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FURTHER INEQUALITIES OF HERMITE-HADAMARD TYPE FOR TRIGONOMETRICALLY ρ-CONVEX FUNCTIONS

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Abstract. In this paper we establish some new Hermite-Hadamard type inequalities for trigonometrically ρ-convex functions. Applications for special means are also provided.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the Hermite-Hadamard inequality.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, for which we would like to refer the reader to the monograph [9], the recent survey paper [10] and the references therein.

Let I be a finite or infinite open interval of real numbers and $\rho > 0$.

In the following we present the basic definitions and results concerning the class of trigonometrically ρ-convex function, see for example [13], [14] and [3], [5], [6], [12], [15], [17] and [18].

Following [1], we say that a function $f: I \to \mathbb{R}$ is *trigonometrically ρ-convex* on I if for any closed subinterval $[a,b]$ of I with $0 < b-a < \frac{\pi}{\rho}$ we have

$$f(x) \leq \frac{\sin[\rho(b-x)]}{\sin[\rho(b-a)]} f(a) + \frac{\sin[\rho(x-a)]}{\sin[\rho(b-a)]} f(b) \tag{1.2}$$

for all $x \in [a,b]$.

If the inequality (1.2) holds with " $\geq$ ", then the function will be called *trigonometrically ρ-concave* on I.

Geometrically speaking, this means that the graph of f on $[a,b]$ lies nowhere above the ρ-trigonometric function determined by the equation

$$H(x) = H(x;a,b,f) := A\cos(\rho x) + B\sin(\rho x)$$

where A and B are chosen such that $H(a) = f(a)$ and $H(b) = f(b)$.

If we take $x = (1-t)a + tb \in [a,b]$, $t \in [0,1]$, then the condition (1.2) becomes

$$f((1-t)a+tb) \leq \frac{\sin[\rho(1-t)(b-a)]}{\sin[\rho(b-a)]} f(a) + \frac{\sin[\rho t(b-a)]}{\sin[\rho(b-a)]} f(b) \tag{1.3}$$

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for any $t \in [0,1]$.

We have the following properties of trigonometrically ρ-convex on I, [1].

(i) A trigonometrically ρ-convex function $f : I \to \mathbb{R}$ has finite right and left derivatives $f'_+(x)$ and $f'_-(x)$ at every point $x \in I$ and $f'_-(x) \leq f'_+(x)$. The function f is differentiable on I with the exception of an at most countable set.

(ii) A necessary and sufficient condition for the function $f : I \to \mathbb{R}$ to be trigonometrically ρ-convex function on I is that it satisfies the *gradient inequality*

$$f(y) \geq f(x)\cos[\rho(y-x)] + K_{x,f}\sin[\rho(y-x)] \tag{1.4}$$

for any $x,\ y \in I$ where $K_{x,f} \in \left[f'_-(x), f'_+(x)\right]$. If f is differentiable at the point x then $K_{x,f} = f'(x)$.

(iii) A necessary and sufficient condition for the function f to be a trigonometrically ρ-convex in I, is that the function

$$\varphi(x) = f'(x) + \rho^2 \int_a^x f(t)\,dt$$

is nondecreasing on I, where $a \in I$.

(iv) Let $f : I \to \mathbb{R}$ be a two times continuously differentiable function on I. Then f is trigonometrically ρ-convex on I if and only if for all $x \in I$ we have

$$f''(x) + \rho^2 f(x) \geq 0. \tag{1.5}$$

For other properties of trigonometrically ρ-convex functions, see [1].

As general examples of trigonometrically ρ-convex functions we can give the indicator function

$$h_F(\theta) := \limsup_{r\to\infty} \frac{\log\left|F\left(re^{i\theta}\right)\right|}{r^\rho},\ \theta \in (\alpha,\beta),$$

where F is an entire function of order $\rho \in (0,\infty)$.

If $0 < \beta - \alpha < \frac{\pi}{\rho}$, then, it was shown in 1908 by Phragmén and Lindelöf, see [13], that h_F is trigonometrically ρ-convex on (α,β).

Using the condition (1.5) one can also observe that any nonnegative twice differentiable and convex function on I is also trigonometrically ρ-convex on I for any $\rho > 0$.

There exists also concave functions on an interval that are trigonometrically ρ-convex on that interval for some $\rho > 0$.

Consider for example $f(x) = \cos x$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then

$$f''(x) + \rho^2 f(x) = -\cos x + \rho^2 \cos x = \left(\rho^2 - 1\right)\cos x,$$

which shows that it is trigonometrically ρ-convex on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for all $\rho > 1$ and trigonometrically ρ-concave for $\rho \in (0,1)$.

Consider the function $f : (0,\infty) \to (0,\infty)$, $f(x) = x^p$ with $p \in \mathbb{R}\setminus\{0\}$. If $p \in (-\infty,0)\cup[1,\infty)$ the function is convex and therefore trigonometrically ρ-convex for any $\rho > 0$. If $p \in (0,1)$ then the function is concave and

$$f''(x) + \rho^2 f(x) = \rho^2 x^p - p(1-p)x^{p-2} = \rho^2 x^{p-2}\left(x^2 - \frac{p(1-p)}{\rho^2}\right),\ x > 0.$$

This shows that for $p \in (0,1)$ and $\rho > 0$ the function $f(x) = x^p$ is trigonometrically ρ-convex on $\left(\frac{1}{\rho}\sqrt{p(1-p)}, \infty\right)$ and trigonometrically ρ-concave on $\left(0, \frac{1}{\rho}\sqrt{p(1-p)}\right)$.

Consider the concave function $f : (0,\infty) \to \mathbb{R}$, $f(x) = \ln x$. We observe that

$$g(x) := f''(x) + \rho^2 f(x) = \rho^2 \ln x - \frac{1}{x^2},\ x > 0.$$

We have $g'(x) = \frac{2+\rho^2 x^2}{x^2} > 0$ for $x > 0$ and $\lim_{x\to 0+} g(x) = -\infty$, $\lim_{x\to\infty} g(x) = \infty$, showing that the function g is strictly increasing on $(0,\infty)$ and the equation $g(x) = 0$ has a unique solution. Therefore $g(x) < 0$ for $x \in (0, x_\rho)$ and $g(x) > 0$ for $x \in (x_\rho, \infty)$, where x_ρ is the unique solution of the equation $\ln x = \frac{1}{\rho^2 x^2}$.

In conclusion, if $\rho > 0$, then the function $f(x) = \ln x$ is trigonometrically ρ-concave on $(0, x_\rho)$ and trigonometrically ρ-convex on (x_ρ, ∞).

The following Hermite-Hadamard type inequality that was obtained in 2013 in [2].

Theorem 1. *Assume that the function $f : I \to \mathbb{R}$ is trigonometrically ρ-convex on I. Then for any $a, b \in I$ with $0 < b - a < \frac{\pi}{\rho}$ we have*

$$\frac{2}{\rho} f\left(\frac{a+b}{2}\right) \sin\left[\frac{\rho(b-a)}{2}\right] \le \int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{\rho} \tan\left[\frac{\rho(b-a)}{2}\right]. \tag{1.6}$$

The inequality (1.6) for $\rho = 1$ was obtained in 2004 by M. Bessenyei in his Ph.D. Thesis [4, Corollary 2.13] in the context of Chebyshev system $(\cos, \sin)$. For a simpler proof than provided in [2] and the following related results, see [11]:

Theorem 2. *Assume that the function $f : I \to \mathbb{R}$ is trigonometrically ρ-convex on I. Then for any $a, b \in I$ with $0 < b - a < \frac{\pi}{\rho}$ we have*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x) \sec\left[\rho\left(x - \frac{a+b}{2}\right)\right] dx \le \frac{f(a)+f(b)}{2} \tag{1.7}$$

and

$$\begin{aligned} \frac{1}{2}\left[b - a + \frac{1}{\rho}\sin[\rho(b-a)]\right] f\left(\frac{a+b}{2}\right) &\le \int_a^b f(x) \cos\left[\rho\left(x - \frac{a+b}{2}\right)\right] dx \\ &\le \frac{b-a+\frac{1}{\rho}\sin[\rho(b-a)]}{2\cos\left[\frac{\rho(b-a)}{2}\right]} \frac{f(a)+f(b)}{2}. \end{aligned} \tag{1.8}$$

Motivated by the above results, in this paper we establish some new Hermite-Hadamard type inequalities for trigonometrically ρ-convex functions. Applications for special means are also provided.

2. The Main Results

We have:

Theorem 3. *Assume that the function $f : I \to \mathbb{R}$ is trigonometrically ρ-convex on I. Then for any $a, b \in I$ with $0 < b - a < \frac{\pi}{\rho}$ we have*

$$\frac{f(a)+f(b)}{2}(b-a) + \frac{1}{2}\rho^2 \int_a^b (b-t)(t-a) f(t)\,dt \ge \int_a^b f(t)\,dt. \tag{2.1}$$

Proof. By the property (iii) from introduction, we have for $y \geq x$ that

$$f'(y) + \rho^2 \int_a^y f(t)\,dt \geq f'(x) + \rho^2 \int_a^x f(t)\,dt,$$

namely

$$f'(y) - f'(x) + \rho^2 \int_x^y f(t)\,dt \geq 0.$$

By multiplying with $y - x \geq 0$, we get

$$(f'(y) - f'(x))(y - x) + \rho^2 (y - x) \int_x^y f(t)\,dt \geq 0. \tag{2.2}$$

If we assume that $x \geq y$, then similarly, we have

$$(f'(x) - f'(y))(x - y) + \rho^2 (x - y) \int_y^x f(t)\,dt \geq 0$$

namely, the inequality (2.2) also holds in this case.

Therefore, for a.e. $x, y \in [a,b]$ we have the inequality (2.2). If we integrate the inequality (2.2) on $[a,b]^2$, then we get

$$\int_a^b \int_a^b (f'(y) - f'(x))(y - x)\,dxdy + \rho^2 \int_a^b \int_a^b (y - x) \left(\int_x^y f(t)\,dt \right) dxdy \geq 0. \tag{2.3}$$

Now, by using the *Korkine identity*

$$\begin{aligned} &\frac{1}{2} \int_a^b \int_a^b (g(x) - g(y))(h(x) - h(y))\,dxdy \\ &= (b-a) \int_a^b g(x) h(x)\,dx - \int_a^b g(x)\,dx \int_a^b h(x)\,dx, \end{aligned}$$

that can be easily proved by multiplying in the left side and integrating, we have

$$\begin{aligned} &\frac{1}{2} \int_a^b \int_a^b (f'(y) - f'(x))(y - x)\,dxdy \\ &= (b-a) \int_a^b x f'(x)\,dx - \int_a^b x dx \int_a^b f'(x)\,dx \\ &= (b-a) \left[b f(b) - a f(a) - \int_a^b f(x)\,dx \right] - \frac{b^2 - a^2}{2} (f(b) - f(a)) \\ &= (b-a) \left[b f(b) - a f(a) - \frac{b+a}{2} (f(b) - f(a)) - \int_a^b f(x)\,dx \right] \\ &= (b-a) \left[\frac{b f(b) - a f(a) - a f(b) + b f(a)}{2} - \int_a^b f(x)\,dx \right] \\ &= (b-a) \left[\frac{f(b) + f(a)}{2} (b - a) - \int_a^b f(x)\,dx \right]. \end{aligned} \tag{2.4}$$

By denoting $F(x) := \int_a^x f(s)\,ds$, then in a similar manner, we have

$$\begin{aligned}
(2.5)\qquad & \frac{1}{2}\int_a^b\int_a^b (y-x)\left(\int_x^y f(t)\,dt\right)dxdy \\
&= \frac{1}{2}\int_a^b\int_a^b (y-x)\left(F(x)-F(y)\right)dxdy \\
&= (b-a)\int_a^b xF(x)\,dx - \int_a^b xdx\int_a^b F(x)\,dx \\
&= (b-a)\int_a^b xF(x)\,dx - \frac{b^2-a^2}{2}\int_a^b F(x)\,dx \\
&= (b-a)\left[\int_a^b xF(x)\,dx - \frac{b+a}{2}\int_a^b F(x)\,dx\right].
\end{aligned}$$

Moreover, using integration by parts, we have

$$\begin{aligned}
\int_a^b xF(x)\,dx &= \frac{1}{2}\int_a^b F(x)\,d\left(x^2\right) = \frac{1}{2}\left[F(x)\,x^2\Big|_a^b - \int_a^b x^2 d\left(F(x)\right)\right] \\
&= \frac{1}{2}\left[F(b)\,b^2\Big|_a^b - \int_a^b x^2 f(x)\,dx\right] \\
&= \frac{1}{2}\left[b^2\int_a^b f(x)\,dx - \int_a^b x^2 f(x)\,dx\right] \\
&= \frac{1}{2}\int_a^b \left(b^2-x^2\right) f(x)\,dx
\end{aligned}$$

and

$$\begin{aligned}
\int_a^b F(x)\,dx &= F(x)\,x\Big|_a^b - \int_a^b xf(x)\,dx \\
&= b\int_a^b f(x)\,dx - \int_a^b xf(x)\,dx = \int_a^b (b-x)\,f(x)\,dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \int_a^b xF(x)\,dx - \frac{b+a}{2}\int_a^b F(x)\,dx \\
&= \frac{1}{2}\int_a^b \left(b^2-x^2\right) f(x)\,dx - \frac{b+a}{2}\int_a^b (b-x)\,f(x)\,dx \\
&= \frac{1}{2}\int_a^b (b-x)(b+x-b-a)\,f(x)\,dx \\
&= \frac{1}{2}\int_a^b (b-x)(x-a)\,f(x)\,dx
\end{aligned}$$

and by (2.5) we get

$$(2.6)\qquad \frac{1}{2}\int_a^b\int_a^b (y-x)\left(\int_x^y f(t)\,dt\right)dxdy = \frac{1}{4}\int_a^b (b-x)(x-a)\,f(x)\,dx.$$

By using the inequality (2.3) and the equalities (2.4) and (2.6), we get the desired result (2.1). □

As we have shown in Introduction, there are many examples of concave functions that are also trigonometrically ρ-convex functions for some $\rho > 0$ and on some apropriately chosen intervals.

Corollary 1. *Assume that the function $f : I \to \mathbb{R}$ is concave and trigonometrically ρ-convex on I. Then for any $a, b \in I$ with $0 < b - a < \frac{\pi}{\rho}$ we have*

$$\frac{1}{2}\rho^2 \int_a^b (b-t)(t-a) f(t)\, dt \geq \int_a^b f(t)\, dt - \frac{f(a) + f(b)}{2}(b-a) \geq 0. \tag{2.7}$$

We have:

Theorem 4. *Assume that the function $f : I \to \mathbb{R}$ is trigonometrically ρ-convex on I. Then for any $a, b \in I$ with $0 < b - a < \frac{\pi}{\rho}$ we have*

$$\frac{f(b) + f(a)}{2} + \frac{1}{2}\rho^2 \left[\int_a^{\frac{a+b}{2}} (t-a) f(t)\, dt + \int_{\frac{a+b}{2}}^b (b-t) f(t)\, dt\right] \geq f\left(\frac{a+b}{2}\right). \tag{2.8}$$

Proof. Consider the function $F : I \to \mathbb{R}$, defined by

$$F(x) = \rho^2 \int_a^x (x-t) f(t)\, dt + f(x) - f(a),\ x \in I.$$

The function F is differentiable on I with the exception of an at most countable set and

$$\begin{aligned} F'(x) &= \left(\rho^2 x \int_a^x f(t)\, dt - \rho^2 \int_a^x t f(t)\, dt + f(x) - f(a)\right)' \\ &= \rho^2 \int_a^x f(t)\, dt + \rho^2 x f(x) - \rho^2 x f(x) + f'(x) \\ &= \rho^2 \int_a^x f(t)\, dx + f'(x). \end{aligned}$$

Using property (iii) from Introduction we conclude that the function F is convex on I, then for any $a, b \in I$ with $0 < b - a < \frac{\pi}{\rho}$ we have

$$\frac{1}{2}[F(a) + F(b)] \geq F\left(\frac{a+b}{2}\right),$$

namely

$$\begin{aligned} &\frac{1}{2}\left[\rho^2 \int_a^b (b-t) f(t)\, dt + f(b) - f(a)\right] \\ &\geq \rho^2 \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right) f(t)\, dt + f\left(\frac{a+b}{2}\right) - f(a). \end{aligned}$$

This is equivalent to

$$
\begin{aligned}
&\frac{f(b)+f(a)}{2}-f\left(\frac{a+b}{2}\right) \\
&\geq \rho^2 \int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right) f(t)\, dt-\frac{1}{2} \rho^2 \int_a^b(b-t) f(t)\, dt \\
&=\rho^2\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right) f(t)\, dt-\frac{1}{2} \int_a^b(b-t) f(t)\, dt\right] \\
&=\rho^2\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right) f(t)\, dt-\frac{1}{2} \int_a^{\frac{a+b}{2}}(b-t) f(t)\, dt-\frac{1}{2} \int_{\frac{a+b}{2}}^b(b-t) f(t)\, dt\right] \\
&=\rho^2\left[\frac{1}{2} \int_a^{\frac{a+b}{2}}(a-t) f(t)\, dt-\frac{1}{2} \int_{\frac{a+b}{2}}^b(b-t) f(t)\, dt\right] \\
&=\frac{1}{2} \rho^2\left[\int_a^{\frac{a+b}{2}}(a-t) f(t)\, dt-\int_{\frac{a+b}{2}}^b(b-t) f(t)\, dt\right]
\end{aligned}
$$

and the inequality (2.8) is proved. □

Corollary 2. *Assume that the function $f: I \to \mathbb{R}$ is concave and trigonometrically ρ-convex on I. Then for any $a, b \in I$ with $0 < b - a < \frac{\pi}{\rho}$ we have*

$$
\begin{aligned}
\frac{1}{2} \rho^2\left[\int_a^{\frac{a+b}{2}}(t-a) f(t)\, dt+\int_{\frac{a+b}{2}}^b(b-t) f(t)\, dt\right] \\
\geq f\left(\frac{a+b}{2}\right)-\frac{f(b)+f(a)}{2} \geq 0.
\end{aligned} \tag{2.9}
$$

3. Some Inequalities for Special Means

Recall the following special means:

(1) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}, \qquad a, b \geq 0;$$

(2) The geometric mean:

$$G = G(a,b) := \sqrt{ab}, \qquad a, b \geq 0;$$

(3) The harmonic mean:

$$H = H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}, \qquad a, b \geq 0;$$

(4) The logarithmic mean:

$$L = L(a,b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases} \qquad a, b > 0;$$

(5) The identric mean:

$$I := I(a,b) = \begin{cases} a & \text{if } a = b \\ & & a, b > 0; \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases}$$

(6) The p-logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a \neq b; \\ a & \text{if } a = b \end{cases}$$

where $p \in \mathbb{R} \setminus \{-1, 0\}$ and $a,\ b > 0$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$.

In particular, we have the inequalities

$$H \leq G \leq L \leq I \leq A. \tag{3.1}$$

Assume that the function $f : I \to \mathbb{R}$ is concave and *trigonometrically ρ-convex* on I. Then for any $a,\ b \in I$ with $0 < b - a < \frac{\pi}{\rho}$ we have by (2.7) that

$$\frac{1}{2}\rho^2 \frac{1}{b-a} \int_a^b (b-t)(t-a) f(t)\, dt \geq \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{f(a) + f(b)}{2} \geq 0. \tag{3.2}$$

Consider the function $f : (0, \infty) \to (0, \infty)$, $f(x) = x^p$ with $p \in \mathbb{R} \setminus \{0\}$. If $p \in (0,1)$, then the function is concave and trigonometrically ρ-convex on $\left(\frac{1}{\rho}\sqrt{p(1-p)}, \infty\right)$. Let $a,\ b \in \left(\frac{1}{\rho}\sqrt{p(1-p)}, \infty\right)$ with $0 < b - a < \frac{\pi}{\rho}$.

We have

$$\frac{1}{b-a} \int_a^b t^p dt = L_p^p(a,b)$$

and

$$\begin{aligned} &\frac{1}{2(b-a)} \int_a^b (b-t)(t-a) t^p dt \\ &= \frac{1}{2(b-a)} \int_a^b \left[(a+b)t - ab - t^2\right] t^p dt \\ &= A(a,b) \frac{1}{b-a} \int_a^b t^{p+1} dt - \frac{1}{2} G^2(a,b) \frac{1}{b-a} \int_a^b t^p dt - \frac{1}{2} \frac{1}{b-a} \int_a^b t^{p+2} dt \\ &= A(a,b) L_p^p(a,b) - \frac{1}{2} G^2(a,b) L_p^p(a,b) - \frac{1}{2} L_{p+2}^{p+2}(a,b), \end{aligned}$$

therefore by (3.2) we get

$$\begin{aligned} \rho^2 \left[A(a,b) L_p^p(a,b) - \frac{1}{2} G^2(a,b) L_p^p(a,b) - \frac{1}{2} L_{p+2}^{p+2}(a,b)\right] \\ \geq L_p^p(a,b) - A(a^p, b^p) \geq 0 \end{aligned} \tag{3.3}$$

for any $a,\ b \in \left(\frac{1}{\rho}\sqrt{p(1-p)}, \infty\right)$ with $0 < b - a < \frac{\pi}{\rho}$.

We also have

$$\begin{aligned}
&\frac{1}{2}\int_{a}^{\frac{a+b}{2}}(t-a)\,t^{p}dt+\frac{1}{2}\int_{\frac{a+b}{2}}^{b}(b-t)\,t^{p}dt \\
&=\frac{1}{2}\int_{a}^{\frac{a+b}{2}}t^{p+1}dt-\frac{1}{2}a\int_{a}^{\frac{a+b}{2}}t^{p}dt+\frac{1}{2}b\int_{\frac{a+b}{2}}^{b}t^{p}-\frac{1}{2}\int_{\frac{a+b}{2}}^{b}t^{p+1}dt \\
&=\frac{1}{2}\frac{A(a,b)-a}{A(a,b)-a}\int_{a}^{\frac{a+b}{2}}t^{p+1}dt-\frac{1}{2}a\frac{A(a,b)-a}{A(a,b)-a}\int_{a}^{\frac{a+b}{2}}t^{p}dt \\
&+\frac{1}{2}b\frac{b-A(a,b)}{b-A(a,b)}\int_{\frac{a+b}{2}}^{b}t^{p}-\frac{1}{2}\frac{b-A(a,b)}{b-A(a,b)}\int_{\frac{a+b}{2}}^{b}t^{p+1}dt \\
&=\frac{1}{4}(b-a)\,L^{p+1}(a,A(a,b))-\frac{1}{4}(b-a)\,aL^{p}(a,A(a,b)) \\
&+\frac{1}{4}b\,(b-a)\,L^{p}(A(a,b),b)-\frac{1}{4}(b-a)\,L^{p+1}(A(a,b),b) \\
&=\frac{1}{4}(b-a)\left[L^{p+1}(a,A(a,b))-L^{p+1}(A(a,b),b)\right] \\
&+\frac{1}{4}(b-a)\left[bL^{p}(A(a,b),b)-aL^{p}(a,A(a,b))\right]
\end{aligned}$$

and by (2.8) written for the function $f:(0,\infty)\to(0,\infty)$, $f(x)=x^{p}$, we get

$$\begin{aligned}
(3.4)\quad &\frac{1}{4}(b-a)\,\rho^{2}\left[L^{p+1}(a,A(a,b))-L^{p+1}(A(a,b),b)\right] \\
&+\frac{1}{4}(b-a)\,\rho^{2}\left[bL^{p}(A(a,b),b)-aL^{p}(a,A(a,b))\right] \\
&\geq A^{p}(a,b)-A(a^{p},b^{p})\geq 0
\end{aligned}$$

for any $a,\ b\in\left(\frac{1}{\rho}\sqrt{p(1-p)},\infty\right)$ with $0<b-a<\frac{\pi}{\rho}$.

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