GENERALIZED TRAPEZOID INEQUALITY FOR THE COMPLEX INTEGRAL OF HOLOMORPHIC FUNCTIONS ON CONVEX DOMAINS

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ABSTRACT. In this paper we extend the generalized trapezoid inequality to the complex integral, by providing upper bounds for the quantity

$$\left|\left[\lambda f(u)+(1-\lambda) f(w)\right](y-x)-\int_{\gamma} f(z)\, dz\right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t\in[a,b]$, $x=z(a)$ and $y=z(b)$ while f is holomorphic in G, convex domain, $\gamma\subset G$, u, $w\in D$ and $\lambda\in\mathbb{C}$. Applications for some particular functions of interest are also given.

1. Introduction

Inequalities providing upper bounds for the quantity

$$\left|(t-a) f(a)+(b-t) f(b)-\int_{a}^{b} f(s)\, ds\right|, \qquad t\in[a,b] \tag{1.1}$$

are known in the literature as *generalized trapezoid inequalities* and it has been shown in [2] that

$$\begin{aligned}\left|(t-a) f(a)+(b-t) f(b)-\int_{a}^{b} f(s)\, ds\right| \\ \leq\left[\frac{1}{2}+\left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right](b-a) \bigvee_{a}^{b}(f)\end{aligned} \tag{1.2}$$

for any $t\in[a,b]$, provided that f is of bounded variation on $[a,b]$. The constant $\frac{1}{2}$ is the best possible.

If f is *absolutely continuous* on $[a,b]$, then (see [1, p. 93])

$$\left|(t-a) f(a)+(b-t) f(b)-\int_{a}^{b} f(s)\, ds\right| \tag{1.3}$$

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$$\leq \begin{cases} \left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2\|f'\|_\infty & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{(q+1)^{1/q}}\left[\left(\frac{t-a}{b-a}\right)^{q+1}+\left(\frac{b-t}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{1+1/q}\|f'\|_p & \text{if } f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}+\left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right](b-a)\|f'\|_1 \end{cases}$$

for any $t \in [a,b]$. The constants $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{(q+1)^{1/q}}$ are the best possible.

For other recent results on the trapezoid inequality, see [3], [4], [8], [9], [10] and [12].

In order to extend this result for the complex integral, we need some preparations as follows.

Suppose γ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a)=u$ and $z(b)=w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w}=\gamma$ as

$$\int_\gamma f(z)\,dz=\int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are *piecewise smooth.* Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma)=\int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.4}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right| \leq \int_\gamma |f(z)|\,|dz| \leq \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.5}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \geq 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\,|dz|\right)^{1/p}.$$

For $p=1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|\,.$$

If p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}\,.$$

We have the following recent result for functions of complex variable [6]:

Theorem 1. *Let f be holomorphic in G, an open domain and suppose $\gamma\subset G$ is a smooth path from $z(a)=u$ to $z(b)=w$. If $v=z(x)$ with $x\in(a,b)$, then $\gamma_{u,w}=\gamma_{u,v}\cup\gamma_{v,w}$,*

$$\left|(v-u)f(u)+(w-v)f(w)-\int_\gamma f(z)\,dz\right| \le \|f'\|_{\gamma_{u,v};\infty}\int_{\gamma_{u,v}}|z-v|\,|dz| + \|f'\|_{\gamma_{v,w};\infty}\int_{\gamma_{v,w}}|z-v|\,|dz| \le \|f'\|_{\gamma_{u,w};\infty}\int_{\gamma_{u,w}}|z-v|\,|dz|\,, \tag{1.6}$$

and

$$\left|(v-u)f(u)+(w-v)f(w)-\int_\gamma f(z)\,dz\right| \le \|f'\|_{\gamma_{u,v};1}\max_{z\in\gamma_{u,v}}|z-v| + \|f'\|_{\gamma_{v,w};1}\max_{z\in\gamma_{v,w}}|z-v| \le \|f'\|_{\gamma_{u,w};1}\max_{z\in\gamma_{u,w}}|z-v|\,. \tag{1.7}$$

If p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\left|(v-u)f(u)+(w-v)f(w)-\int_\gamma f(z)\,dz\right| \le \|f'\|_{\gamma_{u,v};p}\left(\int_{\gamma_{u,v}}|z-v|^q\,|dz|\right)^{1/q} + \|f'\|_{\gamma_{v,w};p}\left(\int_{\gamma_{v,w}}|z-v|^q\,|dz|\right)^{1/q} \le \|f'\|_{\gamma_{u,w};p}\left(\int_{\gamma_{u,w}}|z-v|^q\,|dz|\right)^{1/q}. \tag{1.8}$$

In this paper we extend the generalized trapezoid inequality to the complex integral, by providing upper bounds for the quantity

$$\left|[\lambda f(u)+(1-\lambda)f(w)](y-x)-\int_\gamma f(z)\,dz\right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t\in[a,b]$, $x=z(a)$ and $y=z(b)$ while f is holomorphic in G, convex domain, $\gamma\subset G$, and u, $w\in D$ and $\lambda\in\mathbb{C}$. Applications for some particular functions of interest are also given.

2. Generalized Trapezoid Inequalities

We have:

Theorem 2. *Let $f: D \subseteq \mathbb{C} \to \mathbb{C}$ be a holomorphic function on the convex domain D and suppose $\gamma \subset D$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a,b]$, $x = z(a)$ and $y = z(b)$. If $\lambda \in \mathbb{C}$ and $u, w \in D$ then*

$$\left| [\lambda f(u) + (1-\lambda) f(w)](y-x) - \int_\gamma f(z)\,dz \right| \leq |\lambda| \int_\gamma |u-z| \left| \int_0^1 f'((1-t)z + tu)\,dt \right| |dz| + |1-\lambda| \int_\gamma |w-z| \left| \int_0^1 f'((1-t)z+tw)\,dt \right| |dz| =: A(\lambda) \tag{2.1}$$

and

$$\left| [\lambda f(u) + (1-\lambda) f(w)](y-x) - \int_\gamma f(z)\,dz \right| \leq |\lambda| \int_0^1 \left| \int_\gamma (u-z) f'((1-t)z+tu)\,dz \right| dt + |1-\lambda| \int_0^1 \left| \int_\gamma (w-z) f'((1-t)z+tw)\,dz \right| dt = B(\lambda). \tag{2.2}$$

Proof. Due to the convexity of D, for any $z, v \in D$ we can define the function $\varphi_{z,v} : [0,1] \to \mathbb{R}$ by $\varphi_{z,v}(t) := f((1-t)z + tv)$. The function $\varphi_{z,v}$ is differentiable on $(0,1)$ and

$$\frac{d\varphi_{z,v}(t)}{dt} = (v-z) f'((1-t)z+tv) \text{ for } t \in (0,1).$$

We have

$$f(v) - f(z) = \varphi_{z,v}(1) - \varphi_{z,v}(0) = \int_0^1 \frac{d\varphi_{z,v}(t)}{dt} dt = (v-z) \int_0^1 f'((1-t)z+tv)\,dt$$

namely

$$f(v) = f(z) + (v-z) \int_0^1 f'((1-t)z+tv)\,dt \tag{2.3}$$

for any $z, v \in D$.

Therefore, by (2.3) we get

$$f(u) = f(z) + (u-z) \int_0^1 f'((1-t)z+tu)\,dt \tag{2.4}$$

and

$$f(w) = f(z) + (w-z) \int_0^1 f'((1-t)z+tw)\,dt \tag{2.5}$$

for any $z \in D$.

If we multiply (2.4) and (2.5) by λ and $1-\lambda$ and add, we get

$$\begin{aligned}(2.6)\quad & \lambda f(u)+(1-\lambda) f(w)-f(z) \\ & =\lambda(u-z) \int_0^1 f'((1-t) z+t u) d t+(1-\lambda)(w-z) \int_0^1 f'((1-t) z+t w) d t\end{aligned}$$

for any $z \in D$ and $\lambda \in \mathbb{C}$.

Now, if we integrate this equality over z in γ and also use Fubini's theorem, we get the following equality of interest

$$\begin{aligned}(2.7)\quad & [\lambda f(u)+(1-\lambda) f(w)](y-x)-\int_\gamma f(z) d z \\ & =\lambda \int_\gamma(u-z)\left(\int_0^1 f'((1-t) z+t u) d t\right) d z \\ & +(1-\lambda) \int_\gamma(w-z)\left(\int_0^1 f'((1-t) z+t w) d t\right) d z \\ & =\lambda \int_0^1\left(\int_\gamma(u-z) f'((1-t) z+t u) d z\right) d t \\ & \qquad +(1-\lambda) \int_0^1\left(\int_\gamma(w-z) f'((1-t) z+t w) d z\right) d t\end{aligned}$$

for any $\lambda \in \mathbb{C}$.

Taking the modulus in the first equality in (2.7) we get

$$\begin{aligned}(2.8)\quad & \left|[\lambda f(u)+(1-\lambda) f(w)](y-x)-\int_\gamma f(z) d z\right| \\ & \leq|\lambda|\left|\int_\gamma(u-z)\left(\int_0^1 f'((1-t) z+t u) d t\right) d z\right| \\ & +|1-\lambda|\left|\int_\gamma(w-z)\left(\int_0^1 f'((1-t) z+t w) d t\right) d z\right| \\ & \leq|\lambda| \int_\gamma|u-z|\left|\int_0^1 f'((1-t) z+t u) d t\right||d z| \\ & \qquad +|1-\lambda| \int_\gamma|w-z|\left|\int_0^1 f'((1-t) z+t w) d t\right||d z|=A(\lambda),\end{aligned}$$

which, by (2.8) proves the inequality (2.1).

Taking the modulus in the second equality in (2.7), we get

$$\begin{aligned}
&\left|\left[\lambda f(u)+(1-\lambda) f(w)\right](y-x)-\int_\gamma f(z)\,dz\right| \\
&\leq |\lambda|\left|\int_0^1\left(\int_\gamma (u-z) f'((1-t)z+tu)\,dz\right)dt\right| \\
&\quad+|1-\lambda|\left|\int_0^1\left(\int_\gamma (w-z) f'((1-t)z+tw)\,dz\right)dt\right| \\
&\leq |\lambda|\int_0^1\left|\int_\gamma (u-z) f'((1-t)z+tu)\,dz\right|dt \\
&\quad+|1-\lambda|\int_0^1\left|\int_\gamma (w-z) f'((1-t)z+tw)\,dz\right|dt=B(\lambda),
\end{aligned}$$

which proves the inequality (2.2) □

Remark 1. *Using Hölder's inequality we also have*

$$\begin{aligned}
&\int_\gamma |u-z|\left|\int_0^1 f'((1-t)z+tu)\,dt\right||dz| \\
&\leq \begin{cases}
\max_{z\in\gamma}|u-z|\int_\gamma\left|\int_0^1 f'((1-t)z+tu)\,dt\right||dz|; \\
\\
\left(\int_\gamma |u-z|^p|dz|\right)^{1/p}\left(\int_\gamma\left|\int_0^1 f'((1-t)z+tu)\,dt\right|^q|dz|\right)^{1/q} \\
p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\
\\
\max_{z\in\gamma}\left|\int_0^1 f'((1-t)z+tu)\,dt\right|\int_\gamma |u-z||dz|,
\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\int_\gamma |w-z|\left|\int_0^1 f'((1-t)z+tw)\,dt\right||dz| \\
&\leq \begin{cases}
\max_{z\in\gamma}|w-z|\int_\gamma\left|\int_0^1 f'((1-t)z+tw)\,dt\right||dz|; \\
\\
\left(\int_\gamma |w-z|^p|dz|\right)^{1/p}\left(\int_\gamma\left|\int_0^1 f'((1-t)z+tw)\,dt\right|^q|dz|\right)^{1/q} \\
p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\
\\
\max_{z\in\gamma}\left|\int_0^1 f'((1-t)z+tw)\,dt\right|\int_\gamma |w-z||dz|,
\end{cases}
\end{aligned}$$

which give the following upper bounds for $A(\lambda)$

$$A(\lambda) \leq |\lambda| \begin{cases} \max_{z\in\gamma}|u-z| \int_\gamma \left|\int_0^1 f'((1-t)z+tu)\,dt\right| |dz|; \\ \\ \left(\int_\gamma |u-z|^p |dz|\right)^{1/p} \left(\int_\gamma \left|\int_0^1 f'((1-t)z+tu)\,dt\right|^q |dz|\right)^{1/q} \\ p,\ q>1 \ \text{with } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \max_{z\in\gamma}\left|\int_0^1 f'((1-t)z+tu)\,dt\right| \int_\gamma |u-z|\,|dz|, \end{cases}$$
$$+|1-\lambda| \begin{cases} \max_{z\in\gamma}|w-z| \int_\gamma \left|\int_0^1 f'((1-t)z+tw)\,dt\right| |dz|; \\ \\ \left(\int_\gamma |w-z|^p |dz|\right)^{1/p} \left(\int_\gamma \left|\int_0^1 f'((1-t)z+tw)\,dt\right|^q |dz|\right)^{1/q} \\ p,\ q>1 \ \text{with } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \max_{z\in\gamma}\left|\int_0^1 f'((1-t)z+tw)\,dt\right| \int_\gamma |w-z|\,|dz|, \end{cases} \tag{2.9}$$

for any $\lambda \in \mathbb{C}$.

Using Hölder's inequality we also have

$$\left|\int_\gamma (u-z) f'((1-t)z+tu)\,dz\right| \leq \begin{cases} \max_{z\in\gamma}|u-z| \int_\gamma |f'((1-t)z+tu)|\,|dz| \\ \\ \left(\int_\gamma |u-z|^p |dz|\right)^{1/p} \left(\int_\gamma |f'((1-t)z+tu)|^q |dz|\right)^{1/q} \\ p,\ q>1 \ \text{with } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \max_{z\in\gamma}|f'((1-t)z+tu)| \int_\gamma |u-z|\,|dz| \end{cases}$$

and

$$\left|\int_\gamma (w-z) f'((1-t)z+tw)\,dz\right| \leq \begin{cases} \max_{z\in\gamma}|w-z| \int_\gamma |f'((1-t)z+tw)|\,|dz| \\ \\ \left(\int_\gamma |w-z|^p |dz|\right)^{1/p} \left(\int_\gamma |f'((1-t)z+tw)|^q |dz|\right)^{1/q} \\ p,\ q>1 \ \text{with } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \max_{z\in\gamma}|f'((1-t)z+tw)| \int_\gamma |w-z|\,|dz| \end{cases}$$

which by integration over $t \in [0,1]$ produce

$$\int_0^1 \left| \int_\gamma (u-z) f'((1-t)z+tu)\, dz \right| dt$$
$$\leq \begin{cases} \max_{z\in\gamma} |u-z| \int_0^1 \left(\int_\gamma |f'((1-t)z+tu)| \, |dz| \right) dt \\ \\ \left(\int_\gamma |u-z|^p \, |dz| \right)^{1/p} \int_0^1 \left(\int_\gamma |f'((1-t)z+tu)|^q \, |dz| \right)^{1/q} dt \\ p,\ q>1 \ \text{with}\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \int_\gamma |u-z| \, |dz| \int_0^1 \left(\max_{z\in\gamma} |f'((1-t)z+tu)| \right) dt \end{cases}$$

and

$$\int_0^1 \left| \int_\gamma (w-z) f'((1-t)z+tw)\, dz \right| dt$$
$$\leq \begin{cases} \max_{z\in\gamma} |w-z| \int_0^1 \left(\int_\gamma |f'((1-t)z+tw)| \, |dz| \right) dt \\ \\ \left(\int_\gamma |w-z|^p \, |dz| \right)^{1/p} \int_0^1 \left(\int_\gamma |f'((1-t)z+tw)|^q \, |dz| \right)^{1/q} dt \\ p,\ q>1 \ \text{with}\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \int_\gamma |w-z| \, |dz| \int_0^1 \left(\max_{z\in\gamma} |f'((1-t)z+tw)| \right) dt. \end{cases}$$

Therefore, we also have the upper bounds for $B(\lambda)$

$$B(\lambda) \leq |\lambda| \begin{cases} \max_{z\in\gamma} |u-z| \int_0^1 \left(\int_\gamma |f'((1-t)z+tu)| \, |dz| \right) dt \\ \\ \left(\int_\gamma |u-z|^p \, |dz| \right)^{1/p} \int_0^1 \left(\int_\gamma |f'((1-t)z+tu)|^q \, |dz| \right)^{1/q} dt \\ p,\ q>1 \ \text{with}\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \int_\gamma |u-z| \, |dz| \int_0^1 \left(\max_{z\in\gamma} |f'((1-t)z+tu)| \right) dt \end{cases}$$
$$+ |1-\lambda| \begin{cases} \max_{z\in\gamma} |w-z| \int_0^1 \left(\int_\gamma |f'((1-t)z+tw)| \, |dz| \right) dt \\ \\ \left(\int_\gamma |w-z|^p \, |dz| \right)^{1/p} \int_0^1 \left(\int_\gamma |f'((1-t)z+tw)|^q \, |dz| \right)^{1/q} dt \\ p,\ q>1 \ \text{with}\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \int_\gamma |w-z| \, |dz| \int_0^1 \left(\max_{z\in\gamma} |f'((1-t)z+tw)| \right) dt, \end{cases} \tag{2.10}$$

for any $\lambda \in \mathbb{C}$.

Corollary 1. *With the assumptions of Theorem 2 and if* $\|f'\|_{D,\infty} := \sup_{z\in D}|f'(z)| < \infty$, *then we also have*

$$\left|[\lambda f(u) + (1-\lambda) f(w)](y-x) - \int_\gamma f(z)\,dz\right| \tag{2.11}$$
$$\leq \|f'\|_{D,\infty}\left[|\lambda|\int_\gamma |u-z|\,|dz| + |1-\lambda|\int_\gamma |w-z|\,|dz|\right]$$

for any $\lambda \in \mathbb{C}$.

Proof. It follows by (2.1) observing that

$$\left|\int_0^1 f'((1-t)z + tu)\,dt\right| \leq \int_0^1 |f'((1-t)z+tu)|\,dt \leq \sup_{z\in D}|f'(z)|\int_0^1 dt$$
$$= \|f'\|_{D,\infty}$$

and, similarly

$$\left|\int_0^1 f'((1-t)z+tw)\,dt\right| \leq \|f'\|_{D,\infty}.$$

□

In the case of some convexity properties for the modulus of the derivative, other upper bounds can be derived as follows.

Corollary 2. *With the assumptions of Theorem 2 and if* $|f'|$ *is convex on* D, *then*

$$\left|[\lambda f(u) + (1-\lambda) f(w)](y-x) - \int_\gamma f(z)\,dz\right| \tag{2.12}$$
$$\leq \frac{1}{2}|\lambda|\left(\int_\gamma |u-z|\,|f'(z)|\,|dz| + |f'(u)|\int_\gamma |u-z|\,|dz|\right)$$
$$+\frac{1}{2}|1-\lambda|\left(\int_\gamma |w-z|\,|f'(z)|\,|dz| + |f'(w)|\int_\gamma |w-z|\,|dz|\right)$$
$$\leq \|f'\|_{D,\infty}\left[|\lambda|\int_\gamma |u-z|\,|dz| + |1-\lambda|\int_\gamma |w-z|\,|dz|\right]$$

provided

$$\|f'\|_{D,\infty} := \sup_{z\in D}|f'(z)| < \infty.$$

Proof. If $g:[0,1]\to\mathbb{R}$ is convex, then the following inequality is well known in the literature as Hermite-Hadamard inequality

$$\int_0^1 g(t)\,dt \leq \frac{g(0)+g(1)}{2}.$$

Let $v\in D$ and $z\in\gamma$. By Hermite-Hadamard inequality for the convex function $[0,1]\ni t\to |f'((1-t)z+tv)|$ we have

$$\int_0^1 |f'((1-t)z+tv)|\,dt \leq \frac{1}{2}[|f'(z)| + |f'(v)|],$$

which implies that

$$\int_0^1 |f'((1-t)z+tu)|\,dt \leq \frac{1}{2}[|f'(z)| + |f'(u)|],$$

and

$$\int_0^1 |f'((1-t)z+tw)|\,dt \le \frac{1}{2}\left[|f'(z)|+|f'(w)|\right].$$

Therefore

$$\begin{aligned}\int_\gamma |u-z| \left|\int_0^1 f'((1-t)z+tu)\,dt\right| |dz| &\le \int_\gamma |u-z| \left(\int_0^1 |f'((1-t)z+tu)|\,dt\right)|dz| \\ &\le \int_\gamma |u-z| \left(\frac{1}{2}\left[|f'(z)|+|f'(u)|\right]\right)|dz| \\ &= \frac{1}{2}\left(\int_\gamma |u-z|\,|f'(z)|\,|dz| + |f'(u)|\int_\gamma |u-z|\,|dz|\right)\end{aligned}$$

and, similarly

$$\int_\gamma |w-z| \left|\int_0^1 f'((1-t)z+tw)\,dt\right| |dz| \le \frac{1}{2}\left(\int_\gamma |w-z|\,|f'(z)|\,|dz| + |f'(w)|\int_\gamma |w-z|\,|dz|\right),$$

which, by (2.1), produces the first inequality in (2.12).

The last part is obvious. □

We also have:

Corollary 3. *With the assumptions of Theorem 2 and if $|f'|^q$ with $q>1$ is convex on D, then*

$$\begin{aligned}(2.13)\quad &\left|\left[\lambda f(u)+(1-\lambda)f(w)\right](y-x) - \int_\gamma f(z)\,dz\right| \\ &\le \frac{1}{2^{1/q}}|\lambda|\left(\int_\gamma |u-z|^p\,|dz|\right)^{1/p}\left(\int_\gamma |f'(z)|^q\,|dz| + |f'(u)|^q\,\ell(\gamma)\right)^{1/q} \\ &+ \frac{1}{2^{1/q}}|1-\lambda|\left(\int_\gamma |w-z|^p\,|dz|\right)^{1/p}\left(\int_\gamma |f'(z)|^q\,|dz| + |f'(w)|^q\,\ell(\gamma)\right)^{1/q} \\ &\le \max\{|\lambda|,|1-\lambda|\}\left(\int_\gamma \left(|u-z|^p+|w-z|^p\right)|dz|\right)^{1/p} \\ &\times\left(\int_\gamma |f'(z)|^q\,|dz| + \frac{|f'(u)|^q+|f'(w)|^q}{2}\ell(\gamma)\right)^{1/q},\end{aligned}$$

where $p>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using power inequality for integral and the convexity of $|f'|^q$, with $q>1$, we have

$$\int_0^1 |f'((1-t)z+tv)|\,dt \le \left(\int_0^1 |f'((1-t)z+tv)|^q\,dt\right)^{1/q} \le \left(\frac{|f'(z)|^q+|f'(v)|^q}{2}\right)^{1/q}$$

for $v\in D$ and $z\in\gamma$.

This implies that

$$\begin{aligned}
&\int_\gamma |u-z|\left|\int_0^1 f'((1-t)z+tu)\,dt\right||dz| \\
&\le \int_\gamma |u-z|\left(\int_0^1 |f'((1-t)z+tu)|\,dt\right)|dz| \\
&\le \int_\gamma |u-z|\left(\int_0^1 |f'((1-t)z+tv)|^q\,dt\right)^{1/q}|dz| \\
&\le \left(\int_\gamma |u-z|^p|dz|\right)^{1/p}\left(\int_\gamma \left[\left(\frac{|f'(z)|^q+|f'(u)|^q}{2}\right)^{1/q}\right]^q|dz|\right)^{1/q} \\
&= \left(\int_\gamma |u-z|^p|dz|\right)^{1/p}\left(\int_\gamma \frac{|f'(z)|^q+|f'(u)|^q}{2}|dz|\right)^{1/q} \\
&= \left(\int_\gamma |u-z|^p|dz|\right)^{1/p}\left(\frac{1}{2}\int_\gamma |f'(z)|^q|dz|+\frac{1}{2}|f'(u)|^q\ell(\gamma)\right)^{1/q}
\end{aligned}$$

and, in a similar way

$$\begin{aligned}
&\int_\gamma |w-z|\left|\int_0^1 f'((1-t)z+tw)\,dt\right||dz| \\
&\le \left(\int_\gamma |w-z|^p|dz|\right)^{1/p}\left(\frac{1}{2}\int_\gamma |f'(z)|^q|dz|+\frac{1}{2}|f'(w)|^q\ell(\gamma)\right)^{1/q}.
\end{aligned}$$

By using (2.1) we get the first part (2.13).

The last part follows by Hölder's discrete inequality. □

Let us give now an example for the complex exponential function.

For $z\in\mathbb{C}$ we have

$$\begin{aligned}
|\exp(z)| &= |\exp(\operatorname{Re} z+i\operatorname{Im} z)| = |\exp(\operatorname{Re} z)\exp(i\operatorname{Im} z)| \\
&= |\exp(\operatorname{Re} z)||\exp(i\operatorname{Im} z)| = \exp(\operatorname{Re} z)|\cos(\operatorname{Im} z)+i\sin(\operatorname{Im} z)| \\
&= \exp(\operatorname{Re} z).
\end{aligned}$$

Then for any $t\in[0,1]$ and for any $z,\,w\in\mathbb{C}$ we have

$$\begin{aligned}
|\exp((1-t)z+tw)|^\alpha &= \exp[\alpha(\operatorname{Re}((1-t)z+tw))] \\
&= \exp[(1-t)\alpha\operatorname{Re} z+t\alpha\operatorname{Re} w] \\
&\le (1-t)\exp(\alpha\operatorname{Re} z)+t\exp(\alpha\operatorname{Re} w) \\
&= (1-t)|\exp(z)|^\alpha+t|\exp(w)|^\alpha
\end{aligned}$$

which shows that the function $g(z) = |\exp(z)|^{\alpha}$ is convex for any $\alpha \in \mathbb{R} \setminus \{0\}$.

Suppose $\gamma \subset D$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a,b]$, $x = z(a)$ and $y = z(b)$. We also have for $\gamma = \gamma_{u,w}$ that

$$\int_{\gamma} \exp(z)\, dz = \int_{\gamma_{u,w}} \exp(z)\, dz = \exp(y) - \exp(x).$$

Using the inequality (2.12) for the function $f(z) = \exp z$, we have for u, w, $\lambda \in \mathbb{C}$ that

$$\begin{aligned} &|[\lambda \exp u + (1-\lambda)\exp w](y-x) - \exp(y) + \exp(x)| \\ &\leq \frac{1}{2}|\lambda| \left(\int_{\gamma} |u-z| \exp(\operatorname{Re} z)\, |dz| + \exp(\operatorname{Re} u) \int_{\gamma} |u-z|\, |dz| \right) \\ &+ \frac{1}{2}|1-\lambda| \left(\int_{\gamma} |w-z| \exp(\operatorname{Re} z)\, |dz| + \exp(\operatorname{Re} w) \int_{\gamma} |w-z|\, |dz| \right) \\ &\leq \|\exp\|_{D,\infty} \left[|\lambda| \int_{\gamma} |u-z|\, |dz| + |1-\lambda| \int_{\gamma} |w-z|\, |dz| \right]. \end{aligned} \tag{2.14}$$

From the inequality (2.13) for the function $f(z) = \exp z$, we have for u, w, $\lambda \in \mathbb{C}$ that

$$\begin{aligned} &||[\lambda \exp u + (1-\lambda)\exp w](y-x) - \exp(y) + \exp(x)|| \\ &\leq \frac{1}{2^{1/q}}|\lambda| \left(\int_{\gamma} |u-z|^{p}\, |dz| \right)^{1/p} \left(\int_{\gamma} \exp(q \operatorname{Re} z)\, |dz| + \exp(q \operatorname{Re} u)\, \ell(\gamma) \right)^{1/q} \\ &+ \frac{1}{2^{1/q}}|1-\lambda| \left(\int_{\gamma} |w-z|^{p}\, |dz| \right)^{1/p} \left(\int_{\gamma} \exp(q \operatorname{Re} z)\, |dz| + \exp(q \operatorname{Re} w)\, \ell(\gamma) \right)^{1/q} \\ &\leq \max\{|\lambda|, |1-\lambda|\} \left(\int_{\gamma} (|u-z|^{p} + |w-z|^{p})\, |dz| \right)^{1/p} \\ &\times \left(\int_{\gamma} \exp(q \operatorname{Re} z)\, |dz| + \frac{\exp(q \operatorname{Re} u) + \exp(q \operatorname{Re} w)}{2} \ell(\gamma) \right)^{1/q}, \end{aligned} \tag{2.15}$$

where $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

3. Related Results

Now, by the help of power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficients of the original series, namely, $f_a(z) = \sum_{n=0}^{\infty} |a_n| z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $a_n \geq 0$ then $f_a = f$.

We notice that if

$$f(z)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^n=\ln\frac{1}{1+z},\ z\in D(0,1); \tag{3.1}$$
$$g(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}=\cos z,\ z\in\mathbb{C};$$
$$h(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}=\sin z,\ z\in\mathbb{C};$$
$$l(z)=\sum_{n=0}^{\infty}(-1)^n z^n=\frac{1}{1+z},\ z\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(z)=\sum_{n=1}^{\infty}\frac{1}{n!}z^n=\ln\frac{1}{1-z},\ z\in D(0,1); \tag{3.2}$$
$$g_a(z)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n}=\cosh z,\ z\in\mathbb{C};$$
$$h_a(z)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1}=\sinh z,\ z\in\mathbb{C};$$
$$l_a(z)=\sum_{n=0}^{\infty}z^n=\frac{1}{1-z},\ z\in D(0,1).$$

Theorem 3. *Consider the power series $f(z)=\sum_{n=0}^{\infty}a_n z^n$ that is convergent on the open disk $D(0,R)$ and suppose $\gamma\subset D(0,R)$ is a piecewise smooth path parametrized by $z(t)$, $t\in[a,b]$, $x=z(a)$ and $y=z(b)$. If $u,\ w\in D(0,R)$, then we have the inequalities*

$$\begin{aligned}\left|[\lambda f(u)+(1-\lambda)f(w)](y-x)-\int_\gamma f(z)\,dz\right| &\le \frac{1}{2}|\lambda|\left[\int_\gamma |u-z|\,|f_a'(z)|\,|dz|+|f_a'(u)|\,\ell(\gamma)\right]\\ &\quad+\frac{1}{2}|1-\lambda|\left[\int_\gamma |w-z|\,|f_a'(z)|\,|dz|+|f_a'(w)|\,\ell(\gamma)\right]\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\left|[\lambda f(u)+(1-\lambda)f(w)](y-x)-\int_\gamma f(z)\,dz\right| &\le \frac{1}{2}|\lambda|\left[\int_\gamma |u-z|\,f_a'(|z|)\,|dz|+f_a'(|u|)\,\ell(\gamma)\right]\\ &\quad+\frac{1}{2}|1-\lambda|\left[\int_\gamma |w-z|\,f_a'(|z|)\,|dz|+f_a'(|w|)\,\ell(\gamma)\right]\end{aligned} \tag{3.4}$$

for $\lambda\in\mathbb{C}$.

Proof. We have $f'(z)=\sum_{n=1}^{\infty}na_nz^{n-1}$ and $f_a'(z)=\sum_{n=1}^{\infty}n\left|a_n\right|z^{n-1}$. For $m\geq 1$, by using the generalized triangle inequality we have

$$\left|\sum_{n=1}^{m}na_nz^{n-1}\right|\leq\sum_{n=1}^{m}n\left|a_n\right|z^{n-1}. \tag{3.5}$$

Since the series $\sum_{n=1}^{\infty}na_nz^{n-1}$ and $\sum_{n=1}^{\infty}n\left|a_n\right|z^{n-1}$ are convergent, then by letting $m\to\infty$ in (3.5) we get

$$\left|f'(z)\right|\leq f_a'(|z|) \text{ for any } z\in D\left(0,R\right).$$

We observe that, since f_a' has nonnegative coefficients, then this functions is convex as a real variable functions on the interval $(-R,R)$ and increasing on $[0,R)$.

For $z,v\in D$, consider the function $h_{z,v}:[0,1]\to[0,\infty)$, $h_{z,v}\left(t\right):=f_a'\left(\left|\left(1-t\right)z+tv\right|\right)$. For $\alpha,\beta\in[0,1]$ with $\alpha+\beta=1$ and $t_1,t_2\in[0,1]$ we have

$$\begin{aligned}h_{z,v}\left(\alpha t_1+\beta t_2\right)&=f_a'\left(\left|\left(1-\alpha t_1-\beta t_2\right)z+\alpha t_1+\beta t_2v\right|\right)\\&=f_a'\left[\left|\alpha\left(\left(1-t_1\right)z+t_1v\right)+\beta\left(\left(1-t_2\right)z+t_2v\right)\right|\right]\\&\leq f_a'\left[\alpha\left|\left(1-t_1\right)z+t_1v\right|+\beta\left|\left(1-t_2\right)z+t_2v\right|\right]\\&\leq\alpha f_a'\left(\left|\left(1-t_1\right)z+t_1v\right|\right)+\beta f_a'\left(\left|\left(1-t_2\right)z+t_2v\right|\right),\end{aligned}$$

which shows that $h_{z,v}$ is convex on $[0,1]$.

If we write the Hermite-Hadamard inequality for $h_{z,v}$ on $[0,1]$ then we get

$$\int_0^1 f_a'\left(\left|\left(1-t\right)z+tv\right|\right)dt\leq\frac{\left|f_a'\left(z\right)\right|+\left|f_a'\left(v\right)\right|}{2}$$

for any $z,v\in D$, which implies that

$$\begin{aligned}&\int_\gamma\left|u-z\right|\left(\int_0^1\left|f'\left(\left(1-t\right)z+tu\right)\right|dt\right)\left|dz\right|\\&\leq\int_\gamma\left|u-z\right|\left(\int_0^1 f_a'\left(\left|\left(1-t\right)z+tu\right|\right)dt\right)\left|dz\right|\\&\leq\int_\gamma\left|u-z\right|\frac{\left|f_a'\left(z\right)\right|+\left|f_a'\left(u\right)\right|}{2}\left|dz\right|\\&=\frac{1}{2}\left[\int_\gamma\left|u-z\right|\left|f_a'\left(z\right)\right|\left|dz\right|+\left|f_a'\left(u\right)\right|\ell\left(\gamma\right)\right]\end{aligned}$$

and

$$\begin{aligned}&\int_\gamma\left|w-z\right|\left(\int_0^1\left|f'\left(\left(1-t\right)z+tw\right)\right|dt\right)\left|dz\right|\\&\leq\int_\gamma\left|w-z\right|\left(\int_0^1 f_a'\left(\left|\left(1-t\right)z+tw\right|\right)dt\right)\left|dz\right|\\&\leq\int_\gamma\left|w-z\right|\frac{\left|f_a'\left(z\right)\right|+\left|f_a'\left(w\right)\right|}{2}\left|dz\right|\\&=\frac{1}{2}\left[\int_\gamma\left|w-z\right|\left|f_a'\left(z\right)\right|\left|dz\right|+\left|f_a'\left(w\right)\right|\ell\left(\gamma\right)\right],\end{aligned}$$

the inequality (3.3) is proved.

We also have

$$f_a'(|(1-t)z+tv|) \le f_a'((1-t)|z|+t|v|)$$

for any $z, v \in D$ and $t \in [0,1]$ and since the function $p_{z,v}(t) := f_a'((1-t)|z|+t|v|)$ is convex, then by Hermite-Hadamard inequality we have

$$\int_0^1 f_a'(|(1-t)z+tv|)\,dt \le \int_0^1 f_a'((1-t)|z|+t|v|)\,dt \le \frac{f_a'(|z|)+f_a'(|v|)}{2}.$$

This implies that

$$\begin{aligned}&\int_\gamma |u-z|\left(\int_0^1 f_a'(|(1-t)z+tu|)\,dt\right)|dz|\\ &\le \int_\gamma |u-z|\left(\int_0^1 f_a'((1-t)|z|+t|u|)\,dt\right)|dz|\\ &\le \int_\gamma |u-z|\frac{f_a'(|z|)+f_a'(|u|)}{2}|dz| = \frac{1}{2}\left[\int_\gamma |u-z|\,f_a'(|z|)\,|dz| + f_a'(|u|)\,\ell(\gamma)\right],\end{aligned}$$

and

$$\begin{aligned}&\int_\gamma |w-z|\left(\int_0^1 f_a'(|(1-t)z+tw|)\,dt\right)|dz|\\ &\le \int_\gamma |w-z|\left(\int_0^1 f_a'((1-t)|z|+t|w|)\,dt\right)|dz|\\ &\le \int_\gamma |w-z|\frac{f_a'(|z|)+f_a'(|w|)}{2}|dz| = \frac{1}{2}\left[\int_\gamma |w-z|\,f_a'(|z|)\,|dz| + f_a'(|w|)\,\ell(\gamma)\right],\end{aligned}$$

which proves (3.4). □

Remark 2. *If we consider, for instance $f(z)=\sin z$, then $f_a(z)=\sinh z$, $z\in\mathbb{C}$ and by (3.3) and (3.4) we get*

$$\begin{aligned}|[\lambda\sin(u)+(1-\lambda)\sin(w)](y-x)+\cos y-\cos x|\\ \le \frac{1}{2}|\lambda|\left[\int_\gamma |u-z|\,|\cosh(z)|\,|dz| + |\cosh(u)|\,\ell(\gamma)\right]\\ +\frac{1}{2}|1-\lambda|\left[\int_\gamma |w-z|\,|\cosh(z)|\,|dz| + |\cosh(w)|\,\ell(\gamma)\right]\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}|[\lambda\sin(u)+(1-\lambda)\sin(w)](y-x)+\cos y-\cos x|\\ \le \frac{1}{2}|\lambda|\left[\int_\gamma |u-z|\cosh(|z|)\,|dz| + \cosh(|u|)\,\ell(\gamma)\right]\\ +\frac{1}{2}|1-\lambda|\left[\int_\gamma |w-z|\cosh(|z|)\,|dz| + \cosh(|w|)\,\ell(\gamma)\right]\end{aligned} \tag{3.7}$$

for any $u, w, \lambda\in\mathbb{C}$ and $\gamma_{x,y}\subset\mathbb{C}$ a piecewise smooth path.

Corollary 4. *If the power series $f(z)=\sum_{n=0}^{\infty}a_nz^n$ has nonnegative coefficients and is convergent on the open disk $D(0,R)$, then with the other assumptions in Theorem 3 we have*

$$\left|\left[\lambda f(u)+(1-\lambda)f(w)\right](y-x)-\int_\gamma f(z)\,dz\right| \le \frac{1}{2}|\lambda|\left[\int_\gamma |u-z|\,|f'(z)|\,|dz|+|f'(u)|\,\ell(\gamma)\right] + \frac{1}{2}|1-\lambda|\left[\int_\gamma |w-z|\,|f'(z)|\,|dz|+|f'(w)|\,\ell(\gamma)\right] \tag{3.8}$$

and

$$\left|\left[\lambda f(u)+(1-\lambda)f(w)\right](y-x)-\int_\gamma f(z)\,dz\right| \le \frac{1}{2}|\lambda|\left[\int_\gamma |u-z|\,f'(|z|)\,|dz|+f'(|u|)\,\ell(\gamma)\right] + \frac{1}{2}|1-\lambda|\left[\int_\gamma |w-z|\,f'(|z|)\,|dz|+f'(|w|)\,\ell(\gamma)\right]. \tag{3.9}$$

Important examples of functions as power series representations with nonnegative coefficients in addition to the ones from (3.2), are:

$$\exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n,\ \ z\in\mathbb{C}; \tag{3.10}$$

$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ \ z\in D(0,1);$$

$$\sin^{-1}(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(2n+1)\,n!}z^{2n+1},\ \ z\in D(0,1);$$

$$\tanh^{-1}(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ \ z\in D(0,1);$$

$$_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\,\Gamma(n+\beta)\,\Gamma(\gamma)}{n!\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0,$$
$$z\in D(0,1);$$

where Γ is *Gamma function.*

If we write the inequalities (3.8) and (3.9) for the function $f(z)=\ln(1-z)^{-1}$, $z\in D(0,1)$, then we get

$$\left|\left[\lambda\ln(1-u)^{-1}+(1-\lambda)\ln(1-w)^{-1}\right](y-x)-\int_\gamma \ln(1-z)^{-1}\,dz\right| \le \frac{1}{2}|\lambda|\left[\int_\gamma |u-z|\left|(1-z)^{-1}\right||dz|+\left|(1-u)^{-1}\right|\ell(\gamma)\right] + \frac{1}{2}|1-\lambda|\left[\int_\gamma |w-z|\left|(1-z)^{-1}\right||dz|+\left|(1-w)^{-1}\right|\ell(\gamma)\right] \tag{3.11}$$

and

$$\left|\left[\lambda \ln (1-u)^{-1}+(1-\lambda) \ln (1-w)^{-1}\right](y-x)-\int_{\gamma} \ln (1-z)^{-1} d z\right| \tag{3.12}$$
$$\leq \frac{1}{2}|\lambda|\left[\int_{\gamma}|u-z|\left|(1-|z|)^{-1}\right||d z|+\left|(1-|u|)^{-1}\right| \ell(\gamma)\right]$$
$$+\frac{1}{2}|1-\lambda|\left[\int_{\gamma}|w-z|\left|(1-|z|)^{-1}\right||d z|+\left|(1-|w|)^{-1}\right| \ell(\gamma)\right]$$

where $u,\, w \in D(0,1)$ and $\gamma_{x,y} \subset D(0,1)$.

References

[1] P. CERONE and S. S. DRAGOMIR, Trapezoidal-type rules from an inequalities point of view, in *Handbook of Analytic-Computational Methods in Applied Mathematics*, G. Anastassiou (Ed.), CRC Press, NY, 2000, 65-134.

[2] P. CERONE, S. S. DRAGOMIR and C. E. M. PEARCE, A generalised trapezoid inequality for functions of bounded variation, *Turkish J. Math.,* **24**(2) (2000), 147-163.

[3] S. S. DRAGOMIR, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Ineq. Pure & Appl. Math.,* **3**(2) (2002), Art. 31. [ONLINE: http://jipam.vu.edu.au/article.php?sid=183].

[4] S. S. DRAGOMIR, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Ineq. Pure & Appl. Math.,* **3**(3) (2002), Art. 35. [ONLINE: http://jipam.vu.edu.au/article.php?sid=187].

[5] S. S. DRAGOMIR, The Ostrowski integral inequality for mappings of bounded variation, *Bull. Austral. Math. Soc.*, **60** (1999), 495–508.

[6] S. S. DRAGOMIR, An extension of trapezoid inequality to the complex integral, Preprint *RGMIA Res. Rep. Coll.* **21** (2018), Art. 113, 16 pp., [Online https://rgmia.org/papers/v21/v21a113.pdf].

[7] S. S. DRAGOMIR and Th. M. RASSIAS (Eds.), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht, 2002.

[8] A. I. KECHRINIOTIS and N. D. ASSIMAKIS, Generalizations of the trapezoid inequalities based on a new mean value theorem for the remainder in Taylor's formula. *J. Inequal. Pure Appl. Math.* **7** (2006), no. 3, Article 90, 13 pp. (electronic).

[9] Z. LIU, Some inequalities of perturbed trapezoid type. *J. Inequal. Pure Appl. Math.* **7** (2006), no. 2, Article 47, 9 pp. (electronic).

[10] McD. A. MERCER, On perturbed trapezoid inequalities. *J. Inequal. Pure Appl. Math.* **7** (2006), no. 4, Article 118, 7 pp. (electronic).

[11] A. OSTROWSKI, Über die absolutabweichung einer differentiierbaren funktion von ihrem integralmittelwert, *Comment. Math. Helv.,* **10** (1938), 226–227.

[12] N. UJEVIĆ, Error inequalities for a generalized trapezoid rule. *Appl. Math. Lett.* **19** (2006), no. 1, 32–37.

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