ON SOME ČEBYŠEV TYPE INEQUALITIES FOR THE COMPLEX INTEGRAL

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Assume that f and g are continuous on γ, $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$ and the *complex Čebyšev functional* is defined by

$$\mathcal{D}_\gamma(f,g) := \frac{1}{w-u}\int_\gamma f(z)\,g(z)\,dz - \frac{1}{w-u}\int_\gamma f(z)\,dz\frac{1}{w-u}\int_\gamma g(z)\,dz.$$

In this paper we establish some bounds for the magnitude of the functional $\mathcal{D}_\gamma(f,g)$ under Lipschitzian assumptions for the functions f and g and provide a complex version for the well known Čebyšev inequality.

1. Introduction

For two Lebesgue integrable functions f, $g : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt.$$

In 1934, G. Grüss [17] showed that

$$|C(f,g)| \le \frac{1}{4}(M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \le f \le M < \infty, \quad -\infty < n \le g \le N < \infty \quad \text{a.e. on} \quad [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.4) in the sense that it cannot be replaced by a smaller one.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [4], states that

$$|C(f,g)| \le \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{1.3}$$

provided that f', g' exist and are continuous on $[a,b]$ and $\|f'\|_\infty = \sup_{t\in[a,b]}|f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (1.3) also holds if f, $g : [a,b] \to \mathbb{R}$ are assumed to be *absolutely continuous* and f', $g' \in L_\infty[a,b]$ while $\|f'\|_\infty = \operatorname{essup}_{t\in[a,b]}|f'(t)|$.

For other inequality of Grüss' type see [1]-[5], [6]-[16], [18]-[23] and [25]-[28].

In order to extend Grüss' inequality to complex integral we need the following preparations.

1991 *Mathematics Subject Classification.* 26D15, 26D10, 30A10, 30A86.

Key words and phrases. Complex integral, Continuous functions, Holomorphic functions, Čebyšev inequality.

RGMIA Res. Rep. Coll. 21 (2018), Art. 122, 9 pp. Received 08/10/18

Suppose γ is a smooth path parametrized by $z(t)$, $t\in[a,b]$ and f is a complex function which is continuous on γ. Put $z(a)=u$ and $z(b)=w$ with $u,w\in\mathbb{C}$. We define the integral of f on $\gamma_{u,w}=\gamma$ as

$$\int_{\gamma} f(z)\,dz=\int_{\gamma_{u,w}} f(z)\,dz:=\int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t\in[a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz:=\int_{\gamma_{u,v}} f(z)\,dz+\int_{\gamma_{v,w}} f(z)\,dz$$

where $v:=z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz|:=\int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma)=\int_{\gamma_{u,w}} |dz|=\int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, and open domain and suppose $\gamma\subset G$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz=f(w)\,g(w)-f(u)\,g(u)-\int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.4}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_{\gamma} f(z)\,dz\right|\le\int_{\gamma}|f(z)|\,|dz|\le\|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.5}$$

where $\|f\|_{\gamma,\infty}:=\sup_{z\in\gamma}|f(z)|$.

We also define the p-norm with $p\ge1$ by

$$\|f\|_{\gamma,p}:=\left(\int_{\gamma}|f(z)|^p\,|dz|\right)^{1/p}.$$

For $p=1$ we have

$$\|f\|_{\gamma,1}:=\int_{\gamma}|f(z)|\,|dz|.$$

If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1}\le[\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}.$$

Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t\in\gamma$ from $z(a)=u$ to $z(b)=w$ with $w\ne u$. If f and g are continuous on γ, we consider the *complex Čebyšev functional* defined by

$$\mathcal{D}_{\gamma}(f,g):=\frac{1}{w-u}\int_{\gamma} f(z)\,g(z)\,dz-\frac{1}{w-u}\int_{\gamma} f(z)\,dz\frac{1}{w-u}\int_{\gamma} g(z)\,dz.$$

In this paper we establish some bounds for the magnitude of the functional $\mathcal{D}_\gamma(f,g)$ under various assumptions for the functions f and g and provide a complex version for the Čebyšev inequality (1.3).

2. Čebyšev Type Results

We start with the following identity of interest:

Lemma 1. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ, then*

$$\begin{aligned}(2.1)\quad \mathcal{D}_\gamma(f,g) &= \frac{1}{2(w-u)^2}\int_\gamma \left(\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dw\right)dz \\ &= \frac{1}{2(w-u)^2}\int_\gamma \left(\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dz\right)dw \\ &= \frac{1}{2(w-u)^2}\int_\gamma\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dzdw.\end{aligned}$$

Proof. For any $z \in \gamma$ the integral $\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dw$ exists and

$$\begin{aligned}I(z) &:= \int_\gamma (f(z)-f(w))(g(z)-g(w))\,dw \\ &= \int_\gamma (f(z)g(z)+f(w)g(w)-g(z)f(w)-f(z)g(w))\,dw \\ &= f(z)g(z)\int_\gamma dw + \int_\gamma f(w)g(w)\,dw - g(z)\int_\gamma f(w)\,dw - f(z)\int_\gamma g(w)\,dw \\ &= (w-u)f(z)g(z) + \int_\gamma f(w)g(w)\,dw - g(z)\int_\gamma f(w)\,dw - f(z)\int_\gamma g(w)\,dw.\end{aligned}$$

The function $I(z)$ is also continuous on γ, then the integral $\int_\gamma I(z)\,dz$ exists and

$$\begin{aligned}\int_\gamma I(z)\,dz &= \int_\gamma \Big[(w-u)f(z)g(z) + \int_\gamma f(w)g(w)\,dw \\ &\qquad -g(z)\int_\gamma f(w)\,dw - f(z)\int_\gamma g(w)\,dw\Big]dz \\ &= (w-u)\int_\gamma f(z)g(z)\,dz + (w-u)\int_\gamma f(w)g(w)\,dw \\ &\qquad -\int_\gamma f(w)\,dw\int_\gamma g(z)\,dz - \int_\gamma g(w)\,dw\int_\gamma f(z)\,dz \\ &= 2(w-u)\int_\gamma f(z)g(z)\,dz - 2\int_\gamma f(z)\,dz\int_\gamma g(z)\,dz = 2(w-u)^2\,\mathcal{P}_\gamma(f,g),\end{aligned}$$

which proves the first equality in (2.1).

The rest follows in a similar manner and we omit the details. □

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $h : \gamma \to \mathbb{C}$ a continuous function on γ. Define the quantity:

$$\mathcal{P}_\gamma\left(h,\overline{h}\right)=\frac{1}{\ell(\gamma)}\int_\gamma|h(z)|^2|dz|-\left|\frac{1}{\ell(\gamma)}\int_\gamma h(z)|dz|\right|^2 \tag{2.2}$$
$$=\frac{1}{\ell(\gamma)}\int_\gamma\left|h(v)-\frac{1}{\ell(\gamma)}\int_\gamma h(z)|dz|\right|^2|dv|\geq 0.$$

We say that the function $f:G\subset\mathbb{C}\to\mathbb{C}$ is L-h-Lipschitzian on the subset G if

$$|f(z)-f(w)|\leq L|h(z)-h(w)|$$

for any $z,w\in G$. If $h(z)=z$, we recapture the usual concept of L-Lipschitzian functions on G.

Theorem 1. *Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t\in\gamma$ from $z(a)=u$ to $z(b)=w$ with $w\neq u$ and $h:\gamma\to\mathbb{C}$ is continuous, f and g are L_1, L_2-h-Lipschitzian functions on γ, then*

$$|\mathcal{D}_\gamma(f,g)|\leq L_1L_2\frac{\ell^2(\gamma)}{|w-u|^2}\mathcal{P}_\gamma\left(h,\overline{h}\right). \tag{2.3}$$

Proof. Taking the modulus in the first equality in (2.1), we get

$$|\mathcal{D}_\gamma(f,g)|=\frac{1}{2|w-u|^2}\left|\int_\gamma\left(\int_\gamma(f(z)-f(w))(g(z)-g(w))\,dw\right)dz\right| \tag{2.4}$$
$$\leq\frac{1}{2|w-u|^2}\int_\gamma\left|\int_\gamma(f(z)-f(w))(g(z)-g(w))\,dw\right||dz|$$
$$\leq\frac{1}{2|w-u|^2}\int_\gamma\left(\int_\gamma|(f(z)-f(w))(g(z)-g(w))||dw|\right)|dz|$$
$$\leq\frac{L_1L_2}{2|w-u|^2}\int_\gamma\left(\int_\gamma|h(z)-h(w)|^2|dw|\right)|dz|=:A.$$

Now, observe that

$$\int_\gamma\left(\int_\gamma|h(z)-h(w)|^2|dw|\right)|dz| \tag{2.5}$$
$$=\int_\gamma\left(\int_\gamma\left(|h(z)|^2-2\operatorname{Re}\left(h(z)\overline{h(w)}\right)+|h(w)|^2\right)|dw|\right)|dz|$$
$$=\int_\gamma\left(\ell(\gamma)|h(z)|^2-2\operatorname{Re}\left(h(z)\int_\gamma\overline{h(w)}|dw|\right)+\int_\gamma|h(w)|^2|dw|\right)|dz|$$
$$=\ell(\gamma)\int_\gamma|h(z)|^2|dz|-2\operatorname{Re}\left(\int_\gamma h(z)|dz|\int_\gamma\overline{h(w)}|dw|\right)+\ell(\gamma)\int_\gamma|h(w)|^2|dw|$$
$$=2\ell(\gamma)\int_\gamma|h(z)|^2|dz|-2\operatorname{Re}\left(\int_\gamma h(z)|dz|\overline{\left(\int_\gamma h(w)|dw|\right)}\right)$$
$$=2\left[\ell(\gamma)\int_\gamma|h(z)|^2|dz|-\left|\int_\gamma h(z)|dz|\right|^2\right]=2\ell^2(\gamma)\mathcal{P}_\gamma\left(h,\overline{h}\right).$$

Therefore, by (2.5) we get

$$A=L_1L_2\frac{\ell^2(\gamma)}{|w-u|^2}\mathcal{P}_\gamma\left(h,\overline{h}\right)$$

and by (2.4) we get the desired result (2.3). □

Further, for $\gamma \subset \mathbb{C}$ a piecewise smooth path parametrized by $z(t)$ and by taking $h(z) = z$ in (2.2) we can consider the quantity

$$\begin{aligned}(2.6)\quad \mathcal{P}_\gamma &:= \frac{1}{\ell(\gamma)} \int_\gamma |z|^2 |dz| - \left| \frac{1}{\ell(\gamma)} \int_\gamma z |dz| \right|^2 \\ &= \frac{1}{\ell(\gamma)} \int_\gamma \left| v - \frac{1}{\ell(\gamma)} \int_\gamma z |dz| \right|^2 |dv| = \frac{1}{2\ell^2(\gamma)} \int_\gamma \left(\int_\gamma |z-w|^2 |dw| \right) |dz| \geq 0.\end{aligned}$$

Corollary 1. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$ and $h : \gamma \to \mathbb{C}$ is continuous, f and g are L_1, L_2-Lipschitzian functions on γ, then*

$$|\mathcal{D}_\gamma(f,g)| \leq L_1 L_2 \frac{\ell^2(\gamma)}{|w-u|^2} \mathcal{P}_\gamma. \tag{2.7}$$

Remark 1. *Assume that f is L-h-Lipschitzian on γ. For $g = f$ we have*

$$\mathcal{D}_\gamma(f,f) = \frac{1}{w-u} \int_\gamma f^2(z)\, dz - \left(\frac{1}{w-u} \int_\gamma f(z)\, dz \right)^2 \tag{2.8}$$

and by (2.3) we get

$$|\mathcal{D}_\gamma(f,f)| \leq L^2 \frac{\ell^2(\gamma)}{|w-u|^2} \mathcal{P}_\gamma\left(h, \overline{h}\right). \tag{2.9}$$

For $g = \bar{f}$ we have

$$\mathcal{D}_\gamma\left(f,\bar{f}\right) = \frac{1}{w-u} \int_\gamma |f(z)|^2 dz - \frac{1}{w-u} \int_\gamma f(z)\, dz \frac{1}{w-u} \int_\gamma \overline{f(z)} dz \tag{2.10}$$

and by (2.3) we get

$$\left|\mathcal{D}_\gamma\left(f,\bar{f}\right)\right| \leq L^2 \frac{\ell^2(\gamma)}{|w-u|^2} \mathcal{P}_\gamma\left(f, \overline{f}\right). \tag{2.11}$$

If f is L-Lipschitzian on γ, then

$$|\mathcal{D}_\gamma(f,f)| \leq L^2 \frac{\ell^2(\gamma)}{|w-u|^2} \mathcal{P}_\gamma \tag{2.12}$$

and

$$\left|\mathcal{D}_\gamma\left(f,\bar{f}\right)\right| \leq L^2 \frac{\ell^2(\gamma)}{|w-u|^2} \mathcal{P}_\gamma. \tag{2.13}$$

If the path γ is a segment $[u,w]$ connecting two distinct points u and w in $\mathbb{C}$ then we write $\int_\gamma f(z)\, dz$ as $\int_u^w f(z)\, dz$.

Now, if f and g are L_1, L_2-Lipschitzian functions on $[u,w] := \{(1-t)u + tw,\ t \in [0,1]\}$, then by (2.7) we have

$$|\mathcal{D}_\gamma(f,g)| \leq L_1 L_2 \mathcal{P}_{[u,w]},$$

where

$$\mathcal{P}_{[u,w]} = \frac{|w-u|^2}{2|w-u|^2}\int_0^1\left(\int_0^1 |(1-t)u+tw-(1-s)u-sw|^2\,dt\right)ds$$
$$= \frac{1}{2}|w-u|^2\int_0^1\left(\int_0^1 (t-s)^2\,dt\right)ds = \frac{1}{12}|w-u|^2.$$

Therefore,

$$\left|\frac{1}{w-u}\int_\gamma f(z)g(z)\,dz - \frac{1}{w-u}\int_\gamma f(z)\,dz\frac{1}{w-u}\int_\gamma g(z)\,dz\right| \le \frac{1}{12}|w-u|^2 L_1L_2, \tag{2.14}$$

if f and g are L_1, L_2-Lipschitzian functions on $[u,w]$.

If f is L-Lipschitzian on $[u,w]$, then

$$\left|\frac{1}{w-u}\int_\gamma f^2(z)\,dz - \left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)^2\right| \le \frac{1}{12}|w-u|^2 L^2 \tag{2.15}$$

and

$$\left|\frac{1}{w-u}\int_\gamma |f(z)|^2\,dz - \frac{1}{w-u}\int_\gamma f(z)\,dz\frac{1}{w-u}\int_\gamma \overline{f(z)}dz\right| \le \frac{1}{12}|w-u|^2 L^2. \tag{2.16}$$

3. Examples for Circular Paths

Let $[a,b] \subseteq [0,2\pi]$ and the circular path $\gamma_{[a,b],R}$ centered in 0 and with radius $R>0$

$$z(t) = R\exp(it) = R(\cos t + i\sin t),\ t\in[a,b].$$

If $[a,b]=[0,\pi]$ then we get a half circle while for $[a,b]=[0,2\pi]$ we get the full circle.

Since

$$\left|e^{is}-e^{it}\right|^2 = \left|e^{is}\right|^2 - 2\operatorname{Re}\left(e^{i(s-t)}\right) + \left|e^{it}\right|^2$$
$$= 2-2\cos(s-t) = 4\sin^2\left(\frac{s-t}{2}\right)$$

for any t, $s\in\mathbb{R}$, then

$$\left|e^{is}-e^{it}\right|^r = 2^r\left|\sin\left(\frac{s-t}{2}\right)\right|^r \tag{3.1}$$

for any t, $s\in\mathbb{R}$ and $r>0$. In particular,

$$\left|e^{is}-e^{it}\right| = 2\left|\sin\left(\frac{s-t}{2}\right)\right|$$

for any t, $s\in\mathbb{R}$.

If $u=R\exp(ia)$ and $w=R\exp(ib)$ then

$$w-u = R[\exp(ib)-\exp(ia)] = R[\cos b + i\sin b - \cos a - i\sin a]$$
$$= R[\cos b - \cos a + i(\sin b - \sin a)].$$

Since

$$\cos b-\cos a=-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

and

$$\sin b-\sin a=2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right),$$

hence

$$\begin{aligned}w-u&=R\left[-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)+2i\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right)\right]\\&=2R\sin\left(\frac{b-a}{2}\right)\left[-\sin\left(\frac{a+b}{2}\right)+i\cos\left(\frac{a+b}{2}\right)\right]\\&=2Ri\sin\left(\frac{b-a}{2}\right)\left[\cos\left(\frac{a+b}{2}\right)+i\sin\left(\frac{a+b}{2}\right)\right]\\&=2Ri\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right].\end{aligned}$$

If $\gamma=\gamma_{[a,b],R}$, then the *circular complex Čebyšev functional* is defined by

$$\begin{aligned}\mathcal{C}_{[a,b],R}(f,g)&:=\mathcal{D}_{\gamma_{[a,b],R}}(f,g)\\&=\frac{1}{2\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right]}\int_a^b f(R\exp(it))\,g(R\exp(it))\exp(it)\,dt\\&\quad-\frac{1}{4\sin^2\left(\frac{b-a}{2}\right)\exp\left[2\left(\frac{a+b}{2}\right)i\right]}\\&\quad\times\int_a^b f(R\exp(it))\exp(it)\,dt\int_a^b g(R\exp(it))\exp(it)\,dt.\end{aligned}\tag{3.2}$$

If $\gamma=\gamma_{[a,b],R}$, then

$$\begin{aligned}\mathcal{P}_\gamma&:=\frac{1}{2\ell^2(\gamma)}\int_\gamma\left(\int_\gamma|z-w|^2|dw|\right)|dz|\\&=\frac{R^4}{2R^2(b-a)^2}\int_a^b\left(\int_a^b\left|e^{is}-e^{it}\right|^2dt\right)ds\end{aligned}\tag{3.3}$$

$$= \frac{R^4}{2R^2(b-a)^2}\int_a^b\left(\int_a^b[2-2\cos(s-t)]\,dt\right)ds$$

$$= \frac{R^2}{(b-a)^2}\int_a^b\left(\int_a^b[1-\cos(s-t)]\,dt\right)ds$$

$$= \frac{R^2}{(b-a)^2}\int_a^b(b-a-\sin(b-s)-\sin(s-a))\,ds$$

$$= \frac{R^2}{(b-a)^2}\left[(b-a)^2-1+\cos(b-a)+\cos(b-a)-1\right]$$

$$= \frac{R^2}{(b-a)^2}\left[(b-a)^2-2(1-\cos(b-a))\right]$$

$$= \frac{R^2}{(b-a)^2}\left[(b-a)^2-4\sin^2\left(\frac{b-a}{2}\right)\right]$$

$$= \frac{4R^2}{(b-a)^2}\left[\left(\frac{b-a}{2}\right)^2-\sin^2\left(\frac{b-a}{2}\right)\right]$$

We have the following result:

Proposition 1. *Let $\gamma_{[a,b],R}$ be a circular path centered in 0 and with radius $R>0$ and $[a,b]\subset[0,2\pi]$. If f and g are L_1, L_2-Lipschitzian functions on $\gamma_{[a,b],R}$, then*

$$\left|\mathcal{C}_{[a,b],R}(f,g)\right| \le \frac{R^2}{\sin^2\left(\frac{b-a}{2}\right)}\left[\left(\frac{b-a}{2}\right)^2-\sin^2\left(\frac{b-a}{2}\right)\right]L_1L_2. \tag{3.4}$$

References

[1] M. W. Alomari, A companion of Grüss type inequality for Riemann-Stieltjes integral and applications. *Mat. Vesnik* **66** (2014), no. 2, 202–212.

[2] D. Andrica and C. Badea, Grüss' inequality for positive linear functionals. *Period. Math. Hungar.* **19** (1988), no. 2, 155–167.

[3] D. Baleanu, S. D. Purohit and F. Uçar, On Grüss type integral inequality involving the Saigo's fractional integral operators. *J. Comput. Anal. Appl.* **19** (2015), no. 3, 480–489

[4] P. L. Chebyshev, Sur les expressions approximatives des intègrals dèfinis par les outres prises entre les même limites, *Proc. Math. Soc. Charkov,* **2** (1882), 93-98.

[5] P. Cerone, On a Čebyšev-type functional and Grüss-like bounds. *Math. Inequal. Appl.* **9** (2006), no. 1, 87–102.

[6] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications, *Tamkang J. Math.*, **38**(1) (2007), 37-49. Preprint *RGMIA Res. Rep. Coll.,* **5**(2) (2002), Article 14. [ONLINE: `http://rgmia.vu.edu.au/v5n2.html`].

[7] P. Cerone and S. S. Dragomir, Some new Ostrowski-type bounds for the Čebyšev functional and applications. *J. Math. Inequal.* **8** (2014), no. 1, 159–170.

[8] P. Cerone, S. S. Dragomir and J. Roumeliotis, Grüss inequality in terms of Δ-seminorms and applications. *Integral Transforms Spec. Funct.* **14** (2003), no. 3, 205–216.

[9] X. L. Cheng and J. Sun, A note on the perturbed trapezoid inequality, *J. Ineq. Pure and Appl. Math.,* **3**(2) Art. 29, (2002).

[10] S. S. Dragomir, A generalization of Grüss's inequality in inner product spaces and applications. *J. Math. Anal. Appl.* **237** (1999), no. 1, 74–82.

[11] S. S. Dragomir, A Grüss' type integral inequality for mappings of r-Hölder's type and applications for trapezoid formula. *Tamkang J. Math.* **31** (2000), no. 1, 43–47.

[12] S. S. Dragomir, Some integral inequalities of Grüss type. *Indian J. Pure Appl. Math.* **31** (2000), no. 4, 397–415.

[13] S. S. Dragomir, Integral Grüss inequality for mappings with values in Hilbert spaces and applications. *J. Korean Math. Soc.* **38** (2001), no. 6, 1261–1273.
[14] S. S. Dragomir and I. A. Fedotov, An inequality of Grüss' type for Riemann-Stieltjes integral and applications for special means. *Tamkang J. Math.* **29** (1998), no. 4, 287–292.
[15] S. S. Dragomir and I. Gomm, Some integral and discrete versions of the Grüss inequality for real and complex functions and sequences. *Tamsui Oxf. J. Math. Sci.* **19** (2003), no. 1, 67–77.
[16] A. M. Fink, A treatise on Grüss' inequality. *Analytic and Geometric Inequalities and Applications*, 93–113, Math. Appl., 478, Kluwer Acad. Publ., Dordrecht, 1999.
[17] G. Grüss, Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \int_a^b g(x)dx$, *Math. Z.* , **39**(1935), 215-226.
[18] D. Jankov Maširević and T. K. Pogány, Bounds on Čebyšev functional for $C_\varphi[0,1]$ function class. *J. Anal.* **22** (2014), 107–117.
[19] Z. Liu, Refinement of an inequality of Grüss type for Riemann-Stieltjes integral. *Soochow J. Math.* **30** (2004), no. 4, 483–489.
[20] Z. Liu, Notes on a Grüss type inequality and its application. *Vietnam J. Math.* **35** (2007), no. 2, 121–127.
[21] A. Lupaş, The best constant in an integral inequality, *Mathematica* (Cluj, Romania), **15**(**38**)(2) (1973), 219-222.
[22] A. Mc.D. Mercer and P. R. Mercer, New proofs of the Grüss inequality. *Aust. J. Math. Anal. Appl.* **1** (2004), no. 2, Art. 12, 6 pp.
[23] N. Minculete and L. Ciurdariu, A generalized form of Grüss type inequality and other integral inequalities. *J. Inequal. Appl.* **2014**, 2014:119, 18 pp.
[24] A. M. Ostrowski, On an integral inequality, *Aequat. Math.,* **4** (1970), 358-373.
[25] B. G. Pachpatte, A note on some inequalities analogous to Grüss inequality. *Octogon Math. Mag.* **5** (1997), no. 2, 62–66
[26] J. Pečarić and Š. Ungar, On a inequality of Grüss type. *Math. Commun.* **11** (2006), no. 2, 137–141.
[27] M. Z. Sarikaya and H. Budak, An inequality of Grüss like via variant of Pompeiu's mean value theorem. *Konuralp J. Math.* **3** (2015), no. 1, 29–35.
[28] N. Ujević, A generalization of the pre-Grüss inequality and applications to some quadrature formulae. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 1, Article 13, 9 pp.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa