

By taking the modulus in (2.4) and using the properties of modulus, we get

$$\begin{aligned}(2.5)\quad & \left| f(w)\,\ell(\gamma_{v,w}) + f(u)\,\ell(\gamma_{v,w}) - \int_{\gamma_{u,w}} f(z)\,|dz| \right| \\ &\le \left| \int_x^b \left(\int_x^t |z'(s)|\,ds\right) f'(z(t))\,z'(t)\,dt \right| \\ &+ \left| \int_a^x \left(\int_t^x |z'(s)|\,ds\right) f'(z(t))\,z'(t)\,dt \right| \\ &\le \int_x^b \left(\int_x^t |z'(s)|\,ds\right) |f'(z(t))|\,|z'(t)|\,dt \\ &\qquad + \int_a^x \left(\int_t^x |z'(s)|\,ds\right) |f'(z(t))|\,|z'(t)|\,dt =: B(x)\end{aligned}$$

for $x \in [a,b]$.

We have

$$\int_x^t |z'(s)|\,ds \le \int_x^b |z'(s)|\,ds \text{ for } t \in [x,b]$$

and

$$\int_t^x |z'(s)|\,ds \le \int_a^x |z'(s)|\,ds \text{ for } t \in [a,x],$$

then

$$\begin{aligned} B(x) &\le \left(\int_x^b |z'(s)|\,ds\right) \int_x^b |f'(z(t))|\,|z'(t)|\,dt \\ &\quad + \left(\int_a^x |z'(s)|\,ds\right) \int_a^x |f'(z(t))|\,|z'(t)|\,dt \\ &= \ell(\gamma_{v,w}) \int_{\gamma_{v,w}} |f'(z)|\,|dz| + \ell(\gamma_{u,v}) \int_{\gamma_{u,v}} |f'(z)|\,|dz| \end{aligned}$$

and by (2.5) we get the first inequality in (2.1).

The second part follows by Hölder's inequalities

$$mn + cd \le \begin{cases} \max\{m,c\}(n+d) \\ (m^p + c^p)^{1/p} (n^q + d^q)^{1/q}, \text{ for } p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

where m, n, c, $d \ge 0$. $\square$

Corollary 1. *With the assumption of Theorem 2 and if there exists $m \in \gamma$ such that $\ell\left(\gamma_{u,m}\right) = \ell\left(\gamma_{m,w}\right)$, then*

$$\left|\frac{f(u)+f(w)}{2}\ell(\gamma) - \int_\gamma f(z)\,|dz|\right| \le \frac{1}{2}\ell(\gamma)\,\|f'\|_{\gamma_{u,w};1} \tag{2.6}$$

and if $s \in \gamma$ such that $\int_{\gamma_{u,s}} |f'(z)|\,|dz| = \int_{\gamma_{s,w}} |f'(z)|\,|dz|$, then

$$\left|f(u)\,\ell\left(\gamma_{u,s}\right) + f(w)\,\ell\left(\gamma_{s,w}\right) - \int_\gamma f(z)\,|dz|\right| \le \frac{1}{2}\ell(\gamma)\,\|f'\|_{\gamma_{u,w};1}\,. \tag{2.7}$$

We have also the following result for p-norms:

Theorem 3. *With the assumption of Theorem 2 we have for $p,\ q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ that*

$$\begin{aligned}
&\left|f(u)\,\ell\left(\gamma_{u,v}\right) + f(w)\,\ell\left(\gamma_{v,w}\right) - \int_\gamma f(z)\,|dz|\right| \\
&\le \frac{1}{q+1}\left[\ell^{1+1/q}\left(\gamma_{u,v}\right)\|f'\|_{\gamma_{u,v};p} + \ell^{1+1/q}\left(\gamma_{v,w}\right)\|f'\|_{\gamma_{v,w};p}\right] \\
&\le \frac{1}{q+1}\begin{cases}
\frac{1}{2^{1+1/q}}\left[\ell\left(\gamma_{u,w}\right) + \left|\ell\left(\gamma_{u,v}\right) - \ell\left(\gamma_{v,w}\right)\right|\right]^{1+1/q} \\
\times\left[\|f'\|_{\gamma_{u,v};p} + \|f'\|_{\gamma_{v,w};p}\right], \\
\\
\left[\ell^{q+1}\left(\gamma_{u,v}\right) + \ell^{q+1}\left(\gamma_{v,w}\right)\right]^{1/q}\|f'\|_{\gamma_{u,w};p}, \\
\\
\frac{1}{2^{1/p}}\left[\|f'\|^p_{\gamma_{u,w};p} + \left|\|f'\|^p_{\gamma_{u,v};p} - \|f'\|^p_{\gamma_{v,w};p}\right|\right]^{1/p} \\
\times\left[\ell^{1+1/q}\left(\gamma_{u,v}\right) + \ell^{1+1/q}\left(\gamma_{v,w}\right)\right].
\end{cases}
\end{aligned} \tag{2.8}$$

Proof. Using the weighted Hölder integral inequality for $p,\ q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ we have

$$\begin{aligned}
&\int_a^x |f'(z(t))|\,|z'(t)|\left(\int_t^x |z'(s)|\,ds\right)dt \\
&\le \left(\int_a^x |f'(z(t))|^p\,|z'(t)|\,dt\right)^{1/p}\left(\int_a^x\left(\int_t^x |z'(s)|\,ds\right)^q |z'(t)|\,dt\right)^{1/q} \\
&= \left(\int_a^x |f'(z(t))|^p\,|z'(t)|\,dt\right)^{1/p}\left(\frac{\left(\int_a^x |z'(s)|\,ds\right)^{q+1}}{q+1}\right)^{1/q} \\
&= \frac{\left(\int_a^x |z'(s)|\,ds\right)^{1+1/q}}{(q+1)^{1/q}}\left(\int_a^x |f'(z(t))|^p\,|z'(t)|\,dt\right)^{1/p}
\end{aligned}$$

and

$$\int_x^b |f'(z(t))| |z'(t)| \left(\int_x^t |z'(s)| ds\right) dt$$
$$\leq \left(\int_x^b |f'(z(t))|^p |z'(t)| dt\right)^{1/p} \left(\int_x^b \left(\int_x^t |z'(s)| ds\right)^q |z'(t)| dt\right)^{1/q}$$
$$= \left(\int_x^b |f'(z(t))|^p |z'(t)| dt\right)^{1/p} \left(\frac{\left(\int_x^b |z'(s)| ds\right)^{q+1}}{q+1}\right)^{1/q}$$
$$= \frac{\left(\int_x^b |z'(s)| ds\right)^{1+1/q}}{(q+1)^{1/q}} \left(\int_x^b |f'(z(t))|^p |z'(t)| dt\right)^{1/p}$$

for $x \in (a,b)$.

If we add these two inequalities we get

$$B(x) \leq \frac{\left(\int_a^x |z'(s)| ds\right)^{1+1/q}}{(q+1)^{1/q}} \left(\int_a^x |f'(z(t))|^p |z'(t)| dt\right)^{1/p}$$
$$+ \frac{\left(\int_x^b |z'(s)| ds\right)^{1+1/q}}{(q+1)^{1/q}} \left(\int_x^b |f'(z(t))|^p |z'(t)| dt\right)^{1/p}$$
$$= \frac{1}{q+1} \left[\ell^{1+1/q}(\gamma_{u,v}) \left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \ell^{1+1/q}(\gamma_{v,w}) \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p}\right],$$

which proves the first inequality in (2.8).

We also have

$$\ell^{1+1/q}(\gamma_{u,v}) \left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \ell^{1+1/q}(\gamma_{v,w}) \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p}$$
$$\leq \max\left\{\ell^{1+1/q}(\gamma_{u,v}), \ell^{1+1/q}(\gamma_{v,w})\right\}$$
$$\times \left[\left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p}\right]$$
$$= \left[\max\left\{\ell(\gamma_{u,v}), \ell(\gamma_{v,w})\right\}\right]^{1+1/q} \left[\left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p}\right]$$
$$= \frac{1}{2^{1+1/q}} \left[\ell(\gamma_{u,w}) + \left|\ell(\gamma_{u,v}) - \ell(\gamma_{v,w})\right|\right]^{1+1/q}$$
$$\times \left[\left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p}\right]$$

and

$$\ell^{1+1/q}\left(\gamma_{u,v}\right)\left(\int_{\gamma_{u,v}}\left|f'(z)\right|^{p}|dz|\right)^{1/p}+\ell^{1+1/q}\left(\gamma_{v,w}\right)\left(\int_{\gamma_{v,w}}\left|f'(z)\right|^{p}|dz|\right)^{1/p}$$
$$\leq\max\left\{\left(\int_{\gamma_{u,v}}\left|f'(z)\right|^{p}|dz|\right)^{1/p},\left(\int_{\gamma_{v,w}}\left|f'(z)\right|^{p}|dz|\right)^{1/p}\right\}$$
$$\times\left[\ell^{1+1/q}\left(\gamma_{u,v}\right)+\ell^{1+1/q}\left(\gamma_{v,w}\right)\right]$$
$$=\left[\max\left\{\int_{\gamma_{u,v}}\left|f'(z)\right|^{p}|dz|,\int_{\gamma_{v,w}}\left|f'(z)\right|^{p}|dz|\right\}\right]^{1/p}$$
$$\times\left[\ell^{1+1/q}\left(\gamma_{u,v}\right)+\ell^{1+1/q}\left(\gamma_{v,w}\right)\right]$$
$$=\frac{1}{2^{1/p}}\left[\int_{\gamma_{u,w}}\left|f'(z)\right|^{p}|dz|+\left|\int_{\gamma_{u,v}}\left|f'(z)\right|^{p}|dz|-\int_{\gamma_{v,w}}\left|f'(z)\right|^{p}|dz|\right|\right]^{1/p}$$
$$\times\left[\ell^{1+1/q}\left(\gamma_{u,v}\right)+\ell^{1+1/q}\left(\gamma_{v,w}\right)\right].$$

By Hölder's discrete inequality we have

$$\ell^{1+1/q}\left(\gamma_{u,v}\right)\left(\int_{\gamma_{u,v}}\left|f'(z)\right|^{p}|dz|\right)^{1/p}+\ell^{1+1/q}\left(\gamma_{v,w}\right)\left(\int_{\gamma_{v,w}}\left|f'(z)\right|^{p}|dz|\right)^{1/p}$$
$$\leq\left[\ell^{q+1}\left(\gamma_{u,v}\right)+\ell^{q+1}\left(\gamma_{v,w}\right)\right]^{1/q}\left[\int_{\gamma_{u,v}}\left|f'(z)\right|^{p}|dz|+\int_{\gamma_{v,w}}\left|f'(z)\right|^{p}|dz|\right]^{1/p}$$
$$=\left[\ell^{q+1}\left(\gamma_{u,v}\right)+\ell^{q+1}\left(\gamma_{v,w}\right)\right]^{1/q}\left(\int_{\gamma_{u,w}}\left|f'(z)\right|^{p}|dz|\right)^{1/p},$$

which prove the last part of (2.8). □

We have:

Corollary 2. *With the assumption of Theorem 3 and if there exists $m\in\gamma$ such that $\ell\left(\gamma_{u,m}\right)=\ell\left(\gamma_{m,w}\right)$, then*

$$\left|\frac{f(u)+f(w)}{2}\ell(\gamma)-\int_{\gamma}f(z)|dz|\right|\leq\frac{1}{2^{1+1/q}(q+1)}\ell^{1+1/q}\left(\gamma_{u,w}\right)\left[\left\|f'\right\|_{\gamma_{u,m};p}+\left\|f'\right\|_{\gamma_{m,w};p}\right]\tag{2.9}$$

and if $s\in\gamma$ such that $\int_{\gamma_{u,s}}\left|f'(z)\right|^{p}|dz|=\int_{\gamma_{s,w}}\left|f'(z)\right|^{p}|dz|$, then

$$\left|f(u)\,\ell\left(\gamma_{u,s}\right)+f(w)\,\ell\left(\gamma_{s,w}\right)-\int_{\gamma}f(z)|dz|\right|\leq\frac{1}{2^{1/p}(q+1)}\left[\ell^{1+1/q}\left(\gamma_{u,s}\right)+\ell^{1+1/q}\left(\gamma_{s,w}\right)\right]\left\|f'\right\|_{\gamma_{u,w};p}.\tag{2.10}$$

Finally we have:

Theorem 4. *With the assumption of Theorem 2 we have*

$$\left|f(u)\,\ell\left(\gamma_{u,v}\right)+f(w)\,\ell\left(\gamma_{v,w}\right)-\int_{\gamma} f(z)\,|dz|\right| \tag{2.11}$$

$$\leq \frac{1}{2}\left[\ell^{2}\left(\gamma_{u,v}\right)\|f'\|_{\gamma_{u,v};\infty}+\ell^{2}\left(\gamma_{v,w}\right)\|f'\|_{\gamma_{v,w};\infty}\right]$$

$$\leq \frac{1}{2}\begin{cases} \frac{1}{4}\left[\ell\left(\gamma_{u,w}\right)+\left|\ell\left(\gamma_{u,v}\right)-\ell\left(\gamma_{v,w}\right)\right|\right]^{2} \\ \times\left[\|f'\|_{\gamma_{u,v};\infty}+\|f'\|_{\gamma_{v,w};\infty}\right], \\ \\ \left[\ell^{2q}\left(\gamma_{u,v}\right)+\ell^{2q}\left(\gamma_{v,w}\right)\right]^{1/q}\left(\|f'\|_{\gamma_{u,v};\infty}^{p}+\|f'\|_{\gamma_{v,w};\infty}^{p}\right)^{1/p}, \\ p,\ q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left[\ell^{2}\left(\gamma_{u,v}\right)+\ell^{2}\left(\gamma_{v,w}\right)\right]\|f'\|_{\gamma_{u,w};\infty}. \end{cases}$$

Proof. We have

$$\begin{aligned} &\int_{a}^{x}\left|f'(z(t))\right|\left|z'(t)\right|\left(\int_{t}^{x}\left|z'(s)\right|ds\right)dt \\ &\qquad\leq \max_{t\in[a,x]}\left|f'(z(t))\right|\int_{a}^{x}\left(\int_{t}^{x}\left|z'(s)\right|ds\right)\left|z'(t)\right|dt \\ &\qquad=\frac{1}{2}\max_{t\in[a,x]}\left|f'(z(t))\right|\left(\int_{a}^{x}\left|z'(s)\right|ds\right)^{2}=\frac{1}{2}\ell^{2}\left(\gamma_{u,v}\right)\|f'\|_{\gamma_{u,v};\infty} \end{aligned}$$

and

$$\begin{aligned} &\int_{x}^{b}\left|f'(z(t))\right|\left|z'(t)\right|\left(\int_{x}^{t}\left|z'(s)\right|ds\right)dt \\ &\qquad\leq \max_{t\in[x,b]}\left|f'(z(t))\right|\int_{x}^{b}\left(\int_{x}^{t}\left|z'(s)\right|ds\right)\left|z'(t)\right|dt \\ &\qquad=\frac{1}{2}\max_{t\in[x,b]}\left|f'(z(t))\right|\left(\int_{x}^{b}\left|z'(s)\right|ds\right)^{2}=\frac{1}{2}\ell^{2}\left(\gamma_{v,w}\right)\|f'\|_{\gamma_{v,w};\infty}, \end{aligned}$$

which by addition produce

$$B(x)\leq \frac{1}{2}\left[\ell^{2}\left(\gamma_{u,v}\right)\|f'\|_{\gamma_{u,v};\infty}+\ell^{2}\left(\gamma_{v,w}\right)\|f'\|_{\gamma_{v,w};\infty}\right]$$

and by utilising the inequality (2.5) we get the first part of (2.11).

The second part is obvious and we omit the details. □

Corollary 3. *With the assumption of Theorem 2 and if there exists $m\in\gamma$ such that $\ell\left(\gamma_{u,m}\right)=\ell\left(\gamma_{m,w}\right)$, then*

$$\left|\frac{f(u)+f(w)}{2}\ell(\gamma)-\int_{\gamma} f(z)\,|dz|\right| \tag{2.12}$$

$$\leq \frac{1}{8}\left[\|f'\|_{\gamma_{u,m};\infty}+\|f'\|_{\gamma_{m,w};\infty}\right]\ell^{2}\left(\gamma_{u,w}\right)\leq \frac{1}{4}\|f'\|_{\gamma_{u,w};\infty}\,\ell^{2}\left(\gamma_{u,w}\right).$$

3. Examples for Circular Paths

Let $[a,b] \subseteq [0,2\pi]$ and the circular path $\gamma_{[a,b],R}$ centered in 0 and with radius $R>0$

$$z(t) = R\exp(it) = R(\cos t + i\sin t), \ t \in [a,b].$$

If $[a,b] = [0,\pi]$ then we get a half circle while for $[a,b] = [0,2\pi]$ we get the full circle.

We have

$$z'(t) = Ri\exp(it), \ t \in [a,b]$$

and $|z'(t)| = R$ for $t \in [a,b]$ giving that

$$\ell\left(\gamma_{[a,b],R}\right) = \int_a^b |z'(t)|\,dt = R(b-a).$$

If $x \in [a,b]$ and $v = R\exp(ix)$, then by (2.1) we have

$$\left|R(x-a)f(R\exp(ia)) + R(b-x)f(R\exp(ib)) - R\int_a^b f(R\exp(it))\,dt\right|$$
$$\leq R(x-a)R\int_a^x |f'(R\exp(it))|\,dt + R(b-x)R\int_x^b |f'(R\exp(it))|\,dt$$

that is equivalent to

$$\left|(x-a)f(R\exp(ia)) + (b-x)f(R\exp(ib)) - \int_a^b f(R\exp(it))\,dt\right| \tag{3.1}$$
$$\leq R\left[(x-a)\int_a^x |f'(R\exp(it))|\,dt + (b-x)\int_x^b |f'(R\exp(it))|\,dt\right]$$

for $x \in [a,b]$.

In particular, if we take $x = \frac{a+b}{2}$ in (3.1), then we get

$$\left|\frac{f(R\exp(ia)) + f(R\exp(ib))}{2}(b-a) - \int_a^b f(R\exp(it))\,dt\right| \tag{3.2}$$
$$\leq \frac{1}{2}R(b-a)\int_a^b |f'(R\exp(it))|\,dt.$$

If $m \in [a,b]$ is such that

$$\int_a^m |f'(R\exp(it))|\,dt = \int_m^b |f'(R\exp(it))|\,dt,$$

then from (3.1) we get

$$\left|(m-a)f(R\exp(ia)) + (b-m)f(R\exp(ib)) - \int_a^b f(R\exp(it))\,dt\right| \tag{3.3}$$
$$\leq \frac{1}{2}R(b-a)\int_a^b |f'(R\exp(it))|\,dt.$$

By making use of (2.8) we get

$$\left|R(x-a)f(R\exp(ia))+R(b-x)f(R\exp(ib))-R\int_a^b f(R\exp(it))\,dt\right|$$
$$\leq \frac{1}{q+1}\left[R^{1+1/q}(x-a)^{1+1/q}R^{1/p}\left(\int_a^x |f'(R\exp(it))|^p\,dt\right)^{1/p}\right.$$
$$\left.+R^{1+1/q}(b-x)^{1+1/q}R^{1/p}\left(\int_x^b |f'(R\exp(it))|^p\,dt\right)^{1/p}\right]$$

that is equivalent to

$$\left|(x-a)f(R\exp(ia))+(b-x)f(R\exp(ib))-\int_a^b f(R\exp(it))\,dt\right| \tag{3.4}$$
$$\leq \frac{1}{q+1}R\left[(x-a)^{1+1/q}\left(\int_a^x |f'(R\exp(it))|^p\,dt\right)^{1/p}\right.$$
$$\left.+(b-x)^{1+1/q}\left(\int_x^b |f'(R\exp(it))|^p\,dt\right)^{1/p}\right]$$

for $x\in[a,b]$.

If $c\in[a,b]$ is such that

$$\int_a^c |f'(R\exp(it))|^p\,dt=\int_c^b |f'(R\exp(it))|^p\,dt,$$

then by (3.4) we get

$$\left|(c-a)f(R\exp(ia))+(b-c)f(R\exp(ib))-\int_a^b f(R\exp(it))\,dt\right| \tag{3.5}$$
$$\leq \frac{1}{(q+1)\,2^{1/p}}R\left[(c-a)^{1+1/q}+(b-c)^{1+1/q}\right]\left(\int_a^b |f'(R\exp(it))|^p\,dt\right)^{1/p}.$$

Further, if we use (2.11), then we have

$$\left|R(x-a)f(R\exp(ia))+R(b-x)f(R\exp(ib))-R\int_a^b f(R\exp(it))\,dt\right|$$
$$\leq \frac{1}{2}R^2\left[(x-a)^2\sup_{t\in[a,x]}|f'(R\exp(it))|+(b-x)^2\sup_{t\in[x,b]}|f'(R\exp(it))|\right]$$
$$\leq \frac{1}{2}R^2\left[(x-a)^2+(b-x)^2\right]\sup_{t\in[a,b]}|f'(R\exp(it))|$$

that is equivalent to

$$\left|(x-a)\, f\left(R\exp\left(ia\right)\right)+(b-x)\, f\left(R\exp\left(ib\right)\right)-\int_a^b f\left(R\exp\left(it\right)\right)dt\right| \tag{3.6}$$
$$\leq \frac{1}{2}R\left[(x-a)^2\sup_{t\in[a,x]}\left|f'\left(R\exp\left(it\right)\right)\right|+(b-x)^2\sup_{t\in[x,b]}\left|f'\left(R\exp\left(it\right)\right)\right|\right]$$
$$\leq R\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\sup_{t\in[a,b]}\left|f'\left(R\exp\left(it\right)\right)\right|$$

for $x\in[a,b]$.

In particular, we have

$$\left|(b-a)\, f\left(R\exp\left(\frac{a+b}{2}i\right)\right)-\int_a^b f\left(R\exp\left(it\right)\right)dt\right| \tag{3.7}$$
$$\leq \frac{1}{8}R(b-a)^2\left[\sup_{t\in\left[a,\frac{a+b}{2}\right]}\left|f'\left(R\exp\left(it\right)\right)\right|+\sup_{t\in\left[\frac{a+b}{2},b\right]}\left|f'\left(R\exp\left(it\right)\right)\right|\right]$$
$$\leq \frac{1}{4}R(b-a)^2\sup_{t\in[a,b]}\left|f'\left(R\exp\left(it\right)\right)\right|.$$

We give now examples for some fundamental complex functions.

Consider the function $f(z)=z^n$, $z\in\mathbb{C}$ with $n\geq 1$. Then $f'(z)=nz^{n-1}$,

$$f\left(R\exp\left(ix\right)\right)=R^n\exp\left(nxi\right),$$
$$\left|f'\left(R\exp\left(it\right)\right)\right|=nR^{n-1}\left|\exp\left((n-1)\, it\right)\right|=nR^{n-1},\ t\in[a,b]$$

and

$$\int_a^b f\left(R\exp\left(it\right)\right)dt=R^n\int_a^b\exp\left(nti\right)dt=R^n\frac{\exp\left(nbi\right)-\exp\left(nai\right)}{ni}.$$

Making use of the inequality (3.6) we get

$$\left|(x-a)\, R^n\exp\left(nai\right)+(b-x)\, R^n\exp\left(nbi\right)-R^n\frac{\exp\left(nbi\right)-\exp\left(nai\right)}{ni}\right|$$
$$\leq R\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]nR^{n-1},$$

which is equivalent to

$$\left|(x-a)\exp\left(nai\right)+(b-x)\exp\left(nbi\right)-\frac{\exp\left(nbi\right)-\exp\left(nai\right)}{ni}\right| \tag{3.8}$$
$$\leq n\left[\frac{1}{2}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right],$$

for $x\in[a,b]$.

If we take in (3.8) $x=\frac{a+b}{2}$, then we get

$$\left|\frac{\exp\left(nai\right)+\exp\left(nbi\right)}{2}(b-a)-\frac{\exp\left(nbi\right)-\exp\left(nai\right)}{ni}\right|\leq\frac{1}{4}n(b-a)^2, \tag{3.9}$$

for $n \geq 1$.

Consider the exponential function $f(z) = \exp(z)$, $z \in \mathbb{C}$. Then $f'(z) = \exp(z)$,

$$|f'(R\exp(it))| = |\exp(R(\cos t + i\sin t))| = \exp(R\cos t), \ t \in [a,b]$$

and by the inequality and by the inequality (3.1) we get

$$\left|(x-a)\exp(R\exp(ia)) + (b-x)\exp(R\exp(ib)) - \int_a^b \exp(R\exp(it))\,dt\right| \\ \leq R\left[(x-a)\int_a^x \exp(R\cos t)\,dt + (b-x)\int_x^b \exp(R\cos t)\,dt\right], \tag{3.10}$$

while from the inequality (3.6) we get

$$\left|(x-a)\exp(R\exp(ia)) + (b-x)\exp(R\exp(ib)) - \int_a^b \exp(R\exp(it))\,dt\right| \\ \leq \frac{1}{2}R\left[(x-a)^2\sup_{t\in[a,x]}\exp(R\cos t) + (b-x)^2\sup_{t\in[x,b]}\exp(R\cos t)\right] \\ \leq R\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right]\sup_{t\in[a,b]}\exp(R\cos t) \tag{3.11}$$

for $x \in [a,b]$.

From the inequality (3.4) we get

$$\left|(x-a)\exp(R\exp(ia)) + (b-x)\exp(R\exp(ib)) - \int_a^b \exp(R\exp(it))\,dt\right| \\ \leq \frac{1}{q+1}R\left[(x-a)^{1+1/q}\left(\int_a^x \exp(pR\cos t)\,dt\right)^{1/p}\right. \\ \left. + (b-x)^{1+1/q}\left(\int_x^b \exp(pR\cos t)\,dt\right)^{1/p}\right] \tag{3.12}$$

for $x \in [a,b]$, where $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

If in the inequality (3.10) we take $x = \frac{a+b}{2}$, then we get

$$\left|\frac{\exp(R\exp(ia)) + \exp(R\exp(ib))}{2}(b-a) - \int_a^b \exp(R\exp(it))\,dt\right| \\ \leq \frac{1}{2}R(b-a)\int_a^b \exp(R\cos t)\,dt, \tag{3.13}$$

while from the inequality (2.10) we get

$$\left|\frac{\exp(R\exp(ia))+\exp(R\exp(ib))}{2}(b-a)-\int_a^b\exp(R\exp(it))\,dt\right| \tag{3.14}$$
$$\leq\frac{1}{8}(b-a)^2R\left[\sup_{t\in\left[a,\frac{a+b}{2}\right]}\exp(R\cos t)+\sup_{t\in\left[\frac{a+b}{2},b\right]}\exp(R\cos t)\right]$$
$$\leq\frac{1}{4}R(b-a)^2\sup_{t\in[a,b]}\exp(R\cos t).$$

From (3.12) we have

$$\left|\frac{\exp(R\exp(ia))+\exp(R\exp(ib))}{2}(b-a)-\int_a^b\exp(R\exp(it))\,dt\right| \tag{3.15}$$
$$\leq\frac{1}{(q+1)2^{1+1/q}}R(b-a)^{1+1/q}$$
$$\times\left[\left(\int_a^{\frac{a+b}{2}}\exp(pR\cos t)\,dt\right)^{1/p}+\left(\int_{\frac{a+b}{2}}^b\exp(pR\cos t)\,dt\right)^{1/p}\right].$$

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