OSTROWSKI'S INEQUALITY FOR THE COMPLEX INTEGRAL OF HOLOMORPHIC FUNCTIONS ON CONVEX DOMAINS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we extend the Ostrowski inequality to the complex integral, by providing upper bounds for the quantity

$$\left| f(v)(w-u) - \int_\gamma f(z)\,dz \right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$, $u = z(a)$ and $w = z(b)$ while f is holomorphic in G, convex domain, $\gamma \subset G$ and $z \in G$. Applications for some particular functions of interest are also given.

1. Introduction

In 1938, A. Ostrowski [8], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x \in [a,b]$.

Theorem 1 (Ostrowski, 1938 [8]). *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a), \tag{1.1}$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

For extensions of Ostrowski's inequality in terms of the p-norms of the derivative, see [1], [2] and [3]. For a recent survey on Ostrowski's inequality, see [4].

In order to extend this result for the complex integral, we need some preparations as follows.

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$

1991 *Mathematics Subject Classification.* 26D15, 26D10, 30A10, 30A86.

Key words and phrases. Complex integral, Continuous functions, Holomorphic functions, Ostrowski inequality.

RGMIA Res. Rep. Coll. 21 (2018), Art. 116, 12 pp. Received 27/09/18

and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, and open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.2}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.3}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\,|dz|\right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|.$$

If $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}.$$

In the recent paper [5] we obtained the following result for functions of complex variable:

Theorem 2. *Let f be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(x)$ with $x \in (a,b)$, then $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$,*

$$\begin{aligned} &\left|f(v)(w-u) - \int_\gamma f(z)\,dz\right| \\ &\le \|f'\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-u|\,|dz| + \|f'\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-w|\,|dz| \\ &\le \left[\int_{\gamma_{u,v}} |z-u|\,|dz| + \int_{\gamma_{v,w}} |z-w|\,|dz|\right] \|f'\|_{\gamma_{u,w};\infty} \end{aligned} \tag{1.4}$$

and

$$\left|f(v)(w-u)-\int_{\gamma}f(z)\,dz\right| \tag{1.5}$$
$$\leq \max_{z\in\gamma_{u,v}}|z-u|\,\|f'\|_{\gamma_{u,v};1}+\max_{z\in\gamma_{v,w}}|z-w|\,\|f'\|_{\gamma_{v,w};1}$$
$$\leq \max\left\{\max_{z\in\gamma_{u,v}}|z-u|\,,\,\max_{z\in\gamma_{v,w}}|z-w|\right\}\|f'\|_{\gamma_{u,w};1}\,.$$

If $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\left|f(v)(w-u)-\int_{\gamma}f(z)\,dz\right| \tag{1.6}$$
$$\leq\left(\int_{\gamma_{u,v}}|z-u|^{q}\,|dz|\right)^{1/q}\|f'\|_{\gamma_{u,v};p}+\left(\int_{\gamma_{v,w}}|z-w|^{q}\,|dz|\right)^{1/q}\|f'\|_{\gamma_{v,w};p}$$
$$\leq\left(\int_{\gamma_{u,v}}|z-u|^{q}\,|dz|+\int_{\gamma_{v,w}}|z-w|^{q}\,|dz|\right)^{1/q}\|f'\|_{\gamma_{u,w};p}\,.$$

In this paper we extend the Ostrowski inequality to the complex integral, by providing upper bounds for the quantity

$$\left|f(v)(w-u)-\int_{\gamma}f(z)\,dz\right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t\in[a,b]$, $u=z(a)$ and $w=z(b)$ while f is holomorphic in G, convex domain, $\gamma\subset G$ and $z\in G$. Applications for some particular functions of interest are also given.

2. Ostrowski Type Inequalities

We have:

Theorem 3. *Let* $f:D\subseteq\mathbb{C}\to\mathbb{C}$ *be a holomorphic function on the convex domain* D *and suppose* $\gamma\subset D$ *is a piecewise smooth path parametrized by* $z(t)$, $t\in[a,b]$, $u=z(a)$ *and* $w=z(b)$. *If* $v\in D$, *then*

$$\left|(w-u)f(v)-\int_{\gamma}f(z)\,dz\right| \tag{2.1}$$
$$\leq\int_{\gamma}|v-z|\left|\int_0^1 f'((1-t)z+tv)\,dt\right||dz|$$
$$\leq\begin{cases}\max_{z\in\gamma}|v-z|\int_{\gamma}\left|\int_0^1 f'((1-t)z+tv)\,dt\right||dz|\,;\\[2ex] \left(\int_{\gamma}|v-z|^{p}\,|dz|\right)^{1/p}\left(\int_{\gamma}\left|\int_0^1 f'((1-t)z+tv)\,dt\right|^{q}|dz|\right)^{1/q}\\ p,\ q>1\ \textit{with}\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \int_{\gamma}|v-z|\,|dz|\max_{z\in\gamma}\left|\int_0^1 f'((1-t)z+tv)\,dt\right|\end{cases}$$

and

$$(2.2)\quad \left|(w-u)\,f(v)-\int_\gamma f(z)\,dz\right| \le \int_0^1\left(\int_\gamma |v-z|\,|f'((1-t)\,z+tv)|\,|dz|\right)dt$$

$$\le\begin{cases}\max_{z\in\gamma}|v-z|\int_0^1\left(\int_\gamma |f'((1-t)\,z+tv)|\,|dz|\right)dt;\\[2ex] \left(\int_\gamma |v-z|^p\,|dz|\right)^{1/p}\int_0^1\left(\int_\gamma |f'((1-t)\,z+tv)|^q\,|dz|\right)^{1/q}dt\\ p,\ q>1\ \textit{with}\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \int_\gamma |v-z|\,|dz|\int_0^1 \max_{z\in\gamma}|f'((1-t)\,z+tv)|\,dt.\end{cases}$$

Proof. Due to the convexity of D, for any $z,\ v\in D$ we can define the function $\varphi_{z,v}:[0,1]\to\mathbb{R}$ by $\varphi_{z,v}(t):=f((1-t)\,z+tv)$. The function $\varphi_{z,v}$ is differentiable on $(0,1)$ and

$$\frac{d\varphi_{z,v}(t)}{dt}=(v-z)\,f'((1-t)\,z+tv)\ \text{ for } t\in(0,1).$$

We have

$$f(v)-f(z)=\varphi_{z,v}(1)-\varphi_{z,v}(0)=\int_0^1\frac{d\varphi_{z,v}(t)}{dt}dt = (v-z)\int_0^1 f'((1-t)\,z+tv)\,dt$$

for any $z,\ v\in D$.

Integrating over z on γ we have

$$f(v)\int_\gamma dz-\int_\gamma f(z)\,dz=\int_\gamma (v-z)\left(\int_0^1 f'((1-t)\,z+tv)\,dt\right)dz$$

namely

$$(w-u)\,f(v)-\int_\gamma f(z)\,dz=\int_\gamma (v-z)\left(\int_0^1 f'((1-t)\,z+tv)\,dt\right)dz$$

and by Fubini theorem, we get the equality of interest

$$(2.3)\quad (w-u)\,f(v)-\int_\gamma f(z)\,dz=\int_\gamma (v-z)\left(\int_0^1 f'((1-t)\,z+tv)\,dt\right)dz = \int_0^1\left(\int_\gamma (v-z)\,f'((1-t)\,z+tv)\,dz\right)dt.$$

Taking the modulus in the first equality in (2.3), we get ||

$$(2.4)\quad \left|(w-u)\,f(v)-\int_\gamma f(z)\,dz\right|\le\left|\int_\gamma (v-z)\left(\int_0^1 f'((1-t)\,z+tv)\,dt\right)dz\right| \le \int_\gamma |v-z|\left|\int_0^1 f'((1-t)\,z+tv)\,dt\right||dz|.$$

Using Hölder's inequality we also have

$$\int_\gamma |v-z| \left| \int_0^1 f'((1-t)z+tv)\,dt \right| |dz|$$
$$\leq \begin{cases} \max_{z\in\gamma} |v-z| \int_\gamma \left| \int_0^1 f'((1-t)z+tv)\,dt \right| |dz|; \\ \\ \left(\int_\gamma |v-z|^p |dz| \right)^{1/p} \left(\int_\gamma \left| \int_0^1 f'((1-t)z+tv)\,dt \right|^q |dz| \right)^{1/q} \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \max_{z\in\gamma} \left| \int_0^1 f'((1-t)z+tv)\,dt \right| \int_\gamma |v-z|\,|dz|, \end{cases}$$

which proves the inequality (2.1).

Using the second equality in (2.3) we get

$$\left| (w-u) f(v) - \int_\gamma f(z)\,dz \right| \leq \int_0^1 \left| \int_\gamma (v-z) f'((1-t)z+tv)\,dz \right| dt \tag{2.5}$$
$$\leq \int_0^1 \left(\int_\gamma |v-z|\,|f'((1-t)z+tv)|\,|dz| \right) dt.$$

Using Hölder's inequality, we have

$$\int_\gamma |v-z|\,|f'((1-t)z+tv)|\,|dz| \tag{2.6}$$
$$\leq \begin{cases} \max_{z\in\gamma} |v-z| \int_\gamma |f'((1-t)z+tv)|\,|dz|; \\ \\ \left(\int_\gamma |v-z|^p |dz| \right)^{1/p} \left(\int_\gamma |f'((1-t)z+tv)|^q |dz| \right)^{1/q} \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \int_\gamma |v-z|\,|dz| \max_{z\in\gamma} |f'((1-t)z+tv)| \end{cases}$$

for $t\in[0,1]$.

Integrating the inequality (2.6) over $t\in[0,1]$, we obtain

$$\int_0^1 \left(\int_\gamma |v-z|\,|f'((1-t)z+tv)|\,|dz| \right) dt$$
$$\leq \begin{cases} \max_{z\in\gamma} |v-z| \int_0^1 \left(\int_\gamma |f'((1-t)z+tv)|\,|dz| \right) dt; \\ \\ \left(\int_\gamma |v-z|^p |dz| \right)^{1/p} \int_0^1 \left(\int_\gamma |f'((1-t)z+tv)|^q |dz| \right)^{1/q} dt \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \int_\gamma |v-z|\,|dz| \int_0^1 \max_{z\in\gamma} |f'((1-t)z+tv)|\,dt \end{cases}$$

and by (2.5) we get (2.2). □

Remark 1. *From (2.1) we also have the inequalities in terms of the integrals of modulus*

$$\left|(w-u)f(v)-\int_\gamma f(z)\,dz\right| \tag{2.7}$$
$$\leq \int_\gamma |v-z|\left|\int_0^1 f'((1-t)z+tv)\,dt\right||dz|$$
$$\leq \int_\gamma |v-z|\left(\int_0^1 |f'((1-t)z+tv)|\,dt\right)|dz|$$
$$\leq \begin{cases} \max_{z\in\gamma}|v-z|\int_\gamma\left(\int_0^1|f'((1-t)z+tv)|\,dt\right)|dz|; \\ \\ \left(\int_\gamma |v-z|^p|dz|\right)^{1/p}\left(\int_\gamma\left(\int_0^1|f'((1-t)z+tv)|^q\,dt\right)|dz|\right)^{1/q} \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \int_\gamma|v-z||dz|\max_{z\in\gamma}\int_0^1|f'((1-t)z+tv)|\,dt. \end{cases}$$

If $\|f'\|_{D,\infty}:=\sup_{z\in D}|f'(z)|<\infty$*, then we also have*

$$\left|(w-u)f(v)-\int_\gamma f(z)\,dz\right| \tag{2.8}$$
$$\leq \int_\gamma |v-z|\left|\int_0^1 f'((1-t)z+tv)\,dt\right||dz|$$
$$\leq \int_\gamma |v-z|\left(\int_0^1 |f'((1-t)z+tv)|\,dt\right)|dz| \leq \|f'\|_{D,\infty}\int_\gamma|v-z||dz|.$$

Corollary 1. *With the assumptions of Theorem 3 and if* $|f'|$ *is convex on* D*, then*

$$\left|(w-u)f(v)-\int_\gamma f(z)\,dz\right| \tag{2.9}$$
$$\leq \frac{1}{2}\left[\int_\gamma |v-z||f'(z)||dz|+|f'(v)|\int_\gamma|v-z||dz|\right]$$
$$\leq \frac{1}{2}\left[\|f'\|_{D,\infty}+|f'(v)|\right]\int_\gamma|v-z||dz|\leq\|f'\|_{D,\infty}\int_\gamma|v-z||dz|$$

provided

$$\|f'\|_{D,\infty}:=\sup_{z\in D}|f'(z)|<\infty.$$

Proof. If $g:[0,1]\to\mathbb{R}$ is convex, then the following inequality is well known in the literature as Hermite-Hadamard inequality

$$\int_0^1 g(t)\,dt\leq\frac{g(0)+g(1)}{2}.$$

Let $v\in D$ and $z\in\gamma$. By Hermite-Hadamard inequality for the convex function $[0,1]\ni t\to|f'((1-t)z+tv)|$ we have

$$\int_0^1|f'((1-t)z+tv)|\,dt\leq\frac{1}{2}\left[|f'(z)|+|f'(v)|\right],$$

which implies that

$$\begin{aligned}&\int_{\gamma}|v-z|\left(\int_{0}^{1}\left|f'\left((1-t)z+tv\right)\right|dt\right)|dz|\\&\leq\frac{1}{2}\int_{\gamma}|v-z|\left[\left|f'(z)\right|+\left|f'(v)\right|\right]|dz|\\&=\frac{1}{2}\left[\int_{\gamma}|v-z|\left|f'(z)\right||dz|+\left|f'(v)\right|\int_{\gamma}|v-z||dz|\right]\end{aligned}$$

and by (2.7) we get (2.9). □

We also have:

Corollary 2. *With the assumptions of Theorem 3 and if* $|f'|^q$ *with* $q>1$ *is convex on* D, *then*

$$\left|(w-u)f(v)-\int_{\gamma}f(z)\,dz\right|\leq\frac{1}{2^{1/q}}\left(\int_{\gamma}|v-z|^{p}|dz|\right)^{1/p}\left(\left|f'(v)\right|^{q}\ell(\gamma)+\int_{\gamma}\left|f'(z)\right|^{q}|dz|\right)^{1/q},\tag{2.10}$$

where $p>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using power inequality for integral and the convexity of $|f'|^q$, with $q>1$, we have

$$\begin{aligned}\int_{0}^{1}\left|f'\left((1-t)z+tv\right)\right|dt&\leq\left(\int_{0}^{1}\left|f'\left((1-t)z+tv\right)\right|^{q}dt\right)^{1/q}\\&\leq\left(\frac{\left|f'(z)\right|^{q}+\left|f'(v)\right|^{q}}{2}\right)^{1/q}\end{aligned}$$

for $v\in D$ and $z\in\gamma$.

Using (2.7) we get

$$\begin{aligned}&\left|(w-u)f(v)-\int_{\gamma}f(z)\,dz\right|\\&\leq\int_{\gamma}|v-z|\left(\int_{0}^{1}\left|f'\left((1-t)z+tv\right)\right|dt\right)|dz|\\&\leq\int_{\gamma}|v-z|\left(\frac{\left|f'(z)\right|^{q}+\left|f'(v)\right|^{q}}{2}\right)^{1/q}|dz|\\&\leq\left(\int_{\gamma}|v-z|^{p}|dz|\right)^{1/p}\left(\int_{\gamma}\left[\left(\frac{\left|f'(z)\right|^{q}+\left|f'(v)\right|^{q}}{2}\right)^{1/q}\right]^{q}|dz|\right)^{1/q}\\&=\left(\int_{\gamma}|v-z|^{p}|dz|\right)^{1/p}\left(\int_{\gamma}\frac{\left|f'(z)\right|^{q}+\left|f'(v)\right|^{q}}{2}|dz|\right)^{1/q}\\&=\left(\int_{\gamma}|v-z|^{p}|dz|\right)^{1/p}\left(\frac{1}{2}\int_{\gamma}\left|f'(z)\right|^{q}|dz|+\frac{1}{2}\left|f'(v)\right|^{q}\ell(\gamma)\right)^{1/q}\end{aligned}\tag{2.11}$$

for $v\in D$. □

For $z\in\mathbb{C}$ we have

$$\begin{aligned}
|\exp(z)| &= |\exp(\operatorname{Re} z + i\operatorname{Im} z)| = |\exp(\operatorname{Re} z)\exp(i\operatorname{Im} z)| \\
&= |\exp(\operatorname{Re} z)|\,|\exp(i\operatorname{Im} z)| = \exp(\operatorname{Re} z)\,|\cos(\operatorname{Im} z) + i\sin(\operatorname{Im} z)| \\
&= \exp(\operatorname{Re} z).
\end{aligned}$$

Then for any $t\in[0,1]$ and for any $z,\ w\in\mathbb{C}$ we have

$$\begin{aligned}
|\exp((1-t)z+tw)|^{\alpha} &= \exp[\alpha(\operatorname{Re}((1-t)z+tw))] \\
&= \exp[(1-t)\alpha\operatorname{Re} z + t\alpha\operatorname{Re} w] \\
&\le (1-t)\exp(\alpha\operatorname{Re} z) + t\exp(\alpha\operatorname{Re} w) \\
&= (1-t)|\exp(z)|^{\alpha} + t|\exp(w)|^{\alpha}
\end{aligned}$$

which shows that the function $g(z)=|\exp(z)|^{\alpha}$ is convex for any $\alpha\in\mathbb{R}\setminus\{0\}$.

Suppose $\gamma\subset D$ is a piecewise smooth path parametrized by $z(t),\ t\in[a,b]$, $u=z(a)$ and $w=z(b)$. We also have for $\gamma=\gamma_{u,w}$ that

$$\int_{\gamma}\exp(z)\,dz = \int_{\gamma_{u,w}}\exp(z)\,dz = \exp(w)-\exp(u).$$

Using the inequality (2.9) we get

$$\begin{aligned}
&|(w-u)\exp(v)-\exp(w)+\exp(u)| \\
&\quad\le \frac{1}{2}\left[\int_{\gamma}|v-z|\exp(\operatorname{Re} z)|dz| + \exp(\operatorname{Re} v)\int_{\gamma}|v-z|\,|dz|\right] \\
&\le \frac{1}{2}\left[\|\exp\|_{D,\infty} + \exp(\operatorname{Re} v)\right]\int_{\gamma}|v-z|\,|dz| \le \|\exp\|_{D,\infty}\int_{\gamma}|v-z|\,|dz|
\end{aligned}\tag{2.12}$$

for any $v\in\mathbb{C}$.

By using the inequality (2.10), we get

$$\begin{aligned}
&|(w-u)\exp(v)-\exp(w)+\exp(u)| \\
&\le \frac{1}{2^{1/q}}\left(\int_{\gamma}|v-z|^{p}|dz|\right)^{1/p}\left(\exp(q\operatorname{Re} v)\,\ell(\gamma) + \int_{\gamma}\exp(q\operatorname{Re} z)|dz|\right)^{1/q},
\end{aligned}\tag{2.13}$$

where $p>1$ with $\frac{1}{p}+\frac{1}{q}=1$, for any $v\in\mathbb{C}$.

3. Related Results

Now, by the help of power series $f(z)=\sum_{n=0}^{\infty}a_n z^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficients of the original series, namely, $f_a(z)=\sum_{n=0}^{\infty}|a_n|z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $a_n\ge 0$ then $f_a=f$.

We notice that if

$$f(z)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^n=\ln\frac{1}{1+z},\ z\in D(0,1);\tag{3.1}$$

$$g(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}=\cos z,\ z\in\mathbb{C};$$

$$h(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}=\sin z,\ z\in\mathbb{C};$$

$$l(z)=\sum_{n=0}^{\infty}(-1)^n z^n=\frac{1}{1+z},\ z\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(z)=\sum_{n=1}^{\infty}\frac{1}{n!}z^n=\ln\frac{1}{1-z},\ z\in D(0,1);\tag{3.2}$$

$$g_a(z)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n}=\cosh z,\ z\in\mathbb{C};$$

$$h_a(z)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1}=\sinh z,\ z\in\mathbb{C};$$

$$l_a(z)=\sum_{n=0}^{\infty}z^n=\frac{1}{1-z},\ z\in D(0,1).$$

Theorem 4. *Consider the power series $f(z)=\sum_{n=0}^{\infty}a_nz^n$ that is convergent on the open disk $D(0,R)$ and suppose $\gamma\subset D(0,R)$ is a piecewise smooth path parametrized by $z(t)$, $t\in[a,b]$, $u=z(a)$ and $w=z(b)$. If $v\in D(0,R)$, then we have the inequalities*

$$\left|(w-u)f(v)-\int_\gamma f(z)\,dz\right|\le\frac{1}{2}\left[|f_a'(v)|\,\ell(\gamma)+\int_\gamma|v-z|\,|f_a'(z)|\,|dz|\right]\tag{3.3}$$

and

$$\left|(w-u)f(v)-\int_\gamma f(z)\,dz\right|\le\frac{1}{2}\left[f_a'(|v|)\,\ell(\gamma)+\int_\gamma|v-z|\,f_a'(|z|)\,|dz|\right].\tag{3.4}$$

Proof. We have $f'(z)=\sum_{n=1}^{\infty}na_nz^{n-1}$ and $f_a'(z)=\sum_{n=1}^{\infty}n|a_n|z^{n-1}$. For $m\ge1$, by using the generalized triangle inequality we have

$$\left|\sum_{n=1}^{m}na_nz^{n-1}\right|\le\sum_{n=1}^{m}n|a_n|z^{n-1}.\tag{3.5}$$

Since the series $\sum_{n=1}^{\infty}na_nz^{n-1}$ and $\sum_{n=1}^{\infty}n|a_n|z^{n-1}$ are convergent, then by letting $m\to\infty$ in (3.5) we get

$$|f'(z)|\le f_a'(|z|)\text{ for any }z\in D(0,R).$$

We observe that, since f_a' has nonnegative coefficients, then this functions is convex as a real variable functions on the interval $(-R,R)$ and increasing on $[0,R)$.

For $z, v \in D$, consider the function $h_{z,v} : [0,1] \to [0,\infty)$, $h_{z,v}(t) := f_a'(|(1-t)z + tv|)$. For $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ and $t_1, t_2 \in [0,1]$ we have

$$\begin{aligned} h_{z,v}(\alpha t_1 + \beta t_2) &= f_a'(|(1 - \alpha t_1 - \beta t_2) z + \alpha t_1 + \beta t_2 v|) \\ &= f_a'[|\alpha((1-t_1)z + t_1 v) + \beta((1-t_2)z + t_2 v)|] \\ &\le f_a'[\alpha|(1-t_1)z + t_1 v| + \beta|(1-t_2)z + t_2 v|] \\ &\le \alpha f_a'(|(1-t_1)z + t_1 v|) + \beta f_a'(|(1-t_2)z + t_2 v|), \end{aligned}$$

which shows that $h_{z,v}$ is convex on $[0,1]$.

If we write the Hermite-Hadamard inequality for $h_{z,v}$ on $[0,1]$ then we get

$$\int_0^1 f_a'(|(1-t)z + tv|)\,dt \le \frac{|f_a'(z)| + |f_a'(v)|}{2}$$

for any $z, v \in D$, which implies that

$$\begin{aligned} &\int_\gamma |v - z| \left(\int_0^1 |f'((1-t)z + tv)|\,dt \right) |dz| \\ &\le \int_\gamma |v - z| \left(\int_0^1 f_a'(|(1-t)z + tv|)\,dt \right) |dz| \\ &\le \int_\gamma |v - z| \frac{|f_a'(z)| + |f_a'(v)|}{2} |dz| \\ &= \frac{1}{2} \left[\int_\gamma |v - z| |f_a'(z)| |dz| + |f_a'(v)| \ell(\gamma) \right] \end{aligned}$$

and the inequality (3.3) is proved.

We also have

$$f_a'(|(1-t)z + tv|) \le f_a'((1-t)|z| + t|v|)$$

for any $z, v \in D$ and $t \in [0,1]$ and since the function $p_{z,v}(t) := f_a'((1-t)|z| + t|v|)$ is convex, then by Hermite-Hadamard inequality we have

$$\int_0^1 f_a'(|(1-t)z + tv|)\,dt \le \int_0^1 f_a'((1-t)|z| + t|v|)\,dt \le \frac{f_a'(|z|) + f_a'(|v|)}{2}.$$

This implies that

$$\begin{aligned} &\int_\gamma |v - z| \left(\int_0^1 f_a'(|(1-t)z + tv|)\,dt \right) |dz| \\ &\le \int_\gamma |v - z| \left(\int_0^1 f_a'((1-t)|z| + t|v|)\,dt \right) |dz| \\ &\le \int_\gamma |v - z| \frac{f_a'(|z|) + f_a'(|v|)}{2} |dz| = \frac{1}{2} \left[\int_\gamma |v - z| f_a'(|z|) |dz| + f_a'(|v|) \ell(\gamma) \right], \end{aligned}$$

which proves (3.4). □

Remark 2. *If we consider, for instance $f(z) = \sin z$, then $f_a(z) = \sinh z$, $z \in \mathbb{C}$ and by (3.3) and (3.4) we get*

$$|(w-u)\sin v + \cos w - \cos u| \le \frac{1}{2} \left[|\cosh v| \ell(\gamma) + \int_\gamma |v - z| |\cosh z| |dz| \right] \tag{3.6}$$

and

$$|(w-u)\sin v+\cos w-\cos u|\leq\frac{1}{2}\left[\cosh(|v|)\,\ell(\gamma)+\int_{\gamma}|v-z|\cosh(|z|)\,|dz|\right], \tag{3.7}$$

for any $v\in\mathbb{C}$ *and* $\gamma_{u,w}\subset\mathbb{C}$ *a piecewise smooth path.*

Corollary 3. *If the power series* $f(z)=\sum_{n=0}^{\infty}a_nz^n$ *has nonnegative coefficients and is convergent on the open disk* $D(0,R)$*, then with the other assumptions in Theorem 4 we have*

$$\left|(w-u)f(v)-\int_{\gamma}f(z)\,dz\right|\leq\frac{1}{2}\left[|f'(v)|\,\ell(\gamma)+\int_{\gamma}|v-z|\,|f'(z)|\,|dz|\right] \tag{3.8}$$

and

$$\left|(w-u)f(v)-\int_{\gamma}f(z)\,dz\right|\leq\frac{1}{2}\left[f'(|v|)\,\ell(\gamma)+\int_{\gamma}|v-z|\,f'(|z|)\,|dz|\right]. \tag{3.9}$$

Important examples of functions as power series representations with nonnegative coefficients in addition to the ones from (3.2), are:

$$\exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n,\ z\in\mathbb{C}; \tag{3.10}$$

$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1);$$

$$\sin^{-1}(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(2n+1)\,n!}z^{2n+1},\ z\in D(0,1);$$

$$\tanh^{-1}(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1);$$

$${}_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\,\Gamma(n+\beta)\,\Gamma(\gamma)}{n!\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0,$$
$$z\in D(0,1);$$

where Γ is *Gamma function.*

If we write the inequalities (3.8) and (3.9) for the function $f(z)=\ln(1-z)^{-1}$, $z\in D(0,1)$, then we get

$$\left|(w-u)\ln(1-v)^{-1}-\int_{\gamma}\ln(1-z)^{-1}\,dz\right| \leq\frac{1}{2}\left[\left|(1-v)^{-1}\right|\ell(\gamma)+\int_{\gamma}|v-z|\left|(1-z)^{-1}\right||dz|\right] \tag{3.11}$$

and

$$\left|(w-u)\ln(1-v)^{-1}-\int_{\gamma}\ln(1-z)^{-1}\,dz\right| \leq\frac{1}{2}\left[(1-|v|)^{-1}\ell(\gamma)+\int_{\gamma}|v-z|\,(1-|z|)^{-1}\,|dz|\right], \tag{3.12}$$

where $v\in D(0,1)$ and $\gamma_{u,w}\subset D(0,1)$.

References

[1] S. S. Dragomir, A refinement of Ostrowski's inequality for absolutely continuous functions whose derivatives belong to L_∞ and applications. *Libertas Math.* **22** (2002), 49–63.

[2] S. S. Dragomir, A refinement of Ostrowski's inequality for absolutely continuous functions and applications. *Acta Math. Vietnam.* **27** (2002), no. 2, 203–217.

[3] S. S. Dragomir, A functional generalization of Ostrowski inequality via Montgomery identity, *Acta Math. Univ. Comenian.* (N.S.) **84** (2015), no. 1, 63–78. Preprint *RGMIA Res. Rep. Coll.* **16** (2013), Art. 65, pp. 15 [Online http://rgmia.org/papers/v16/v16a65.pdf].

[4] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results. *Aust. J. Math. Anal. Appl.* **14** (2017), no. 1, Art. 1, 283 pp.

[5] S. S. Dragomir, An extension of Ostrowski's inequality to the complex integral, Preprint *RGMIA Res. Rep. Coll.* **18** (2018), Art. 112, 17 pp. [Online https://rgmia.org/papers/v21/v21a112.pdf].

[6] S. S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and for some numerical quadrature rules, *Appl. Math. Lett.,* **11** (1) (1998), 105-109.

[7] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1994.

[8] A. Ostrowski, Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert, *Comment. Math. Helv.,* **10** (1938), 226-227.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa