ON THE FRACTIONAL INTEGRAL INEQUALITIES BY THE WAY OF DOUBLE INTEGRALS

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ABSTRACT. This study presents some of the latest results annexed to Hermite Hadamard inequality by utilizing Riemann-Liouville fractional derivatives by the way of double integrals. Another aim of this article is to discuss some of the recent developments on Hermite Hadamards type inequalities.

1. INTRODUCTION

The usefulness of inequalities involving convex functions is realized from the very beginning and is now widely acknowledged as one of the prime driving forces behind the development of several modern branches of mathematics and has been given considerable attention. One of the most famous inequalities for convex functions is HermiteHadamard inequality, stated as [8]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

Both inequalities hold in the reversed direction for f to be concave.

It is well known that the Hermite$-$Hadamard inequality plays an important role in nonlinear analysis. In the recent years, this classical inequality has been improved and generalized in a number of ways and a large number of research papers have been written on this inequality, [8, 10, 11, 12, 13].

In recent paper, [14] Sarikaya et. al. proved a variant of Hermite$-$Hadamard's inequalities in fractional integral forms as follows:

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Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a, b]$. If f is convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.2}$$

Remark 1. *For $\alpha = 1$, inequality (1.2) reduces to inequality (1.1).*

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 1. *Let $f \in L[a, b]$, the Reimann-Liouville integrals $J_{a^+}^{\alpha}$ and $J_{b^-}^{\alpha}$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a^+}^{\alpha} = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > \alpha$$

and

$$J_{b^-}^{\alpha} = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < \alpha$$

respectively. Here, $\Gamma(\alpha) = \int_0^{\infty} e^{-t} u^{\alpha-1} du$ is the Gamma function and $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Properties concerning this operator can be found in [9].

In this article, we establish some new estimates of left and right Hermite−Hadamard inequality in the form of fractional integrals by the way of double integrals for functions whose absolute values of first derivatives are convex and concave.

2. Main Results

I this section, first we will establish the identities with name as lemma 1 and lemma 1 and further utilizing these two lemmas we laid down some results which estimates the left and right side of Hermite−Hadamard inequality.

Lemma 1. *Suppose $f : \mathbb{I}^0 \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping over $\mathbb{I}^0$ (interior of $a, b \in \mathbb{I}$) for $a < b$. If $|f'|^q$ be a convex function and $\alpha \in (0, 1)$, then it gives*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \\ &\quad \leq \frac{b-a}{2} \int_0^1 \int_0^1 [f'(ta+(1-t)b) + f'(sb+(1-s)a)](s^{\alpha} - t^{\alpha})dtds \end{aligned} \tag{2.3}$$

Proof. Let

$$I_1 = \int_0^1 \int_0^1 (s^\alpha - t^\alpha) f'(ta + (1-t)b) dt ds$$
$$= \frac{\alpha}{(\alpha-1)(b-a)} f(a) + \frac{1}{(\alpha+1)(b-a)} f(b) - \frac{\alpha}{b-a} \int_0^1 t^{\alpha-1} f(ta+(1-t)b) dt \quad (2.4)$$

Similarly

$$I_2 = \int_0^1 \int_0^1 (s^\alpha - t^\alpha) f'(sb + (1-s)a) dt ds$$
$$= \frac{\alpha}{(\alpha+1)(b-a)} f(b) + \frac{1}{(\alpha+1)(b-a)} f(a) - \frac{\alpha}{(b-a)^\alpha} \int_0^1 s^{\alpha-1} f(sb+(1-s)a) ds \quad (2.5)$$

By adding (2.4) and (2.5), we have

$$\frac{(b-a)}{2}(I_1 + I_2) = \frac{f(a)+f(b)}{2} - \frac{\gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]$$

Which completes the proof. □

Theorem 2. *Suppose $f : \mathbb{I}^0 \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping over $\mathbb{I}^0$ (interior of $a, b \in \mathbb{I}$) for $a < b$. If $|f'|^q$ be a convex function and $\alpha \in (0,1)$, then it gives.*

$$\left| \frac{f(a)+f(b)}{2} - \frac{\gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \right| \leq \frac{\alpha(b-a)}{(\alpha+1)(\alpha+2)} \left[|f'(a)| + |f'(b)|\right] \quad (2.6)$$

Proof. Lemma 1 can be rephrase as

$$\left| \frac{f(a)+f(b)}{2} - \frac{\gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \right|$$
$$\leq \frac{b-a}{2} \int_0^1 \int_0^1 \left| \left[f'(ta + (1-t)b) + f'(sb + (1-s)a)\right](s^\alpha - t^\alpha) \right| dt ds \quad (2.7)$$

By applying convexity,

$$\left| \frac{f(a)+f(b)}{2} - \frac{\gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \right|$$
$$\leq \frac{b-a}{2} \int_0^1 \int_0^1 |(s^\alpha - t^\alpha)| \left(t|f'(a)| + (1-t)|f'(b)|\right) dt ds$$
$$+ \frac{b-a}{2} \int_0^1 \int_0^1 |(s^\alpha - t^\alpha)| \left(s|f'(b)| + (1-s)|f'(a)|\right) dt ds \quad (2.8)$$

Here

$$\int_0^1 \int_0^1 t|(s^\alpha - t^\alpha)| dt ds = \int_0^1 \int_0^1 s|(s^\alpha - t^\alpha)| dt ds = \frac{\alpha(3\alpha+5)}{2(\alpha+1)(\alpha+2)(\alpha+3)}$$

And

$$\int_0^1\int_0^1 (1-t)|(s^\alpha - t^\alpha)|dtds = \int_0^1\int_0^1 (1-s)|(s^\alpha - t^\alpha)|dtds = \frac{\alpha^2+7\alpha}{2(\alpha+1)(\alpha+2)(\alpha+3)}$$

Now equation (2.8) becomes

$$\left|\frac{f(a)+f(b)}{2} - \frac{\gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]\right|$$
$$\le \frac{b-a}{2}\int_0^1\int_0^1 t|(s^\alpha - t^\alpha)||f'(a)|dtds + \frac{b-a}{2}\int_0^1\int_0^1 (1-t)|(s^\alpha - t^\alpha)||f'(b)|dtds$$
$$+ \frac{b-a}{2}\int_0^1\int_0^1 s|(s^\alpha - t^\alpha)|dtds + \frac{b-a}{2}\int_0^1\int_0^1 (1-s)|(s^\alpha - t^\alpha)|dtds$$
$$\le \frac{\alpha(b-a)}{(\alpha+1)(\alpha+2)}\left[|f'(a)| + |f'(b)|\right]$$

Which completes the proof. □

Theorem 3. *Suppose $f : \mathbb{I}^0 \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping over $\mathbb{I}^0$ (interior of $a, b \in \mathbb{I}$) for $a < b$. If $|f'|^q$ be a convex function and $\alpha \in (0,1)$, then it gives*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]\right| \le \frac{2^{\frac{1}{p}}(b-a)}{(\alpha p+1)^{\frac{1}{p}}(q+1)^{\frac{1}{q}}}\left(|f'a)|^q - |f'(b)|^q\right)^{\frac{1}{q}} \tag{2.9}$$

Proof. By applying Holder's inequality on lemma 1, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{\gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]\right|$$
$$\le (b-a)\int_0^1\int_0^1 |(s^\alpha - t^\alpha)|\,|f'(ta+(1-t)b)|\,dtds$$
$$\le (b-a)\left(\int_0^1\int_0^1 |(s^\alpha - t^\alpha)|^p\,dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1 |f'(ta+(1-t)b)|^q dtds\right)^{\frac{1}{q}} \tag{2.10}$$

utilizing the coming inequality in (2.10), we get (2.11)

$$|x^\alpha - y^\alpha|^p \le |x^\alpha + y^\alpha|^p \le |x^\alpha|^p + |y^\alpha|^p$$

$$\int_0^1\int_0^1 |s^\alpha - t^\alpha|^p dtds \le \int_0^1\int_0^1 (|s^\alpha|^p + |t^\alpha|^p)dtds \le \frac{2}{\alpha p+1} \tag{2.11}$$

Since $|f'|^q$ is convex, we have

$$\int_0^1\int_0^1 |f'(ta+(1-t)b)|^q dtds \le \frac{1}{q+1}(|f'(a)|^q - |f'(b)|^q)$$

Now by using these values in (2.10), we catch

$$\left|\frac{f(a)+f(b)}{2} - \frac{\gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]\right| \le \frac{2^{\frac{1}{p}}(b-a)}{(\alpha p+1)^{\frac{1}{p}}(q+1)^{\frac{1}{q}}}\left(|f'(a)|^q - |f'(b)|^q\right)^{\frac{1}{q}}$$

Which completes the proof. □

Theorem 4. *Suppose* $f : \mathbb{I}^0 \subseteq \mathbb{R} \to \mathbb{R}$ *be differentiable mapping over* $\mathbb{I}^0$ *(interior of* $a, b \in \mathbb{I}$*) for* $a < b$*. If* $|f'|^q$ *be a convex function and* $\alpha \in (0,1)$*, then it gives.*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\right]\right| \leq (b-a)(ln2)^{\frac{1}{q}}\left(\frac{2}{\alpha p+1}\right)^{\frac{1}{p}}\left(|f'a)|^q + |f'(b)|^q\right)^{\frac{1}{q}} \quad (2.12)$$

Proof. By applying Holder's inequality on lemma 1, we catch

$$\left|\frac{f(a)+f(b)}{2} - \frac{\gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\right]\right| \leq (b-a)\left(\int_0^1\int_0^1 |(s^\alpha - t^\alpha)|^p\, dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1 |f'(ta+(1-t)b)|^q\, dtds\right)^{\frac{1}{q}} \quad (2.13)$$

By s-convexity, we get

$$\int_0^1\int_0^1 |f'(ta+(1-t)b)|^q dtds \leq \int_0^1\int_0^1 t^s|f'(a)|^q dtds + \int_0^1\int_0^1 (1-t)^s|f'(b)|^q dtds \leq \ln 2\left(|f'(a)|^q + |f'(b)|^q\right)$$

And

$$\int_0^1\int_0^1 |s^\alpha - t^\alpha|^p dtds \leq \frac{2}{\alpha p+1}$$

Now by utilizing above values in (2.13), we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{\gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\right]\right| \leq (b-a)(\ln 2)^{\frac{1}{p}}\left(\frac{2}{\alpha p+1}\right)^{\frac{1}{p}}\left(|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}}$$

This completes the proof. □

Theorem 5. *Suppose* $f : \mathbb{I}^0 \subseteq \mathbb{R} \to \mathbb{R}$ *be differentiable mapping over* $\mathbb{I}^0$ *(interior of* $a, b \in \mathbb{I}$*) for* $a < b$*. If* $|f'|^q$ *be a convex function and* $\alpha \in (0,1)$*, then it contributes*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\right]\right| \leq (b-a)\left(\frac{1}{2\ln 2}\right)^{\frac{1}{q}}\left(\frac{2}{\alpha p+1}\right)^{\frac{1}{p}}\left|f'\left(\frac{a+b}{2}\right)\right| \quad (2.14)$$

Proof. By utilizing Holder's inequality on lemma 1

$$\left|\frac{f(a)+f(b)}{2}-\frac{\gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|$$
$$\leq (b-a)\left(\int_0^1\int_0^1 |(s^{\alpha}-t^{\alpha})|^p\,dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1 |f'(ta+(1-t)b)|^q\,dtds\right)^{\frac{1}{q}} \quad (2.15)$$

By s-concavity, we have

$$\int_0^1\int_0^1 |f'(ta+(1-t)b)|^q\,dtds\leq\left|f'\left(\frac{a+b}{2}\right)\right|^q\int_0^1\int_0^1 2^{s-1}dtds\leq\left|f'\left(\frac{a+b}{2}\right)\right|^q\left(\frac{1}{2\ln 2}\right)$$

And

$$\int_0^1\int_0^1 |s^{\alpha}-t^{\alpha}|^p dtds\leq\frac{2}{\alpha p+1}$$

By utilizing above values in (2.15), we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|$$
$$\leq (b-a)\left(\frac{2}{\alpha p+1}\right)^{\frac{1}{p}}\left|f'(\frac{a+b}{2})\right|\left(\frac{1}{2ln2}\right)^{\frac{1}{q}}$$

It can also be written as.

$$\left|\frac{f(a)+f(b)}{2}-\frac{\gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|$$
$$\leq (b-a)\left(\frac{1}{2ln2}\right)^{\frac{1}{q}}\left(\frac{2}{\alpha p+1}\right)^{\frac{1}{p}}\left|f'\left(\frac{a+b}{2}\right)\right|$$

Which completes the proof. □

Lemma 2. *Suppose $f:\mathbb{I}^0\subseteq\mathbb{R}\to\mathbb{R}$ be differentiable mapping over $\mathbb{I}^0$ (interior of $a,b\in\mathbb{I}$) for $a<b$. If $|f'|^q$ be a convex function and $\alpha\in(0,1)$, then it gives.*

$$f(\frac{a+b}{2})-\frac{\gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]=\frac{b-a}{2^{\alpha+2}}\sum_{k=1}^{4}I_k \quad (2.16)$$

where,

$$I_1=\int_0^1\int_0^1 (t^{\alpha}-s^{\alpha})f'\left(t\frac{a+b}{2}+(1-t)a\right)dtds$$
$$I_2=\int_0^1\int_0^1 (t^{\alpha}-s^{\alpha})f'\left(s\frac{a+b}{2}+(1-s)a\right)dtds$$
$$I_3=\int_0^1\int_0^1 \left[2^{\alpha}+s^{\alpha}-(1+t)^{\alpha}\right]f'\left(ta+(1-t)\frac{a+b}{2}\right)dtds$$
$$I_4=\int_0^1\int_0^1 \left[(1+s)^{\alpha}-2^{\alpha}-t^{\alpha}\right]f'\left(sb+(1-s)\frac{a+b}{2}\right)dtds$$

Proof.

$$
\begin{aligned}
I_1 &= \int_0^1 \int_0^1 (t^\alpha - s^\alpha) f'(t\frac{a+b}{2} + (1-t)a) dtds \\
&= \frac{2\alpha}{(b-a)(\alpha+1)} f(\frac{a+b}{2}) + \frac{2}{(b-a)(\alpha+1)} f(a) \\
&\qquad - \frac{2\alpha}{b-a} \int_a^{\frac{a+b}{2}} 2^{\alpha-1} \frac{(U-a)^{\alpha-1}}{(b-a)^{\alpha-1}} f(u).2\frac{du}{b-a} \\
I_2 &= \int_0^1 \int_0^1 (s^\alpha - t^\alpha) f'(t\frac{a+b}{2} + (1-t)b) dtds \\
&= \frac{2\alpha}{(b-a)(\alpha+1)} f(\frac{a+b}{2}) + \frac{2}{(b-a)(\alpha+1)} f(b) \\
&\qquad - \frac{2\alpha}{b-a} \int_{\frac{a+b}{2}}^{b} 2^{\alpha-1} (\frac{b-U}{b-a})^{\alpha-1} f(u).2\frac{du}{b-a} \\
I_3 &= \int_0^1 \int_0^1 [2^\alpha + s^\alpha - (1+t)^\alpha] f'(ta + (1-t)\frac{a+b}{2}) dtds \\
&= \frac{-2}{(b-a)(\alpha+1)} f(a) + \frac{2}{b-a}(2^\alpha - \frac{\alpha}{\alpha+1}) f(\frac{a+b}{2}) \\
&\qquad - \frac{2\alpha}{b-a} \int_a^{\frac{a+b}{2}} (\frac{b-u}{b-a})^{\alpha-1} f(u) \frac{2du}{b-a} \\
I_4 &= \int_0^1 \int_0^1 [(1+t)^\alpha - 2^\alpha - s^\alpha] f'(tb + (1-t)\frac{a+b}{2}) dtds \\
&= \frac{-2}{(b-a)(\alpha+1)} f(b) + \frac{2}{b-a}(2^\alpha - \frac{\alpha}{\alpha+1}) f(\frac{a+b}{2}) \\
&\qquad - \frac{2\alpha}{b-a} \int_{\frac{a+b}{2}}^{b} 2^{\alpha-1} (\frac{U-a}{b-a})^{\alpha-1} f(u) \frac{2du}{b-a}
\end{aligned}
$$

this gives

$$
\sum_{i=1}^{i=4} I_i = \frac{4.2^\alpha}{b-a} f(\frac{a+b}{2}) - \frac{2^{\alpha+2}\alpha\gamma(\alpha)}{2(b-a)^{\alpha+1}} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]
$$

It can also be written as

$$
\frac{b-a}{2^{\alpha+2}} \sum_{i=1}^{i=4} I_i = f\left(\frac{a+b}{2}\right) - \frac{\gamma(\alpha+1)}{(2b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]
$$

Which completes the proof. □

Theorem 6. *Suppose $f : \mathbb{I}^0 \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping over $\mathbb{I}^0$ (interior of $a, b \in \mathbb{I}$) for $a < b$. If $|f'|^q$ be a convex function and $\alpha \in (0,1)$, then it gives*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{\gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right]\right| \\
&\quad = \frac{b-a}{2^{\alpha+2}}\left[\frac{6\alpha^2+12\alpha-4.2^{\alpha}(\alpha+3)}{2(\alpha+1)(\alpha+2)(\alpha+3)}\right]\left|f'\left(\frac{a+b}{2}\right)\right| \\
&\qquad + \frac{b-a}{2^{\alpha+2}}\left[\frac{2\alpha^2+10\alpha+2^{\alpha}(\alpha+3)(\alpha^2-\alpha+2)}{2(\alpha+1)(\alpha+2)(\alpha+3)}\right]|f'(a)| \\
&\qquad\quad + \frac{b-a}{2^{\alpha+2}}\left[\frac{6\alpha^2+12\alpha-4.2^{\alpha}(\alpha+3)}{2(\alpha+1)(\alpha+2)(\alpha+3)}\right]|f'(b)| \qquad (2.17)
\end{aligned}$$

Proof. From lemma 2

$$\left|f\left(\frac{a+b}{2}\right) - \frac{\gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right]\right| = \frac{b-a}{2^{\alpha+2}}\sum_{k=1}^{4}|I_k| \qquad (2.18)$$

where

$$\begin{aligned}
|I_1| &= \left|\int_0^1\int_0^1 (t^{\alpha}-s^{\alpha})f'\left(t\frac{a+b}{2}+(1-t)a\right)dtds\right| \\
&\le \int_0^1\int_0^1 |t^{\alpha}-s^{\alpha}|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|dtds
\end{aligned}$$

By using convexity of $|f'|$, we have

$$\begin{aligned}
|I_1| &\le \int_0^1\int_0^1 |t^{\alpha}-s^{\alpha}|\left(t\left|f'\left(\frac{a+b}{2}\right)\right| + (1-t)|f'(a)|\right)dtds \\
&\le \int_0^1\int_0^1 t|t^{\alpha}-s^{\alpha}|\left|f'\left(\frac{a+b}{2}\right)\right|dtds + \int_0^1\int_0^1 (1-t)|t^{\alpha}-s^{\alpha}||f'(a)|dtds \qquad (2.19)
\end{aligned}$$

Here

$$\int_0^1\int_0^1 t|t^{\alpha}-s^{\alpha}|dtds = \frac{3\alpha^2+5\alpha}{2(\alpha+1)(\alpha+2)(\alpha+3)}$$

And

$$\int_0^1\int_0^1 (1-t)|t^{\alpha}-s^{\alpha}|dtds = \frac{\alpha^2+7\alpha}{2(\alpha+1)(\alpha+2)(\alpha+3)}$$

By putting values in (2.19), we have

$$|I_1| \le \frac{3\alpha^2+5\alpha}{2(\alpha+1)(\alpha+2)(\alpha+3)}\left|f'\left(\frac{a+b}{2}\right)\right| + \frac{\alpha^2+7\alpha}{2(\alpha+1)(\alpha+2)(\alpha+3)}|f'(a)|$$

Now

$$\begin{aligned}
|I_2| &= \left|\int_0^1\int_0^1 (t^{\alpha}-s^{\alpha})f'\left(s\frac{a+b}{2}+(1-s)b\right)dtds\right| \\
&\le \int_0^1\int_0^1 |t^{\alpha}-s^{\alpha}|\left|f'\left(s\frac{a+b}{2}+(1-s)b\right)\right|dtds
\end{aligned}$$

By using convexity of $|f'|$, we have

$$\begin{aligned} |I_2| &\leq \int_0^1 \int_0^1 |t^\alpha - s^\alpha| \left(s \left| f'\left(\frac{a+b}{2}\right)\right| + (1-s)|f'(b)| \right) dtds \\ &\leq \int_0^1\int_0^1 s|t^\alpha - s^\alpha| \left| f'\left(\frac{a+b}{2}\right)\right| dtds + \int_0^1\int_0^1 (1-s)|t^\alpha - s^\alpha||f'(b)|dtds \end{aligned} \tag{2.20}$$

Here

$$\int_0^1 \int_0^1 s|t^\alpha - s^\alpha|dtds = \frac{3\alpha^2+5\alpha}{2(\alpha+1)(\alpha+2)(\alpha+3)}$$

And

$$\int_0^1 \int_0^1 (1-s)|t^\alpha - s^\alpha|dtds = \frac{\alpha^2+7\alpha}{2(\alpha+1)(\alpha+2)(\alpha+3)}$$

By utilizing values in (2.20), we have

$$|I_2| \leq \frac{3\alpha^2+5\alpha}{2(\alpha+1)(\alpha+2)(\alpha+3)} \left| f'\left(\frac{a+b}{2}\right)\right| + \frac{\alpha^2+7\alpha}{2(\alpha+1)(\alpha+2)(\alpha+3)} |f'(b)|$$

Now

$$\begin{aligned} |I_3| &= |\int_0^1 \int_0^1 [2^\alpha + s^\alpha - (1+t)^\alpha] \left| f'\left(ta + (1-t)\frac{a+b}{2}\right) dtds \right| \\ &\leq \int_0^1 \int_0^1 |2^\alpha + s^\alpha - (1+t)^\alpha| \left| f'\left(ta + (1-t)\frac{a+b}{2}\right)\right| dtds \end{aligned}$$

By convexity of $|f'|$, we have

$$\begin{aligned} |I_3| &\leq \int_0^1 \int_0^1 |2^\alpha + s^\alpha - (1+t)^\alpha| \left(t|f'(a)| + (1-t)\left| f'\left(\frac{a+b}{2}\right)\right|\right) dtds \\ &\leq \int_0^1\int_0^1 t(2^\alpha + s^\alpha - (1+t)^\alpha)|f'(a)|dtds + \int_0^1\int_0^1 (1-t)(2^\alpha + s^\alpha - (1+t)^\alpha) \left| f'\left(\frac{a+b}{2}\right)\right| dtds \end{aligned}$$

From (2.21)

$$\begin{aligned} \int_0^1 \int_0^1 t(2^\alpha + s^\alpha - (1+t)^\alpha)dtds &= \int_0^1 \left[(2^\alpha + s^\alpha) \int_0^1 tdt - \int_0^1 t(1+t)^\alpha dt \right] ds \\ &= \int_0^1 \left[\frac{2^\alpha + s^\alpha}{2} - \frac{2^{\alpha+1}}{\alpha+1} + \frac{2^{\alpha+2}}{(\alpha+1)(\alpha+2)} - \frac{1}{(\alpha+1)(\alpha+2)} \right] ds \\ &= \frac{2^\alpha(\alpha^2-\alpha+2)+\alpha}{2(\alpha+1)(\alpha+2)} \end{aligned}$$

Also from equation 2.21

$$\int_0^1\int_0^1 (1-t)(2^\alpha+s^\alpha-(1+t)^\alpha)dtds$$
$$=\int_0^1\int_0^1 \left[2^\alpha+s^\alpha-(1+t)^\alpha\right]dtds-\int_0^1\int_0^1 t(2^\alpha+s^\alpha-(1+t)^\alpha)dtds$$
$$=\int_0^1\left[2^\alpha+s^\alpha-\frac{2^{\alpha+1}}{\alpha+1}+\frac{1}{\alpha+1}\right]ds-\frac{2^\alpha(\alpha^2-\alpha+2)+\alpha}{2(\alpha+1)(\alpha+2)}$$
$$=\frac{2^\alpha(\alpha-1)+2}{\alpha+1}-\frac{2^\alpha(\alpha^2-\alpha+2)+\alpha}{2(\alpha+1)(\alpha+2)}$$
$$=\frac{2^\alpha(\alpha^2+3\alpha-6)+3\alpha+8}{2(\alpha+1)(\alpha+2)}$$

By putting values in equation (2.21), we have

$$|I_3|\leq\frac{2^\alpha(\alpha^2-\alpha+2)+\alpha}{2(\alpha+1)(\alpha+2)}|f'(a)|+\frac{2^\alpha(\alpha^2+3\alpha-6)+3\alpha+8}{2(\alpha+1)(\alpha+2)}|f'(\frac{a+b}{2})|$$

Now

$$|I_4|=\left|\int_0^1\int_0^1\left[(1+s)^\alpha-2^\alpha-t^\alpha\right]f'(sb+(1-s)\frac{a+b}{2})dtds\right|$$
$$\leq\int_0^1\int_0^1 |(1+s)^\alpha-2^\alpha-t^\alpha|\left|f'(sb+(1-s)\frac{a+b}{2})\right|dtds$$

By convexity of $|f'|$ we have

$$|I_4|\leq\int_0^1\int_0^1 |(1+s)^\alpha-2^\alpha-t^\alpha|(s|f'(b)|+(1-s)|f'(\frac{a+b}{2})|)dtds$$
$$\leq\int_0^1\int_0^1 s((1+s)^\alpha-2^\alpha-t^\alpha)|f'(b)|dtds$$
$$+\int_0^1\int_0^1 (1-s)((1-s)^\alpha-2^\alpha-t^\alpha)|f'(\frac{a+b}{2})|dtds \tag{2.21}$$

From equation (2.21), we have

$$\int_0^1\int_0^1 s((1+s)^\alpha-2^\alpha-t^\alpha)dtds=\frac{2^\alpha(-\alpha^2+\alpha-2)-\alpha}{2(\alpha+1)(\alpha+2)}$$

Also from equation (2.21), we have

$$\int_0^1\int_0^1 (1-s)((1-s)^\alpha-2^\alpha-t^\alpha)dtds=\frac{2^\alpha(-\alpha^2-3\alpha+2)-3\alpha-6}{2(\alpha+1)(\alpha+2)}$$

By putting values in equation (2.21), we have

$$|I_4|\leq\frac{2^\alpha(-\alpha^2+\alpha-2)-\alpha}{2(\alpha+1)(\alpha+2)}|f'(b)|+\frac{2^\alpha(-\alpha^2-3\alpha+2)-3\alpha-6}{2(\alpha+1)(\alpha+2)}|f'(\frac{a+b}{2})|$$

Now

$$\sum_{k=1}^{4} I_k = |I_1| + |I_2| + |I_3| + |I_4| = \frac{3\alpha^2+5\alpha}{2(\alpha+1)(\alpha+2)(\alpha+3)}|f'(\frac{a+b}{2})|$$
$$+\frac{\alpha^2+7\alpha}{2(\alpha+1)(\alpha+2)(\alpha+3)}|f'(a)| + \frac{3\alpha^2+5\alpha}{2(\alpha+1)(\alpha+2)(\alpha+3)}|f'(\frac{a+b}{2})|$$
$$+\frac{\alpha^2+7\alpha}{2(\alpha+1)(\alpha+2)(\alpha+3)}|f'(b)| + \frac{2^\alpha(\alpha^2-\alpha+2)+\alpha}{2(\alpha+1)(\alpha+2)}|f'(a)|$$
$$+\frac{2^\alpha(\alpha^2+3\alpha-6)+3\alpha+8}{2(\alpha+1)(\alpha+2)}|f'(\frac{a+b}{2})| + \frac{2^\alpha(-\alpha^2+\alpha-2)-\alpha}{2(\alpha+1)(\alpha+2)}|f'(b)|$$
$$+\frac{2^\alpha(-\alpha^2-3\alpha+2)-3\alpha-6}{2(\alpha+1)(\alpha+2)}|f'(\frac{a+b}{2})|$$
$$= \left[\frac{6\alpha^2+12\alpha-4.2^\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)(\alpha+3)}\right]|f'(\frac{a+b}{2})|$$
$$+\left[\frac{2\alpha^2+10\alpha+2^\alpha(\alpha+3)(\alpha^2-\alpha+2)}{2(\alpha+1)(\alpha+2)(\alpha+3)}\right]|f'(a)|$$
$$+\left[\frac{6\alpha^2+12\alpha-4.2^\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)(\alpha+3)}\right]|f'(b)|$$

Now equation (2.18) becomes

$$\left|f(\frac{a+b}{2}) - \frac{\gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\right]\right|$$
$$= \frac{b-a}{2^{\alpha+2}}\left[\frac{6\alpha^2+12\alpha-4.2^\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)(\alpha+3)}\right]|f'(\frac{a+b}{2})|$$
$$+\frac{b-a}{2^{\alpha+2}}\left[\frac{2\alpha^2+10\alpha+2^\alpha(\alpha+3)(\alpha^2-\alpha+2)}{2(\alpha+1)(\alpha+2)(\alpha+3)}\right]|f'(a)|$$
$$+\frac{b-a}{2^{\alpha+2}}\left[\frac{6\alpha^2+12\alpha-4.2^\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)(\alpha+3)}\right]|f'(b)|$$

Which completes the proof. □

Theorem 7. *Suppose $f : \mathbb{I}^0 \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping over $\mathbb{I}^0$ (interior of $a, b \in \mathbb{I}$) for $a < b$. If $|f'|^q$ be a convex function and $\alpha \in (0,1)$, then it gives*

$$\left|f(\frac{a+b}{2}) - \frac{\gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\right]\right|$$
$$\leq \frac{b-a}{2^{\alpha+2}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\left(\frac{2}{\alpha p+1}\right)^{\frac{1}{p}} + 2^\alpha\left(\frac{\alpha p+3}{\alpha p+1}\right)^{\frac{1}{p}}\right)\left(|f'(a)|^q + |f'\left(\frac{a+b}{2}\right)|^q\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{2^{\alpha+2}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\left(\frac{2}{\alpha p+1}\right)^{\frac{1}{p}} + 2^\alpha\left(\frac{\alpha p+3}{\alpha p+1}\right)^{\frac{1}{p}}\right)\left(|f'(b)|^q + |f'\left(\frac{a+b}{2}\right)|^q\right)^{\frac{1}{q}} \quad (2.22)$$

Proof. From lemma 2

$$\left|f(\frac{a+b}{2})-\frac{\gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|=\frac{b-a}{2^{\alpha+2}}\sum_{k=1}^{4}|I_k| \tag{2.23}$$

where,

$$\begin{aligned}|I_1|&=\left|\int_0^1\int_0^1(t^{\alpha}-s^{\alpha})f'(t\frac{a+b}{2}+(1-t)a)dtds\right|\\&\le\int_0^1\int_0^1|t^{\alpha}-s^{\alpha}|\left|f'(t\frac{a+b}{2}+(1-t)a)\right|dtds\end{aligned}$$

By applying Holder's inequality, we have

$$|I_1|\le\left(\int_0^1\int_0^1|t^{\alpha}-s^{\alpha}|^p\,dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|f'(t\frac{a+b}{2}+(1-t)a)\right|^q dtds\right)^{\frac{1}{q}}$$

By convexity of $|f'|$, we have

$$|I_1|\le\left(\int_0^1\int_0^1|t^{\alpha}-s^{\alpha}|^p\,dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left(t^q\left|f'(\frac{a+b}{2})\right|^q+(1-t)^q\,|f'(a)|^q\right)dtds\right)^{\frac{1}{q}} \tag{2.24}$$

From equation (2.24), we have

$$\int_0^1\int_0^1|t^{\alpha}-s^{\alpha}|^p\,dtds\le\frac{2}{\alpha p+1}$$

Also from equation (2.24), we have

$$\begin{aligned}&\int_0^1\int_0^1\left(t^q\left|f'(\frac{a+b}{2})\right|^q+(1-t)^q\,|f'(a)|^q\right)dtds\\&\qquad\le\frac{1}{q+1}\left(\left|f'(\frac{a+b}{2})\right|^q+|f'(a)|^q\right)\end{aligned}$$

Putting values in equation (2.24), we have

$$|I_1|\le\left(\frac{2}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\left|f'(\frac{a+b}{2})\right|^q+|f'(a)|^q\right)^{\frac{1}{q}}$$

Now,

$$\begin{aligned}|I_2|&=\left|\int_0^1\int_0^1(t^{\alpha}-s^{\alpha})f'(s\frac{a+b}{2}+(1-s)b)dtds\right|\\&\le\int_0^1\int_0^1|t^{\alpha}-s^{\alpha}|\left|f'\left(s\frac{a+b}{2}+(1-s)b\right)\right|dtds\end{aligned}$$

By applying Holder's inequality, we have

$$|I_2|\le\left(\int_0^1\int_0^1|t^{\alpha}-s^{\alpha}|^p\,dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|f'(s\frac{a+b}{2}+(1-s)b)\right|^q dtds\right)^{\frac{1}{q}}$$

By convexity of $|f'|$, we have

$$|I_2| \leq \left(\int_0^1 \int_0^1 |t^\alpha - s^\alpha|^p \, dtds\right)^{\frac{1}{p}} (\int_0^1 \int_0^1 (s^q|f'(\frac{a+b}{2})|^q + (1-s)^q|f'(b)|^q)dtds)^{\frac{1}{q}} \quad (2.25)$$

From equation (2.25), we have

$$\int_0^1 \int_0^1 |t^\alpha - s^\alpha|^p \, dtds \leq \frac{2}{\alpha p + 1}$$

Also from equation (2.25), we have

$$\int_0^1 \int_0^1 \left(s^q \left|f'(\frac{a+b}{2})\right|^q + (1-s)^q \left|f'(a)\right|^q\right) dtds \leq \frac{1}{q+1} \left(\left|f'(\frac{a+b}{2}\right|^q + |f'(b)|^q\right)$$

Putting values in equation (2.25), we have

$$|I_2| \leq \left(\frac{2}{\alpha p + 1}\right)^{\frac{1}{p}} \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\left|f'(\frac{a+b}{2})\right|^q + |f'(b)|^q\right)^{\frac{1}{q}}$$

Now,

$$|I_3| = \left|\int_0^1 \int_0^1 [2^\alpha + s^\alpha - (1+t)^\alpha] \, f'(ta + (1-t)\frac{a+b}{2})dtds\right| \leq \int_0^1 \int_0^1 |2^\alpha + s^\alpha - (1+t)^\alpha| \left|f'(ta + (1-t)\frac{a+b}{2})\right| dtds$$

By applying Holder's inequality, we have

$$|I_3| \leq \left(\int_0^1 \int_0^1 |2^\alpha + s^\alpha - (1+t)^\alpha|^p \, dtds\right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left|f'(ta + (1-t)\frac{a+b}{2})\right|^q dtds\right)^{\frac{1}{q}}$$

By convexity of $|f'|$, we have

$$|I_3| \leq \left(\int_0^1 \int_0^1 |2^\alpha + s^\alpha - (1+t)^\alpha|^p \, dtds\right)^{\frac{1}{p}} \times \left(\int_0^1 \int_0^1 \left(t^q \left|f'(a)\right|^q + (1-t)^q \left|f'\left(\frac{a+b}{2}\right)\right|^q\right) dtds\right) \quad (2.26)$$

From equation (2.26), we have

$$\int_0^1 \int_0^1 |2^\alpha + s^\alpha - (1+t)^\alpha|^p \, dtds \leq \int_0^1 \int_0^1 |2^\alpha + s^\alpha + (1+t)^\alpha|^p \, dtds \leq 2^{\alpha p} (\frac{\alpha p + 3}{\alpha p + 1})$$

Also from equation (2.26), we have

$$\int_0^1\int_0^1 \left(t^q\left|f'(a)\right|^q + (1-t)^q\left|f'(\frac{a+b}{2})\right|^q\right) dtds$$
$$\le \frac{1}{q+1}(|f'(a)|^q + |f'(\frac{a+b}{2})|^q)$$

Putting values in equation (2.26), we have

$$|I_3| \le 2^\alpha\left(\frac{\alpha p+3}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(|f'(a)|^q + \left|f'(\frac{a+b}{2})\right|^q\right)^{\frac{1}{q}}$$

Now,

$$|I_4| = \left|\int_0^1\int_0^1 \left[(1+s)^\alpha - 2^\alpha - t^\alpha\right] f'(sb+(1-s)\frac{a+b}{2})dtds\right|$$
$$\le \int_0^1\int_0^1 \left|(1+s)^\alpha - 2^\alpha - t^\alpha\right|\left|f'(sb+(1-s)\frac{a+b}{2})\right| dtds$$

By applying Holder's inequality, we have.

$$|I_4| \le \left(\int_0^1\int_0^1 \left|(1+s)^\alpha - 2^\alpha - t^\alpha\right|^p dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1 \left|f'(sb+(1-s)\frac{a+b}{2})\right|^q dtds\right)^{\frac{1}{q}}$$

By convexity of $|f'|$, we have.

$$|I_4| \le \left(\int_0^1\int_0^1 |(1+s)^\alpha - 2^\alpha - t^\alpha|^p dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1 \left(s^q\left|f'(b)\right|^q + (1-s)^q\left|f'\left(\frac{a+b}{2}\right)\right|^q\right) dtds\right)^{\frac{1}{q}} \tag{2.27}$$

From equation (2.27), we have

$$\int_0^1\int_0^1 |(1+s)^\alpha - (2^\alpha + t^\alpha)|^p dtds \le \int_0^1\int_0^1 |(1+s)^\alpha - (2^\alpha + t^\alpha)|^p dtds$$
$$\le 2^{\alpha p}\left(\frac{\alpha p+3}{\alpha p+1}\right)$$

Also from equation (2.27), we have

$$\int_0^1\int_0^1 \left(s^q\left|f'(b)\right|^q + (1-s)^q\left|f'(\frac{a+b}{2})\right|^q\right) dtds$$
$$\le \frac{1}{q+1}\left(|f'(b)|^q + \left|f'(\frac{a+b}{2})\right|^q\right)$$

Putting values in equation (2.27), we have

$$|I_4| \le 2^\alpha\left(\frac{\alpha p+3}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(|f'(b)|^q + \left|f'(\frac{a+b}{2})\right|^q\right)^{\frac{1}{q}}$$

Now,

$$\sum_{k=1}^{4} I_k = |I_1| + |I_2| + |I_3| + |I_4|$$
$$= (\frac{2}{\alpha p+1})^{\frac{1}{p}} (\frac{1}{q+1})^{\frac{1}{q}} (|f'(\frac{a+b}{2})|^q + |f'(a)|^q)^{\frac{1}{q}}$$
$$+(\frac{2}{\alpha p+1})^{\frac{1}{p}} (\frac{1}{q+1})^{\frac{1}{q}} (|f'(\frac{a+b}{2})|^q + |f'(b)|^q)^{\frac{1}{q}}$$
$$+2^{\alpha}(\frac{\alpha p+3}{\alpha p+1})^{\frac{1}{p}} (\frac{1}{q+1})^{\frac{1}{q}} (|f'(a)|^q + |f'(\frac{a+b}{2})|^q)^{\frac{1}{q}}$$
$$+2^{\alpha}(\frac{\alpha p+3}{\alpha p+1})^{\frac{1}{p}} (\frac{1}{q+1})^{\frac{1}{q}} (|f'(b)|^q + |f'(\frac{a+b}{2})|^q)^{\frac{1}{q}}$$

Which implies that,

$$\sum_{k=1}^{4} I_k = \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\left(\frac{2}{\alpha p+1}\right)^{\frac{1}{p}} + 2^{\alpha}\left(\frac{\alpha p+3}{\alpha p+1}\right)^{\frac{1}{p}}\right) \times$$
$$\left[\left(|f'(a)|^q + |f'(\frac{a+b}{2})|^q\right)^{\frac{1}{q}} + \left(|f'(b)|^q + |f'(\frac{a+b}{2})|^q\right)^{\frac{1}{q}}\right]$$

Putting values in equation (2.23), we have

$$\left|f(\frac{a+b}{2}) - \frac{\gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right]\right|$$
$$\leq \frac{b-a}{2^{\alpha+2}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\left(\frac{2}{\alpha p+1}\right)^{\frac{1}{p}} + 2^{\alpha}\left(\frac{\alpha p+3}{\alpha p+1}\right)^{\frac{1}{p}}\right) \left(|f'(a)|^q + |f'\left(\frac{a+b}{2}\right)|^q\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{2^{\alpha+2}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\left(\frac{2}{\alpha p+1}\right)^{\frac{1}{p}} + 2^{\alpha}\left(\frac{\alpha p+3}{\alpha p+1}\right)^{\frac{1}{p}}\right) \left(|f'(b)|^q + |f'\left(\frac{a+b}{2}\right)|^q\right)^{\frac{1}{q}}$$

Which completes the proof. □

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