GRÜSS' TYPE INEQUALITIES FOR THE COMPLEX INTEGRAL ON PATHS

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Abstract. Assume that f and g are continuous on γ, $\gamma \subset \mathbb{C}$ is a piecewise smooth path and the *Čebyšev functional on paths* is defined by

$$\mathcal{P}_{\gamma}(f,g) := \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) g(z) |dz| - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \frac{1}{\ell(\gamma)} \int_{\gamma} g(z) |dz|.$$

In this paper we establish some bounds for the magnitude of the functional $\mathcal{P}_{\gamma}(f,g)$ under various assumptions for the functions f and g and provide a complex version of the Grüss inequality.

1. Introduction

For two Lebesgue integrable functions f, $g : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$C(f,g) := \frac{1}{b-a} \int_a^b f(t) g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \frac{1}{b-a} \int_a^b g(t) dt.$$

In 1934, G. Grüss [17] showed that

$$|C(f,g)| \leq \frac{1}{4} (M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on} \quad [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.6) in the sense that it cannot be replaced by a smaller one.

In [6], P. Cerone and S.S. Dragomir proved the following inequalities:

$$|C(f,g)| \leq \begin{cases} \inf_{\gamma \in \mathbb{R}} \|g-\gamma\|_{\infty} \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt, \\ \\ \inf_{\gamma \in \mathbb{R}} \|g-\gamma\|_{q} \frac{1}{b-a} \left(\int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right|^p dt \right)^{\frac{1}{p}} \\ \qquad \text{where } p>1,\ 1/p+1/q=1. \end{cases} \tag{1.3}$$

For $\gamma = 0$, we get from the first inequality in (1.3)

$$|C(f,g)| \leq \|g\|_{\infty} \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \tag{1.4}$$

for which the constant 1 cannot be replaced by a smaller constant.

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If $m \leq g \leq M$ for a.e. $x \in [a,b]$, then $\left\| g - \frac{m+M}{2} \right\|_{\infty} \leq \frac{1}{2}(M-m)$ and by the first inequality in (1.3) we can deduce the following result obtained by Cheng and Sun [9]

$$|C(f,g)| \leq \frac{1}{2}(M-m)\frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt. \tag{1.5}$$

The constant $\frac{1}{2}$ is best in (1.5) as shown by Cerone and Dragomir in [7].

For other inequality of Grüss' type see [1]-[5], [7]-[16], [18]-[23] and [25]-[28].

In order to extend Grüss' inequality to complex integral with respect to arc-length we need the following preparations.

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, and open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.6}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left| \int_\gamma f(z)\,dz \right| \leq \int_\gamma |f(z)|\,|dz| \leq \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.7}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \geq 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\,|dz| \right)^{1/p}.$$

For $p=1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|\,.$$

If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}\,.$$

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)\,,\ t\in\gamma$ from $z(a)=u$ to $z(b)=w$. If f and g are continuous on γ, we consider the *Čebyšev functional on paths* defined by

$$\mathcal{P}_\gamma(f,g) := \frac{1}{\ell(\gamma)}\int_\gamma f(z)\,g(z)\,|dz| - \frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\,\frac{1}{\ell(\gamma)}\int_\gamma g(z)\,|dz|\,.$$

In this paper we establish some bounds for the magnitude of the functional $\mathcal{P}_\gamma(f,g)$ under various assumptions for the functions f and g and provide a complex version of the Grüss inequality (1.1).

2. Grüss' Type Inequalities

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)\,,\ t\in\gamma$ from $z(a)=u$ to $z(b)=w$. Now, for $\phi,\ \Phi\in\mathbb{C}$, define the sets of complex-valued functions

$$\bar{U}_\gamma(\phi,\Phi) := \left\{f:\gamma\to\mathbb{C}\,|\ \mathrm{Re}\left[(\Phi - f(z))\left(\overline{f(z)}-\overline{\phi}\right)\right]\ge 0 \ \text{ for each }\ z\in\gamma\right\}$$

and

$$\bar{\Delta}_\gamma(\phi,\Phi) := \left\{f:\gamma\to\mathbb{C}\,|\ \left|f(z)-\frac{\phi+\Phi}{2}\right| \le \frac{1}{2}|\Phi-\phi| \ \text{ for each }\ z\in\gamma\right\}.$$

The following representation result may be stated.

Proposition 1. *For any $\phi,\ \Phi\in\mathbb{C},\ \phi\ne\Phi$, we have that $\bar{U}_\gamma(\phi,\Phi)$ and $\bar{\Delta}_\gamma(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_\gamma(\phi,\Phi) = \bar{\Delta}_\gamma(\phi,\Phi)\,. \tag{2.1}$$

Proof. We observe that for any $w\in\mathbb{C}$ we have the equivalence

$$\left|w-\frac{\phi+\Phi}{2}\right| \le \frac{1}{2}|\Phi-\phi|$$

if and only if

$$\mathrm{Re}\left[(\Phi-w)\left(\overline{w}-\overline{\phi}\right)\right]\ge 0.$$

This follows by the equality

$$\frac{1}{4}|\Phi-\phi|^2 - \left|w-\frac{\phi+\Phi}{2}\right|^2 = \mathrm{Re}\left[(\Phi-w)\left(\overline{w}-\overline{\phi}\right)\right]$$

that holds for any $w\in\mathbb{C}$.

The equality (2.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 1. *For any $\phi,\ \Phi\in\mathbb{C},\ \phi\ne\Phi$, we have that*

$$\begin{aligned}\bar{U}_\gamma(\phi,\Phi) = \{f:\gamma\to\mathbb{C}\ |\ &(\mathrm{Re}\,\Phi-\mathrm{Re}\,f(z))(\mathrm{Re}\,f(z)-\mathrm{Re}\,\phi)\\ &+(\mathrm{Im}\,\Phi-\mathrm{Im}\,f(z))(\mathrm{Im}\,f(z)-\mathrm{Im}\,\phi)\ge 0 \textit{ for each } z\in\gamma\}\,.\end{aligned} \tag{2.2}$$

Now, if we assume that $\operatorname{Re}(\Phi) \geq \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \geq \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\bar{S}_{\gamma}(\phi, \Phi) := \{f : \gamma \to \mathbb{C} \mid \operatorname{Re}(\Phi) \geq \operatorname{Re} f(z) \geq \operatorname{Re}(\phi) \text{ and } \operatorname{Im}(\Phi) \geq \operatorname{Im} f(z) \geq \operatorname{Im}(\phi) \text{ for each } z \in \gamma\}. \tag{2.3}$$

One can easily observe that $\bar{S}_{\gamma}(\phi, \Phi)$ is closed, convex and

$$\emptyset \neq \bar{S}_{\gamma}(\phi, \Phi) \subseteq \bar{U}_{\gamma}(\phi, \Phi). \tag{2.4}$$

We have the following simple facts:

Lemma 1. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$. If f is continuous on γ, then for all $\lambda \in \mathbb{C}$ we have*

$$\begin{aligned} \mathcal{P}_{\gamma}\left(f, \overline{f}\right) &= \frac{1}{\ell(\gamma)} \int_{\gamma} |f(z)|^2 |dz| - \left| \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right|^2 \\ &= \frac{1}{\ell(\gamma)} \int_{\gamma} \left(f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right) \left(\overline{f(v)} - \lambda \right) |dv| \end{aligned} \tag{2.5}$$

and, in particular,

$$\mathcal{P}_{\gamma}\left(f, \overline{f}\right) = \frac{1}{\ell(\gamma)} \int_{\gamma} \left| f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right|^2 |dv| \geq 0. \tag{2.6}$$

Proof. We observe that

$$\begin{aligned} &\frac{1}{\ell(\gamma)} \int_{\gamma} \left(f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right) \left(\overline{f(v)} - \lambda \right) |dv| \\ &= \frac{1}{\ell(\gamma)} \int_{\gamma} |f(v)|^2 |dv| - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \frac{1}{\ell(\gamma)} \int_{\gamma} \overline{f(v)} |dv| \\ &- \lambda \frac{1}{\ell(\gamma)} \int_{\gamma} \left(f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right) |dv| \\ &= \frac{1}{\ell(\gamma)} \int_{\gamma} |f(v)|^2 |dv| - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \frac{1}{\ell(\gamma)} \overline{\left(\int_{\gamma} f(v) |dv| \right)} \\ &= \frac{1}{\ell(\gamma)} \int_{\gamma} |f(v)|^2 |dv| - \left| \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right|^2 \end{aligned}$$

for any $\lambda \in \mathbb{C}$, which proves (2.5).

The equality (2.6) follows by (2.5) by taking

$$\lambda = \frac{1}{\ell(\gamma)} \overline{\left(\int_{\gamma} f(v) |dv| \right)}.$$

□

We have:

Theorem 1. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$. If f is continuous on γ and there exists $c \in \mathbb{C}$ and $R > 0$ such that*

$$f \in \overline{D}(c, R) := \{z \in \mathbb{C} \mid |z - c| \leq R\}, \tag{2.7}$$

then

$$0 \leq \mathcal{P}_\gamma\left(f, \overline{f}\right) \leq R\frac{1}{\ell(\gamma)}\left|\int_\gamma f(v) - \frac{1}{\ell(\gamma)}\int_\gamma f(z)|dz|\right||dv| \tag{2.8}$$

and

$$0 \leq \mathcal{P}_\gamma\left(f, \overline{f}\right) \leq R^2. \tag{2.9}$$

Proof. For the equality (2.5) for $\lambda = \overline{c}$ we have

$$\begin{aligned}
\mathcal{P}_\gamma\left(f, \overline{f}\right) &= \left|\frac{1}{\ell(\gamma)}\int_\gamma \left(f(v) - \frac{1}{\ell(\gamma)}\int_\gamma f(z)|dz|\right)\left(\overline{f(v)} - \overline{c}\right)|dv|\right| \\
&\leq \frac{1}{\ell(\gamma)}\left|\int_\gamma f(v) - \frac{1}{\ell(\gamma)}\int_\gamma f(z)|dz|\right|\left|\overline{f(v)} - \overline{c}\right||dv| \\
&= \frac{1}{\ell(\gamma)}\left|\int_\gamma f(v) - \frac{1}{\ell(\gamma)}\int_\gamma f(z)|dz|\right||f(v) - c||dv| \\
&\leq R\frac{1}{\ell(\gamma)}\left|\int_\gamma f(v) - \frac{1}{\ell(\gamma)}\int_\gamma f(z)|dz|\right||dv|,
\end{aligned}$$

which proves (2.8).

Using Cauchy-Bunyakovsky-Schwarz integral inequality, we have

$$\begin{aligned}
&\frac{1}{\ell(\gamma)}\left|\int_\gamma f(v) - \frac{1}{\ell(\gamma)}\int_\gamma f(z)|dz|\right||dv| \\
&\leq \left(\frac{1}{\ell(\gamma)}\int_\gamma\left|f(v) - \frac{1}{\ell(\gamma)}\int_\gamma f(z)|dz|\right|^2|dv|\right)^{1/2} \\
&= \left(\frac{1}{\ell(\gamma)}\int_\gamma |f(z)|^2|dz| - \left|\frac{1}{\ell(\gamma)}\int_\gamma f(z)|dz|\right|^2\right)^{1/2},
\end{aligned} \tag{2.10}$$

where for the last equality we used (2.6).

From (2.8) and (2.10) we have

$$0 \leq \mathcal{P}_\gamma\left(f, \overline{f}\right) \leq R\frac{1}{\ell(\gamma)}\left|\int_\gamma f(v) - \frac{1}{\ell(\gamma)}\int_\gamma f(z)|dz|\right||dv| \leq R\left[\mathcal{P}_\gamma\left(f, \overline{f}\right)\right]^{1/2}$$

which implies that

$$0 \leq \left[\mathcal{P}_\gamma\left(f, \overline{f}\right)\right]^{1/2} \leq R$$

proving the desired result (2.9). □

Corollary 2. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$. If f is continuous on γ and there exists $\phi, \Phi \in \mathbb{C}$, $\phi \neq \Phi$ such that $f \in \bar{\Delta}_\gamma(\phi, \Phi)$, then*

$$0 \leq \mathcal{P}_\gamma\left(f, \overline{f}\right) \leq \frac{1}{2}|\Phi - \phi|\frac{1}{\ell(\gamma)}\left|\int_\gamma f(v) - \frac{1}{\ell(\gamma)}\int_\gamma f(z)|dz|\right||dv| \tag{2.11}$$

and

$$0 \leq \mathcal{P}_\gamma\left(f, \overline{f}\right) \leq \frac{1}{4}|\Phi - \phi|^2. \tag{2.12}$$

The proof follows by Theorem 1 by choosing $c = \frac{\phi+\Phi}{2}$ and $R = \frac{1}{2}|\Phi - \phi|$.

We have the following Grüss' type inequality:

Theorem 2. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$. If f, g are continuous on γ and $\lambda \in \mathbb{C}$, then*

$$|\mathcal{P}_{\gamma}(f,g)| \leq \frac{1}{\ell(\gamma)} \int_{\gamma} |f(v) - \lambda| \left| g(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} g(z) |dz| \right| |dv| \tag{2.13}$$
$$\leq \begin{cases} \max_{v \in \gamma} |f(v) - \lambda| \frac{1}{\ell(\gamma)} \int_{\gamma} \left| g(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} g(z) |dz| \right| |dv|, \\ \\ \left(\frac{1}{\ell(\gamma)} \int_{\gamma} |f(v) - \lambda|^p \right)^{1/p} \left(\frac{1}{\ell(\gamma)} \int_{\gamma} \left| g(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} g(z) |dz| \right|^q |dv| \right)^{1/q} \\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \max_{v \in \gamma} \left| g(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} g(z) |dz| \right| \frac{1}{\ell(\gamma)} \int_{\gamma} |f(v) - \lambda| |dv|. \end{cases}$$

In particular,

$$|\mathcal{P}_{\gamma}(f,f)| \leq \frac{1}{\ell(\gamma)} \int_{\gamma} |f(v) - \lambda| \left| f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right| |dv| \tag{2.14}$$
$$\leq \begin{cases} \max_{v \in \gamma} |f(v) - \lambda| \frac{1}{\ell(\gamma)} \int_{\gamma} \left| f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right| |dv|, \\ \\ \left(\frac{1}{\ell(\gamma)} \int_{\gamma} |f(v) - \lambda|^p \right)^{1/p} \left(\frac{1}{\ell(\gamma)} \int_{\gamma} \left| f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right|^q |dv| \right)^{1/q} \\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \max_{v \in \gamma} \left| f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right| \frac{1}{\ell(\gamma)} \int_{\gamma} |f(v) - \lambda| |dv|, \end{cases}$$

where

$$\mathcal{P}_{\gamma}(f,f) := \frac{1}{\ell(\gamma)} \int_{\gamma} f^2(z) |dz| - \left(\frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right)^2.$$

Proof. We have the following Sonin type identity for the integral with respect to arc-length

$$\frac{1}{\ell(\gamma)} \int_{\gamma} f(z) g(z) |dz| - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \frac{1}{\ell(\gamma)} \int_{\gamma} g(z) |dz| \tag{2.15}$$
$$= \frac{1}{\ell(\gamma)} \int_{\gamma} (f(v) - \lambda) \left[g(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} g(z) |dz| \right] |dv|$$

for any $\lambda \in \mathbb{C}$.

By taking the modulus in (2.15) we get

$$|\mathcal{P}_{\gamma}(f,g)| = \left| \frac{1}{\ell(\gamma)} \int_{\gamma} (f(v) - \lambda) \left[g(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} g(z) |dz| \right] |dv| \right|$$
$$\leq \frac{1}{\ell(\gamma)} \int_{\gamma} |f(v) - \lambda| \left| g(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} g(z) |dz| \right| |dv|.$$

The last inequality follows by Hölder's inequality

$$\frac{1}{\ell(\gamma)}\int_\gamma |f(v)-\lambda|\left|g(v)-\frac{1}{\ell(\gamma)}\int_\gamma g(z)|dz|\right||dv|$$
$$\le \begin{cases} \max_{v\in\gamma}|f(v)-\lambda|\frac{1}{\ell(\gamma)}\int_\gamma\left|g(v)-\frac{1}{\ell(\gamma)}\int_\gamma g(z)|dz|\right||dv| \\ \\ \left(\frac{1}{\ell(\gamma)}\int_\gamma|f(v)-\lambda|^p\right)^{1/p}\left(\frac{1}{\ell(\gamma)}\int_\gamma\left|g(v)-\frac{1}{\ell(\gamma)}\int_\gamma g(z)|dz|\right|^q|dv|\right)^{1/q} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max_{v\in\gamma}\left|g(v)-\frac{1}{\ell(\gamma)}\int_\gamma g(z)|dz|\right|\frac{1}{\ell(\gamma)}\int_\gamma|f(v)-\lambda||dv|. \end{cases}$$

□

We have:

Corollary 3. *Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t\in\gamma$ from $z(a)=u$ to $z(b)=w$. If f is continuous on γ and there exists $c\in\mathbb{C}$ and $R>0$ such that (2.7) is true, then*

$$|\mathcal{P}_\gamma(f,g)|\le R\frac{1}{\ell(\gamma)}\int_\gamma\left|g(v)-\frac{1}{\ell(\gamma)}\int_\gamma g(z)|dz|\right||dv|\le R\left[\mathcal{P}_\gamma(g,\overline{g})\right]^{1/2}. \tag{2.16}$$

In particular,

$$|\mathcal{P}_\gamma(f,f)|\le R\frac{1}{\ell(\gamma)}\int_\gamma\left|f(v)-\frac{1}{\ell(\gamma)}\int_\gamma f(z)|dz|\right||dv|\le R\left[\mathcal{P}_\gamma\left(f,\overline{f}\right)\right]^{1/2}. \tag{2.17}$$

If f is continuous on γ and there exists ϕ, $\Phi\in\mathbb{C}$, $\phi\ne\Phi$ such that $f\in\bar{\Delta}_\gamma(\phi,\Phi)$, then

$$\begin{aligned}|\mathcal{P}_\gamma(f,g)|&\le\frac{1}{2}|\Phi-\phi|\frac{1}{\ell(\gamma)}\int_\gamma\left|g(v)-\frac{1}{\ell(\gamma)}\int_\gamma g(z)|dz|\right||dv|\\&\le\frac{1}{2}|\Phi-\phi|\left[\mathcal{P}_\gamma(g,\overline{g})\right]^{1/2}.\end{aligned} \tag{2.18}$$

In particular,

$$\begin{aligned}|\mathcal{P}_\gamma(f,f)|&\le\frac{1}{2}|\Phi-\phi|\frac{1}{\ell(\gamma)}\int_\gamma\left|f(v)-\frac{1}{\ell(\gamma)}\int_\gamma f(z)|dz|\right||dv|\\&\le\frac{1}{2}|\Phi-\phi|\left[\mathcal{P}_\gamma\left(f,\overline{f}\right)\right]^{1/2}.\end{aligned} \tag{2.19}$$

We have the following Grüss type inequality:

Corollary 4. *Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t\in\gamma$ from $z(a)=u$ to $z(b)=w$. If f and g are continuous on γ and there exists ϕ, Φ, ψ, $\Psi\in\mathbb{C}$, $\phi\ne\Phi$, $\psi\ne\Psi$ such that $f\in\bar{\Delta}_\gamma(\phi,\Phi)$ and $g\in\bar{\Delta}_\gamma(\psi,\Psi)$, then*

$$\begin{aligned}|\mathcal{P}_\gamma(f,g)|&\le\frac{1}{2}|\Phi-\phi|\frac{1}{\ell(\gamma)}\int_\gamma\left|g(v)-\frac{1}{\ell(\gamma)}\int_\gamma g(z)|dz|\right||dv|\\&\le\frac{1}{2}|\Phi-\phi|\left[\mathcal{P}_\gamma(g,\overline{g})\right]^{1/2}\le\frac{1}{4}|\Phi-\phi||\Psi-\psi|\end{aligned} \tag{2.20}$$

and, symmetrically

$$|\mathcal{P}_\gamma(f,g)| \leq \frac{1}{2}|\Psi-\psi| \frac{1}{\ell(\gamma)} \int_\gamma \left| f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z)|dz| \right| |dv| \leq \frac{1}{2}|\Psi-\psi| \left[\mathcal{P}_\gamma\left(f,\overline{f}\right)\right]^{1/2} \leq \frac{1}{4}|\Phi-\phi||\Psi-\psi|. \tag{2.21}$$

Remark 1. *By taking* $\lambda = \frac{1}{\ell(\gamma)}\int_\gamma f(z)|dz|$ *in (2.13) we also get*

$$\begin{aligned}|\mathcal{P}_\gamma(f,g)| &\leq \frac{1}{\ell(\gamma)} \int_\gamma \left| f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z)|dz| \right| \left| g(v) - \frac{1}{\ell(\gamma)} \int_\gamma g(z)|dz| \right| |dv| \\ &\leq \left(\mathcal{P}_\gamma\left(f,\overline{f}\right)\right)^{1/2} \left(\mathcal{P}_\gamma(g,\overline{g})\right)^{1/2}.\end{aligned} \tag{2.22}$$

If we take in (2.13) $\lambda = \frac{f(u)+f(w)}{2}$, *then we get*

$$|\mathcal{P}_\gamma(f,g)| \leq \frac{1}{\ell(\gamma)} \int_\gamma \left| f(v) - \frac{f(u)+f(w)}{2} \right| \left| g(w) - \frac{1}{\ell(\gamma)} \int_\gamma g(z)|dz| \right| |dv|. \tag{2.23}$$

If $m = z\left(\frac{a+b}{2}\right)$, *then by taking* $\lambda = f(m)$ *in (2.13), we obtain*

$$|\mathcal{P}_\gamma(f,g)| \leq \frac{1}{\ell(\gamma)} \int_\gamma |f(v) - f(m)| \left| g(v) - \frac{1}{\ell(\gamma)} \int_\gamma g(z)|dz| \right| |dv|. \tag{2.24}$$

Further, observe that

$$\begin{aligned}&\frac{1}{\ell(\gamma)} \int_\gamma |f(v)-\lambda| \left| g(v) - \frac{1}{\ell(\gamma)} \int_\gamma g(z)|dz| \right| |dv| \\ &= \frac{1}{\ell^2(\gamma)} \int_\gamma |f(v)-\lambda| \left| g(v)\ell(\gamma) - \int_\gamma g(z)|dz| \right| |dv| \\ &= \frac{1}{\ell^2(\gamma)} \int_\gamma |f(v)-\lambda| \left| \int_\gamma [g(v)-g(z)]|dz| \right| |dv| \\ &\leq \frac{1}{\ell^2(\gamma)} \int_\gamma \int_\gamma |f(v)-\lambda||g(v)-g(z)||dz||dv|,\end{aligned}$$

therefore by (2.13) we get

$$\begin{aligned}|\mathcal{P}_\gamma(f,g)| &\leq \frac{1}{\ell(\gamma)} \int_\gamma |f(v)-\lambda| \left| g(v) - \frac{1}{\ell(\gamma)} \int_\gamma g(z)|dz| \right| |dv| \\ &\leq \frac{1}{\ell^2(\gamma)} \int_\gamma \int_\gamma |f(v)-\lambda||g(v)-g(z)||dz||dv|\end{aligned} \tag{2.25}$$

for $\lambda \in \mathbb{C}$.

If we use Hölder's inequality for double integrals, we have

$$
\begin{aligned}
&\frac{1}{\ell^2(\gamma)}\int_\gamma\int_\gamma |f(v)-\lambda|\,|g(v)-g(z)|\,|dz|\,|dv| \\
&\leq \begin{cases}
\max_{v\in\gamma}|f(v)-\lambda|\,\frac{1}{\ell^2(\gamma)}\int_\gamma\int_\gamma |g(v)-g(z)|\,|dz|\,|dv|, \\[2ex]
\left(\frac{1}{\ell^2(\gamma)}\int_\gamma\int_\gamma |f(v)-\lambda|^p\,|dz|\,|dv|\right)^{1/p}\left(\frac{1}{\ell^2(\gamma)}\int_\gamma\int_\gamma |g(v)-g(z)|^q\,|dz|\,|dv|\right)^{1/q} \\
p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\[2ex]
\frac{1}{\ell^2(\gamma)}\int_\gamma\int_\gamma |f(v)-\lambda|\,|dv|\,|dv|\max_{v,z\in\gamma}|g(v)-g(z)|,
\end{cases} \\
&= \begin{cases}
\max_{v\in\gamma}|f(v)-\lambda|\,\frac{1}{\ell^2(\gamma)}\int_\gamma\int_\gamma |g(v)-g(z)|\,|dz|\,|dv|, \\[2ex]
\left(\frac{1}{\ell(\gamma)}\int_\gamma |f(v)-\lambda|^p\,|dv|\right)^{1/p}\left(\frac{1}{\ell^2(\gamma)}\int_\gamma\int_\gamma |g(v)-g(z)|^q\,|dz|\,|dv|\right)^{1/q} \\
p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\[2ex]
\max_{v,z\in\gamma}|g(v)-g(z)|\,\frac{1}{\ell(\gamma)}\int_\gamma |f(v)-\lambda|\,|dv|.
\end{cases}
\end{aligned}
$$

Therefore we can state the following result concerning double integrals as well:

Corollary 5. *Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t\in\gamma$ from $z(a)=u$ to $z(b)=w$. If f, g are continuous on γ and $\lambda\in\mathbb{C}$, then*

$$
\begin{aligned}
(2.26)\quad |\mathcal{P}_\gamma(f,g)| &\leq \frac{1}{\ell(\gamma)}\int_\gamma |f(v)-\lambda|\left|g(v)-\frac{1}{\ell(\gamma)}\int_\gamma g(z)\,|dz|\right|\,|dv| \\
&\leq \begin{cases}
\max_{v\in\gamma}|f(v)-\lambda|\,\frac{1}{\ell^2(\gamma)}\int_\gamma\int_\gamma |g(v)-g(z)|\,|dz|\,|dv|, \\[2ex]
\left(\frac{1}{\ell(\gamma)}\int_\gamma |f(v)-\lambda|^p\,|dv|\right)^{1/p}\left(\frac{1}{\ell^2(\gamma)}\int_\gamma\int_\gamma |g(v)-g(z)|^q\,|dz|\,|dv|\right)^{1/q} \\
p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\[2ex]
\max_{v,z\in\gamma}|g(v)-g(z)|\,\frac{1}{\ell(\gamma)}\int_\gamma |f(v)-\lambda|\,|dv|.
\end{cases}
\end{aligned}
$$

If the path γ is a segment $[u,w]$ connecting two distinct points u and w in $\mathbb{C}$ then we write $\int_\gamma f(z)\,dz$ as $\int_u^w f(z)\,dz$.

If f, g are continuous on $[u,w]$ and $\lambda\in\mathbb{C}$, then

$$
(2.27)\qquad \left|\frac{1}{|w-u|}\int_u^w f(z)\,g(z)\,|dz| - \frac{1}{|w-u|^2}\int_u^w f(z)\,|dz|\int_u^w g(z)\,|dz|\right|
$$

$$\leq \frac{1}{|w-u|}\int_u^w |f(v)-\lambda| \left| g(v) - \frac{1}{|w-u|}\int_u^w g(z)|dz| \right| |dv|$$

$$\leq \begin{cases} \max_{v\in[u,w]} |f(v)-\lambda| \frac{1}{|w-u|}\int_u^w \left| g(v) - \frac{1}{|w-u|}\int_u^w g(z)|dz| \right| |dv|, \\ \\ \left(\frac{1}{|w-u|}\int_u^w |f(v)-\lambda|^p\right)^{1/p} \left(\frac{1}{|w-u|}\int_u^w \left| g(v) - \frac{1}{|w-u|}\int_u^w g(z)|dz| \right|^q |dv|\right)^{1/q} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max_{v\in[u,w]} \left| g(v) - \frac{1}{|w-u|}\int_u^w g(z)|dz| \right| \frac{1}{|w-u|}\int_u^w |f(v)-\lambda| |dv|. \end{cases}$$

In particular,

$$\left| \frac{1}{|w-u|}\int_u^w f^2(z)|dz| - \left(\frac{1}{|w-u|}\int_u^w f(z)|dz|\right)^2 \right| \tag{2.28}$$
$$\leq \frac{1}{|w-u|}\int_u^w |f(v)-\lambda| \left| f(v) - \frac{1}{|w-u|}\int_u^w f(z)|dz| \right| |dv|$$

$$\leq \begin{cases} \max_{v\in[u,w]} |f(v)-\lambda| \frac{1}{|w-u|}\int_u^w \left| f(v) - \frac{1}{|w-u|}\int_u^w f(z)|dz| \right| |dv|, \\ \\ \left(\frac{1}{|w-u|}\int_u^w |f(v)-\lambda|^p\right)^{1/p} \left(\frac{1}{|w-u|}\int_u^w \left| f(v) - \frac{1}{|w-u|}\int_u^w f(z)|dz| \right|^q |dv|\right)^{1/q} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max_{v\in[u,w]} \left| f(v) - \frac{1}{|w-u|}\int_u^w f(z)|dz| \right| \frac{1}{|w-u|}\int_u^w |f(v)-\lambda| |dv|. \end{cases}$$

3. Examples for Circular Paths

Let $[a,b] \subseteq [0,2\pi]$ and the circular path $\gamma_{[a,b],R}$ centered in 0 and with radius $R>0$

$$z(t) = R\exp(it) = R(\cos t + i\sin t),\ t\in[a,b].$$

If $[a,b]=[0,\pi]$ then we get a half circle while for $[a,b]=[0,2\pi]$ we get the full circle.

We have

$$z'(t) = Ri\exp(it),\ t\in[a,b]$$

and $|z'(t)| = R$ for $t\in[a,b]$ giving that

$$\ell\left(\gamma_{[a,b],R}\right) = \int_a^b |z'(t)|\,dt = R(b-a).$$

If f and g are continuous on $\gamma_{[a,b],R}$ and there exists ϕ, Φ, ψ, $\Psi \in \mathbb{C}$, $\phi \neq \Phi$, $\psi \neq \Psi$ such that $f \in \bar{\Delta}_{\gamma_{[a,b],R}}(\phi, \Phi)$ and $g \in \bar{\Delta}_{\gamma_{[a,b],R}}(\psi, \Psi)$, then by (2.20) we get

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(R\exp(it))\,g(R\exp(it))\,dt\right. \\ &\qquad\left.-\frac{1}{b-a}\int_a^b f(R\exp(it))\,dt\frac{1}{b-a}\int_a^b g(R\exp(it))\,dt\right| \\ &\quad\leq \frac{1}{2}|\Phi-\phi|\frac{1}{b-a}\int_a^b\left|g(R\exp(it))-\frac{1}{b-a}\int_a^b g(R\exp(is))\,ds\right|dt \\ &\leq \frac{1}{2}|\Phi-\phi|\left(\frac{1}{b-a}\int_a^b|g(R\exp(it))|^2\,dt-\left|\frac{1}{b-a}\int_a^b g(R\exp(is))\,ds\right|^2\right)^{1/2} \\ &\qquad\qquad\leq \frac{1}{4}|\Phi-\phi|\,|\Psi-\psi|. \end{aligned} \tag{3.1}$$

If γ is the circle $\mathcal{C}(0,R)$ centered in 0 and of radius $R>0$ and f and g are continuous on $\mathcal{C}(0,R)$ and there exists ϕ, Φ, ψ, $\Psi \in \mathbb{C}$, $\phi \neq \Phi$, $\psi \neq \Psi$ such that $f \in \bar{\Delta}_{\mathcal{C}(0,R)}(\phi, \Phi)$ and $g \in \bar{\Delta}_{\mathcal{C}(0,R)}(\psi, \Psi)$, then

$$\begin{aligned} &\left|\frac{1}{2\pi}\int_0^{2\pi} f(R\exp(it))\,g(R\exp(it))\,dt\right. \\ &\qquad\left.-\frac{1}{2\pi}\int_0^{2\pi} f(R\exp(it))\,dt\frac{1}{2\pi}\int_0^{2\pi} g(R\exp(it))\,dt\right| \\ &\quad\leq \frac{1}{2}|\Phi-\phi|\frac{1}{2\pi}\int_0^{2\pi}\left|g(R\exp(it))-\frac{1}{2\pi}\int_0^{2\pi} g(R\exp(is))\,ds\right|dt \\ &\leq \frac{1}{2}|\Phi-\phi|\left(\frac{1}{2\pi}\int_0^{2\pi}|g(R\exp(it))|^2\,dt-\left|\frac{1}{2\pi}\int_0^{2\pi} g(R\exp(is))\,ds\right|^2\right)^{1/2} \\ &\qquad\qquad\leq \frac{1}{4}|\Phi-\phi|\,|\Psi-\psi|. \end{aligned} \tag{3.2}$$

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