HERMITE-HADAMARD INTEGRAL INEQUALITY VIA DELTA-INTEGRAL

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ABSTRACT. The concept of time scale has been used in generalization of some Hermite-Hadamard, Ostrowski and Holder's inequality in [4, 5]. Agarwal in his article [9] established some new description of these inequalities on $\mathbb{T}$. In this article, we procure Hermite-Hadamard type integral inequalities via delta-integral, which not only provides a general statment of known results on time scales, but also gives some other fascenating inequalities.

1. INTRODUCTION

Let ϕ be convex function, then

$$\phi\left(\frac{a_1+a_2}{2}\right) \leq \frac{1}{a_2-a_1}\int_{a_1}^{a_2}\phi(z)dz \leq \frac{\phi(a_1)+\phi(a_2)}{2} \tag{1.1}$$

is the H-H inequality, discussed in [1, 3]. In non-linear analysis, these inequalities play an important role . In recent years there have been elongations, generalizationss and similar type results of H-H type inequalities (see [6, 10]). Some of classical inequalities for mean can come from (1.1). The inequality (1.1) has multiple ises in a variety of settings. This has significant and remarkable groundwork in mathematical analysis.

Theorem 1. *Let $\phi : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{I}^\circ$, $a_1, a_2 \in \mathbb{I}$ with $a_1 < a_2$. If the mapping $|\phi'(z)|$ is convex, then the following inequality holds:*

$$\left|\phi\left(\frac{a_1+a_2}{2}\right) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2}\phi(z)dz\right| \leq (a_2-a_1)\frac{|\phi'(a_1)|+|\phi'(a_2)|}{8} \tag{1.2}$$

$$\left|\frac{\phi(a_1)+\phi(a_2)}{2} - \frac{1}{a_2-a_1}\int_{a_1}^{a_2}\phi(z)dz\right| \leq \frac{M(a_2-a_1)}{4} \tag{1.3}$$

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2. Time Scale

A time scale $\mathbb{T}$ means any closed subset of $\mathbb{R}$. We can give a general definition to discrete and continuous calculus by altering the range of function by an arbitrary $\mathbb{T}$. Whenever we need to study both discrete and continuous analysis we need this concept of time scale, which was given by Stefan and Aulbach. The simplest example of this unification is derivative. If $\phi : \mathbb{T} \to \mathbb{R}$ is differentiable function with derivative $\phi^{\Delta}(v)$, then for $\mathbb{T} = \mathbb{Z}$, $\phi^{\Delta}(v)$ becomes the usual forward difference i.e $\phi^{\Delta}(v) = \Delta\phi(v)$ (The discrete case). For $\mathbb{T} = \mathbb{R}$, $\phi^{\Delta}(v)$ becomes the usual derivative i.e $\phi^{\Delta}(v) = \phi'(v)$ (The continuous case). Stefan defined a time scale as a closed subset of $\mathbb{R}$, If $v : v \in \mathbb{T}$ be a single point in $\mathbb{T}$. $\sigma(v)$ and $\rho(v)$ are defined as forward and backward jump operators. Hilger's derivative or $\Delta-$derivative is defined as $\phi^{\Delta}(v)$; $\phi^{\Delta}(v)$ exist iff for every $\epsilon > 0$ there exist a neighbour U of v s.t

$$\left|\phi^{\sigma}(v) - \phi(w) - \phi^{\Delta}(v)(\sigma(v) - w)\right| \leq \epsilon|\sigma(v) - w| \; for\ all\ w \in U$$

Theorem 2. *Let $\phi : \mathbb{T} \to \mathbb{R}$ be a function and $v \in \mathbb{T}$,*
1. if ϕ is differentiable at v then ϕ is continuous at v.
2. if ϕ is differentiable at v , then

$$\phi^{\sigma}(v) = \phi(v) + \mu(v)\phi^{\Delta}(v)$$

The Jensen's inequality can be stated as follows, which has been recently obtained on time scale via delta-integral[9].

Theorem 3. *For a time scale $\mathbb{T}$ with $a_1, a_2 \in \mathbb{T}$, then if $\psi \in C([a_1, a_2], (a_3, a_4))$ and $\phi \in C((a_3, a_4), \mathbb{R})$ is convex. Then*

$$\phi\left(\frac{\int_{a_1}^{a_2} \psi(v)\Delta v}{a_2 - a_1}\right) \leq \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \phi(\psi(v))\Delta v$$

The calculus on time scale is predominently a new area that interlinks diffrence and differential calculus. The subject was launched by S. Hilger in 20th century. It is now passing hand to hand in many diffrent fields. The study of inequalities on Time scales is most recently attaining considerable attention. It has many application in enomrous areas of sciene.

For concept and introductory results on $\mathbb{T}$, we refer the reader to [2, 7, 8]. This theory has a gigantic potential to make it capable in some mathematical models of real process and phenomenon studied in population dynamics, physics, space, weather and so on. Many reports have been received regarding dynamic inequalities. The new inequalities on discrete calculus and quantum calculus will be deducted by using $\mathbb{T} = \mathbb{Z}\ and\ \mathbb{T} = q^{\mathbb{N}_\circ}$

3. Main Results

Here some estimations of Hermite-Hadamard inequalities via Δ-integral be established.

Theorem 4. *For a time scale $\mathbb{T}$ and $a_1, a_2 \in \mathbb{T}$. If $\phi : [a_1, a_2] \to \mathbb{R}$ be continuous convex, then*

$$\phi(X) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \phi(v)\Delta v \leq \frac{a_2 - X}{a_2 - a_1}\phi(a_1) + \frac{X - a_1}{a_2 - a_1}\phi(a_2) \tag{3.4}$$

where $X = \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} v\Delta v$

Proof. *For every convex function,*

$$\phi(v) \leq \phi(a_1) + \frac{\phi(a_2) - \phi(a_1)}{a_2 - a_1}(v - a_1)$$

By taking Δ-integral over $[a_1, a_2]$, we get

$$\int_{a_1}^{a_2} \phi(v)\Delta v \leq (a_2 - a_1)\phi(a_1) + \frac{\phi(a_2) - \phi(a_1)}{a_2 - a_1}\left(\int_{a_1}^{a_2} v\Delta v - a_1(a_2 - a_1)\right)$$
$$\leq \frac{(a_2 - a_1)}{a_2 - a_1}\phi(a_1) + \frac{\phi(a_2) - \phi(a_1)}{a_2 - a_1}(X - a_1)$$
$$\leq \phi(a_1) + \frac{\phi(a_2) - \phi(a_1)}{a_2 - a_1}(X - a)$$
$$\leq \phi(a_1)\left[1 - \frac{X - a_1}{a_2 - a_1}\right] + \frac{X - a_1}{a_2 - a_1}\phi(a_2)$$

$$\frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \phi(v)\Delta v \leq \frac{a_2 - X}{a_2 - a_1}\phi(a_1) + \frac{X - a_1}{a_2 - a_1}\phi(a_2) \tag{3.5}$$

By theorem 3, taking $\psi(v) = v\ for\ all\ v \in \mathbb{T}$,

$$\phi\left(\frac{\int_{a_1}^{a_2} v\Delta v}{a_2 - a_1}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \phi(v)\Delta v$$

$$\phi(X) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \phi(v)\Delta v \tag{3.6}$$

This completes the proof.

Remark 1. *If we consider $\mathbb{T} = \mathbb{R}$, then inequality (3.4) reduces to inequality (1.1)* ∎

Lemma 1. *For a time scale $\mathbb{T}$ with $a_1, a_2, X, z \in \mathbb{T}$. If $\phi : [a_1, a_2] \to \mathbb{R}$ be $\Delta-$differentiable, then*

$$\phi(X) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \phi(z)\Delta z = \frac{1}{a_2 - a_1} [I_1 + I_2]$$

Where

$$I_1 = (X-a_1)^2 \int_0^1 (1-\sigma(v))\phi^{\Delta}(va_1+(1-v)X)\Delta v,\ I_2 = (a_2-X)^2 \int_0^1 (1-\sigma(v))\phi^{\Delta}(vX+(1-v)a_2)\Delta v$$

and $X = \frac{1}{a_2-a_1} \int_{a_1}^{a_2} v\Delta v$

Proof. *Here*

$$I_1 = (X - a_1)^2 \int_0^1 (1 - \sigma(v))\phi^{\Delta}(va_1 + (1 - v)X)\Delta v$$

$$I_1 = (X - a_1)^2 \left[\frac{\phi(x)}{X - a_1} + \frac{1}{X - a_1} \int_X^{a_1} \frac{\phi(z)\Delta z}{a_1 - X} \right] = (X - a_1)\phi(x) - \int_{a_1}^{X} \phi(z)\Delta z$$

Simiarly,

$$I_2 = (a_2 - X)^2 \int_0^1 (-\sigma(v))\phi^{\Delta}(vX + (1 - v)a_2)\Delta v = (a_2 - X)\phi(x) - \int_X^{a_2} \phi(z)\Delta z$$

Adding I_1 and I_2

$$(a_2 - a_1)\phi(X) - \int_{a_1}^{X} \phi(z)\Delta z - \int_X^{a_2} \phi(z)\Delta z = I_1 + I_2$$

$$(a_2 - a_1)\phi(X) - \int_{a_1}^{X} \phi(z) = I_1 + I_2$$

dividing by $(a_2 - a_1)$, the desired result is obtained. □

Theorem 5. *For a time scale $\mathbb{T}$ with $a_1, a_2 \in \mathbb{T}$. If $\phi : [a_1, a_2] \to \mathbb{R}$ be $\Delta-$differentiable, and ϕ^{Δ} is a convex function, then*

$$\left| \phi(X) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \phi(z)\Delta z \right| \leq \frac{M}{a_2 - a_1} \left[(X - a_1)^2 (1 - h_2(1, 0)) + (a_2 - X)^2 h_2(1, 0) \right] \tag{3.7}$$

where $M = \sup_{z \in \mathbb{T} \cap [a_1, a_2]} \left| \phi^{\Delta}(z) \right|$

Proof. *By using Lemma 1, we have*

$$\left| \phi(X) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \phi(z)\Delta z \right| \leq \frac{1}{a_2 - a_1} \{|I_1| + |I_2|\}$$

$$|I_1| \leq (X - a_1)^2 \int_0^1 (1 - \sigma(v)) \left| \phi^{\Delta}(a_1 v + (1 - v)X) \right| \Delta v$$

by applying convexity on ϕ^{Δ}

$$|I_1| \leq (X - a_1)^2 \left[\int_0^1 (1 - \sigma(v)) \left\{ \left| \phi^{\Delta}(a_1) \right| v + (1 - v) \left| \phi^{\Delta}(x) \right| \right\} \right]$$

$$|I_1| \leq (X - a_1)^2 \left[1 - h_2(1, 0)\right] \tag{3.8}$$

Similarly,

$$|I_2| \leq (d - X)^2 h_2(1, 0) \tag{3.9}$$

which completes proof.

***Remark 2** In above theorem, if we consider $\mathbb{T} = \mathbb{R}$, then inequality(3.8) reduces to inequality (1.2)* □

Theorem 6. *For a time scale $\mathbb{T}$ with $a_1, a_2 \in \mathbb{T}$. If $\phi : [a_1, a_2] \to \mathbb{R}$ be $\Delta-$differentiable, then*

$$\left|\phi(X) - \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \phi(z)\Delta z\right| \leq \frac{1}{a_2 - a_1}\left[(X - a_1)^2 \left\{(h_2(1, 0) - 2h_3(1, 0)) \left|\phi^{\Delta}(a_1)\right|\right.\right.$$
$$\left. + (1 - 2h_2(1, 0) + 2h_3(1, 0)) \left|\phi^{\Delta}(X)\right|\right\} + (a_2 - X)^2 \left\{(h_2(1, 0) - 2h_3(1, 0))\right.$$
$$\left.\left. \times \left|\phi^{\Delta}(X)\right| + (1 - 2h_2(1, 0) + 2h_3(1, 0)) \left|\phi^{\Delta}(a_2)\right|\right\}\right] \tag{3.10}$$

Proof. Proceed as proof of theorem 4.

Remark 3 If we consider $\mathbb{T} = \mathbb{R}$, then(3.11) reduces to (1.2) □

Lemma 2. *For a time scale $\mathbb{T}$ with $a_1, a_2 \in \mathbb{T}$. If $\phi : [a_1, a_2] \to \mathbb{R}$ be $\Delta-$differentiable, then*

$$\frac{a_2 - X}{a_2 - a_1}\phi(a_1) + \frac{X - a_1}{a_2 - a_1}\phi(a_2) - \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \phi(z)\Delta z = \frac{a_2 - a_1}{2}[I_1 + I_2]$$

where

$I_1 = \int_0^1 \left(\frac{X - a_1}{a_2 - a_1} - \sigma(v)\right) \phi^{\Delta}(va_2 + (1-v)a_1)\Delta v$, $I_2 = \int_0^1 \left(\sigma(v) - \frac{a_2 - X}{a_2 - a_1}\right) \phi^{\Delta}(va_2 + (1-v)a_1)\Delta v$
and $X = \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} v\Delta v$

Proof. Proceed as proof of Lemma 1. □

Theorem 7. *For a time scale $\mathbb{T}$ with $a_1, a_2 \in \mathbb{T}$. If $\phi : [a_1, a_2] \to \mathbb{R}$ be $\Delta-$differentiable, then*

$$\left|\frac{a_2 - X}{a_2 - a_1}\phi(a_1) + \frac{X - a_1}{a_2 - a_1}\phi(a_2) - \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \phi(z)\Delta z\right| \leq (a_2 - a_1)\left[h_2\left(\frac{a_2 - X}{a_2 - a_1}, 0\right) + h_2\left(\frac{X - a_1}{a_2 - a_1}, 0\right)\right] \tag{3.11}$$

Where $M = \sup_{z \in \mathbb{T} \cap [a_2, a_1]} \phi^{\Delta}(z)$

Proof. Proceed as proof of Theorem 4.

Remark 4 If $\mathbb{T}=\mathbb{R}$, then (3.12) reduces to (1.3) □

Theorem 8. *Let $\mathbb{T}$ be a time scale with $a_1,\ a_2 \in \mathbb{T}$. Let $\phi : [a_1,\ a_2] \to \mathbb{R}$ be a $\Delta-$differentiable function, then*

$$\left|\frac{a_2-X}{a_2-a_1}\phi(a_1)+\frac{X-a_1}{a_2-a_1}\phi(a_2)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\phi(z)\Delta z\right| \le (a_2-a_1)\left[\left\{h_3\left(\frac{X-a_1}{a_2-a_1},0\right)\right.\right.$$
$$\left.+h_2\left(\frac{a_2-X}{a_2-a_1},0\right)\right\}\left|\phi^{\Delta}(a_1)\right| + \left\{h_3\left(\frac{a_2-X}{a_2-a_1},0\right)+h_2\left(\frac{X-a_1}{a_2-a_1},0\right)\right.$$
$$\left.\left.-h_3\left(\frac{X-a_1}{a_2-a_1},0\right)\right\}\left|\phi^{\Delta}(a_2)\right|\right] \tag{3.12}$$

Proof. *By applying properties of modulus and convexity, then using the following $\Delta-$integrals*

$$\int_0^1\left|\left(\frac{X-a_1}{a_2-a_1}-\sigma(v)\right)\right|v\Delta v=\int_0^{\frac{X-a_1}{a_2-a_1}}\left(\frac{X-a_1}{a_2-a_1}-\sigma(v)\right)v\Delta v+\int_{\frac{X-a_1}{a_2-a_1}}^1\left(v-\frac{X-a_1}{a_2-a_1}\right)v\Delta v$$

Now,

$$\int_0^{\frac{X-a_1}{a_2-a_1}}\left(\frac{X-a_1}{a_2-a_1}-\sigma(v)\right)v\Delta v=\left(\frac{X-a_1}{a_2-a_1}-v\right)\int_X^v v\Delta v\Big|_0^{\frac{X-a_1}{a_2-a_1}}+\int_0^{\frac{X-a_1}{a_2-a_1}}\left(\int_X^v v\Delta v\right)\Delta v$$

$$=\frac{X-a}{a_2-a_1}h_2(x,0)+\int_0^{\frac{X-a_1}{a_2-a_1}}[h_2(v,0)-h_2(x,0)]\Delta v=\int_0^{\frac{X-a_1}{a_2-a_1}}h_2(v,0)\Delta v=h_3\left(\frac{X-a_1}{a_2-a_1},0\right)$$

Also,

$$\int_{\frac{X-a_1}{a_2-a_1}}^1\left(\sigma(v)-\frac{X-a_1}{a_2-a_1}\right)(1-v)\Delta v=\int_0^{\frac{X-a_1}{a_2-a_1}}\left(\frac{X-a_1}{a_2-a_1}-\sigma(v)\right)v\Delta v=h_3\left(\frac{X-a_1}{a_2-a_1},0\right)$$

Similarly,

$$\int_0^{\frac{X-a_1}{a_2-a_1}}\left(\frac{X-a_1}{a_2-a_1}-\sigma(v)\right)(1-v)\Delta v=h_2\left(\frac{X-a_1}{a_2-a_1},0\right)-h_3\left(\frac{X-a_1}{a_2-a_1},0\right)$$

$$\int_{\frac{X-a_1}{a_2-a_1}}^1\left(\sigma(v)-\frac{X-a_1}{a_2-a_1}\right)v\Delta v$$

Analagously,

$$\int_{\frac{X-a_1}{a_2-a_1}}^1\left(\sigma(v)-\frac{X-a_1}{a_2-a_1}\right)v\Delta v=h_2\left(\frac{X-a_1}{a_2-a_1},0\right)-h_3\left(\frac{X-a_1}{a_2-a_1},0\right)$$

$$=\int_0^{\frac{X-a_1}{a_2-a_1}}\left(\frac{X-a_1}{a_2-a_1}-\sigma(v)\right)(1-v)\Delta v$$

And

$$\int_0^{\frac{X-a_1}{a_2-a_1}} \left(\frac{X-a_1}{a_2-a_1} - \sigma(v)\right) = h_3\left(\frac{X-a_1}{a_2-a_1}, 0\right) = \int_{\frac{X-a_1}{a_2-a_1}}^1 \left(\sigma(v) - \frac{X-a_1}{a_2-a_1}\right)(1-v)\Delta v$$

Which completes the proof.

Remark 5 *If we consider $\mathbb{T} = \mathbb{R}$, then the inequality (3.13) reduces to inequality (1.3)* □

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5. Conclusion

we set up generalised H.H on $\mathbb{T}$ which describes the previous inequalities developed in [11,12]. Some nearly new results are also given. These generalizations are understandable in literature as they have instanteneous applications in numerical integrations and special means. our result is capable enough to be derived as as special cases of our result considering different $\mathbb{T}$.

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