GRÜSS' TYPE INEQUALITIES FOR THE COMPLEX INTEGRAL VIA THE SONIN IDENTITY

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Assume that f and g are continuous on γ, $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$ and the *complex Čebyšev functional* is defined by

$$\mathcal{D}_\gamma(f,g) := \frac{1}{w-u}\int_\gamma f(z) g(z) dz - \frac{1}{w-u}\int_\gamma f(z) dz \frac{1}{w-u}\int_\gamma g(z) dz.$$

In this paper, by the use of a Sonin type identiry for the complex integral, we establish some bounds for the magnitude of the functional $\mathcal{D}_\gamma(f,g)$ under various assumptions for the functions f and g.

1. Introduction

For two Lebesgue integrable functions $f, g : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t) g(t) dt - \frac{1}{b-a}\int_a^b f(t) dt \frac{1}{b-a}\int_a^b g(t) dt.$$

In 1934, G. Grüss [17] showed that

$$|C(f,g)| \leq \frac{1}{4}(M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on} \quad [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.6) in the sense that it cannot be replaced by a smaller one.

In [6], P. Cerone and S. S. Dragomir proved the following inequalities:

$$|C(f,g)| \leq \begin{cases} \inf_{\gamma \in \mathbb{R}} \|g-\gamma\|_\infty \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s) ds \right| dt, \\ \\ \inf_{\gamma \in \mathbb{R}} \|g-\gamma\|_q \frac{1}{b-a}\left(\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s) ds \right|^p dt\right)^{\frac{1}{p}}, \\ \qquad\qquad\qquad \text{where } p > 1,\ 1/p + 1/q = 1. \end{cases} \tag{1.3}$$

For $\gamma = 0$, we get from the first inequality in (1.3)

$$|C(f,g)| \leq \|g\|_\infty \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s) ds \right| dt \tag{1.4}$$

1991 *Mathematics Subject Classification.* 26D15, 26D10, 30A10, 30A86.

Key words and phrases. Complex integral, Continuous functions, Holomorphic functions, Grüss inequality.

RGMIA Res. Rep.Coll. 21 (2018), Art. 125, 16 pp. Received 23/10/18

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq g \leq M$ for a.e. $x \in [a,b]$, then $\left\| g - \frac{m+M}{2} \right\|_\infty \leq \frac{1}{2}(M-m)$ and by the first inequality in (1.3) we can deduce the following result obtained by Cheng and Sun [9]

$$|C(f,g)| \leq \frac{1}{2}(M-m)\frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt. \tag{1.5}$$

The constant $\frac{1}{2}$ is best in (1.5) as shown by Cerone and Dragomir in [7].

For other inequality of Grüss' type see [1]-[5], [7]-[16], [18]-[23] and [25]-[28].

In order to extend Grüss' inequality to complex integral we need the following preparations.

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, and open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.6}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left| \int_\gamma f(z)\,dz \right| \leq \int_\gamma |f(z)|\,|dz| \leq \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.7}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \geq 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\,|dz| \right)^{1/p}.$$

For $p=1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|\,.$$

If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}\,.$$

Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z(t)\,,\ t\in\gamma$ from $z(a)=u$ to $z(b)=w$ with $w\neq u$. If f and g are continuous on γ, we consider the *complex Čebyšev functional* defined by

$$\mathcal{D}_\gamma(f,g) := \frac{1}{w-u}\int_\gamma f(z)\,g(z)\,dz - \frac{1}{w-u}\int_\gamma f(z)\,dz\frac{1}{w-u}\int_\gamma g(z)\,dz.$$

In this paper we establish some bounds for the magnitude of the functional $\mathcal{D}_\gamma(f,g)$ under various assumptions for the functions f and g and provide a complex version for the Grüss inequality (1.1).

2. Some Preliminary Facts

Consider the functional

$$\mathcal{S}_{\gamma,2}(f) := \frac{1}{\ell(\gamma)}\int_\gamma \left| f(z) - \frac{1}{w-u}\int_\gamma f(w)\,dw\right|^2 |dz| \tag{2.1}$$

defined for paths γ that are piecewise smooth and for continuous functions $f:\gamma\to\mathbb{C}$.

We have

$$\begin{aligned} 0 &\le \int_\gamma \left| f(z) - \frac{1}{w-u}\int_\gamma f(w)\,dw\right|^2 |dz| \\ &= \int_\gamma \left[|f(z)|^2 - 2\operatorname{Re}\left(\overline{f(z)}\frac{1}{w-u}\int_\gamma f(w)\,dw\right) \right. \\ &\qquad \left. + \frac{1}{|w-u|^2}\left|\int_\gamma f(w)\,dw\right|^2\right] |dz| \\ &= \int_\gamma |f(z)|^2\,|dz| \\ &\quad - 2\operatorname{Re}\left(\int_\gamma \overline{f(z)}\,|dz|\,\frac{1}{w-u}\int_\gamma f(w)\,dw\right) + \frac{\ell(\gamma)}{|w-u|^2}\left|\int_\gamma f(w)\,dw\right|^2, \end{aligned}$$

namely, we have the following inequality of interest:

$$\begin{aligned} \operatorname{Re}&\left(\frac{1}{\ell(\gamma)}\int_\gamma \overline{f(z)}\,|dz|\,\frac{1}{w-u}\int_\gamma f(w)\,dw\right) \\ &\le \frac{1}{2}\left[\frac{1}{\ell(\gamma)}\int_\gamma |f(z)|^2\,|dz| + \left|\frac{1}{w-u}\int_\gamma f(w)\,dw\right|^2\right]. \end{aligned} \tag{2.2}$$

Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $f:\gamma\to\mathbb{C}$ a continuous function on γ. Define the quantity:

$$\mathcal{P}_{\gamma}\left(f,\overline{f}\right)=\frac{1}{\ell\left(\gamma\right)}\int_{\gamma}\left|f\left(z\right)\right|^{2}\left|dz\right|-\left|\frac{1}{\ell\left(\gamma\right)}\int_{\gamma}f\left(z\right)\left|dz\right|\right|^{2} \tag{2.3}$$
$$=\frac{1}{\ell\left(\gamma\right)}\int_{\gamma}\left|f\left(v\right)-\frac{1}{\ell\left(\gamma\right)}\int_{\gamma}f\left(z\right)\left|dz\right|\right|^{2}\left|dv\right|\geq 0.$$

Proposition 1. *Assume that the path γ is piecewise smooth and $f:\gamma\to\mathbb{C}$ is continuous on γ. Then we have*

$$\mathcal{S}_{\gamma,2}\left(f\right)\leq 2\epsilon^{2}\left(\gamma\right)\mathcal{P}_{\gamma}\left(f,\overline{f}\right), \tag{2.4}$$

where $\epsilon\left(\gamma\right):=\frac{\ell(\gamma)}{|w-u|}\geq 1$, can be interpreted as the deviation of the path γ from the segment joining the points u and w in $\mathbb{C}$.

Proof. We have

$$\mathcal{S}_{\gamma,2}\left(f\right)=\frac{1}{\left|w-u\right|^{2}\ell\left(\gamma\right)}\int_{\gamma}\left|\int_{\gamma}\left(f\left(z\right)-f\left(w\right)\right)dw\right|^{2}\left|dz\right|.$$

Using Cauchy-Bunyakovsky-Schwarz inequality, we have

$$\left|\int_{\gamma}\left(f\left(z\right)-f\left(w\right)\right)dw\right|^{2}\leq\int_{\gamma}\left|dw\right|\int_{\gamma}\left|f\left(z\right)-f\left(w\right)\right|^{2}\left|dw\right|$$
$$=\ell\left(\gamma\right)\int_{\gamma}\left|f\left(z\right)-f\left(w\right)\right|^{2}\left|dw\right|.$$

Then we have

$$\int_{\gamma}\left|\int_{\gamma}\left(f\left(z\right)-f\left(w\right)\right)dw\right|^{2}\left|dz\right|\leq\ell\left(\gamma\right)\int_{\gamma}\left(\int_{\gamma}\left|f\left(z\right)-f\left(w\right)\right|^{2}\left|dw\right|\right)\left|dz\right|$$
$$=\ell\left(\gamma\right)\int_{\gamma}\left(\int_{\gamma}\left[\left|f\left(z\right)\right|^{2}-2\operatorname{Re}\left(f\left(z\right)\overline{f\left(w\right)}\right)+\left|f\left(w\right)\right|^{2}\right]\left|dw\right|\right)\left|dz\right|$$
$$=2\ell\left(\gamma\right)\left[\ell\left(\gamma\right)\int_{\gamma}\left|f\left(z\right)\right|^{2}\left|dz\right|-\left|\int_{\gamma}f\left(w\right)\left|dw\right|\right|^{2}\right]$$
$$=2\ell^{3}\left(\gamma\right)\left[\frac{1}{\ell\left(\gamma\right)}\int_{\gamma}\left|f\left(z\right)\right|^{2}\left|dz\right|-\left|\frac{1}{\ell\left(\gamma\right)}\int_{\gamma}f\left(w\right)\left|dw\right|\right|^{2}\right],$$

which implies the desired inequality (2.4). □

We need the following results that provide some Ostrowski type inequalities for functions of complex variable that is of interest in itself:

Lemma 1. *Let f be holomorphic in G, an open domain and suppose $\gamma\subset G$ is a smooth path from $z\left(a\right)=u$ to $z\left(b\right)=w$. If $v=z\left(x\right)$ with $x\in\left(a,b\right)$, then $\gamma_{u,w}=\gamma_{u,v}\cup\gamma_{v,w}$,*

$$\left|f\left(v\right)\left(w-u\right)-\int_{\gamma}f\left(z\right)dz\right| \tag{2.5}$$
$$\leq\left[\int_{\gamma_{u,v}}\left|z-u\right|\left|dz\right|+\int_{\gamma_{v,w}}\left|z-w\right|\left|dz\right|\right]\left\|f'\right\|_{\gamma_{u,w};\infty}$$

and

$$\left|f(v)(w-u)-\int_{\gamma}f(z)\,dz\right| \le \max\left\{\max_{z\in\gamma_{u,v}}|z-u|\,,\ \max_{z\in\gamma_{v,w}}|z-w|\right\}\|f'\|_{\gamma_{u,w};1}\,. \tag{2.6}$$

If $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1,$ *then*

$$\left|f(v)(w-u)-\int_{\gamma}f(z)\,dz\right| \le \left(\int_{\gamma_{u,v}}|z-u|^{q}\,|dz|+\int_{\gamma_{v,w}}|z-w|^{q}\,|dz|\right)^{1/q}\|f'\|_{\gamma_{u,w};p}\,. \tag{2.7}$$

Proof. Using the integration by parts formula (1.6) twice we have

$$\int_{\gamma_{u,v}}(z-u)\,f'(z)\,dz=(v-u)\,f(v)-\int_{\gamma_{u,v}}f(z)\,dz$$

and

$$\int_{\gamma_{v,w}}(z-w)\,f'(z)\,dz=(w-v)\,f(v)-\int_{\gamma_{v,w}}f(z)\,dz.$$

If we add these two equalities, we get

$$\begin{aligned}&\int_{\gamma_{u,v}}(z-u)\,f'(z)\,dz+\int_{\gamma_{v,w}}(z-w)\,f'(z)\,dz\\&=f(v)(w-u)-\int_{\gamma_{u,v}}f(z)\,dz-\int_{\gamma_{v,w}}f(z)\,dz,\end{aligned}$$

which gives the following equality of interest

$$f(v)(w-u)-\int_{\gamma}f(z)\,dz=\int_{\gamma_{u,v}}(z-u)\,f'(z)\,dz+\int_{\gamma_{v,w}}(z-w)\,f'(z)\,dz \tag{2.8}$$

that is a generalization of Montgomery identity for functions of real variables.

Using the properties of modulus and the triangle inequality for the complex integral we have

$$
\begin{aligned}
(2.9)\quad & \left| f(v)(w-u) - \int_\gamma f(z)\,dz \right| \\
&= \left| \int_{\gamma_{u,v}} (z-u) f'(z)\,dz + \int_{\gamma_{v,w}} (z-w) f'(z)\,dz \right| \\
&\leq \left| \int_{\gamma_{u,v}} (z-u) f'(z)\,dz \right| + \left| \int_{\gamma_{v,w}} (z-w) f'(z)\,dz \right| \\
&\leq \int_{\gamma_{u,v}} |z-u| \left| f'(z) \right| |dz| + \int_{\gamma_{v,w}} |z-w| \left| f'(z) \right| |dz| \\
&\leq \|f'\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-u|\,|dz| + \|f'\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-w|\,|dz| \\
&\leq \|f'\|_{\gamma_{u,w};\infty} \left[\int_{\gamma_{u,v}} |z-u|\,|dz| + \int_{\gamma_{v,w}} |z-w|\,|dz| \right],
\end{aligned}
$$

which proves the desired result (2.5).

We also have

$$
\begin{aligned}
& \int_{\gamma_{u,v}} |z-u| \left| f'(z) \right| |dz| + \int_{\gamma_{v,w}} |z-w| \left| f'(z) \right| |dz| \\
&\leq \max_{z\in\gamma_{u,v}} |z-u| \int_{\gamma_{u,v}} \left| f'(z) \right| |dz| + \max_{z\in\gamma_{v,w}} |z-w| \int_{\gamma_{v,w}} \left| f'(z) \right| |dz| \\
&\leq \max \left\{ \max_{z\in\gamma_{u,v}} |z-u|, \max_{z\in\gamma_{v,w}} |z-w| \right\} \left[\int_{\gamma_{u,v}} \left| f'(z) \right| |dz| + \int_{\gamma_{v,w}} \left| f'(z) \right| |dz| \right] \\
&= \max \left\{ \max_{z\in\gamma_{u,v}} |z-u|, \max_{z\in\gamma_{v,w}} |z-w| \right\} \int_{\gamma_{u,w}} \left| f'(z) \right| |dz|
\end{aligned}
$$

and by (2.9) we get (2.6).

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's weighted integral inequality we have

$$
\begin{aligned}
& \int_{\gamma_{u,v}} |z-u| \left| f'(z) \right| |dz| + \int_{\gamma_{v,w}} |z-w| \left| f'(z) \right| |dz| \\
&\leq \left(\int_{\gamma_{u,v}} |z-u|^q\,|dz| \right)^{1/q} \left(\int_{\gamma_{u,v}} \left| f'(z) \right|^p |dz| \right)^{1/p} \\
&\quad + \left(\int_{\gamma_{v,w}} |z-w|^q\,|dz| \right)^{1/q} \left(\int_{\gamma_{v,w}} \left| f'(z) \right|^p |dz| \right)^{1/p} =: B.
\end{aligned}
$$

By the elementary inequality

$$
ab + cd \leq (a^p + c^p)^{1/p} (b^q + d^q)^{1/q},
$$

where a, b, c, $d \geq 0$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we also have

$$B \leq \left(\int_{\gamma_{u,v}} |z-u|^q |dz| + \int_{\gamma_{v,w}} |z-w|^q |dz| \right)^{1/q}$$
$$\times \left(\int_{\gamma_{u,v}} |f'(z)|^p |dz| + \int_{\gamma_{v,w}} |f'(z)|^p |dz| \right)^{1/p}$$
$$= \left(\int_{\gamma_{u,v}} |z-u|^q |dz| + \int_{\gamma_{v,w}} |z-w|^q |dz| \right)^{1/q} \left(\int_{\gamma_{u,w}} |f'(z)|^p |dz| \right)^{1/p},$$

which together with (2.9) gives (2.7). □

Proposition 2. *Let f be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. Then*

$$\mathcal{S}_{\gamma,2}(f) \leq \epsilon^2(\gamma)(\Delta_{\gamma,u,2} + \Delta_{\gamma,w,2}) \frac{1}{\ell(\gamma)} \int_{\gamma_{u,w}} |f'(z)|^2 |dz|, \tag{2.10}$$

where

$$\Delta_{\gamma,u,2} := \frac{1}{\ell^2(\gamma)} \int_\gamma \left(\int_{\gamma_{u,v}} |z-u|^2 |dz| \right) |dv| \tag{2.11}$$
$$= \frac{1}{\ell^2(\gamma)} \int_\gamma \left(\int_{\gamma_{u,v}} |z|^2 |dz| \right) |dv| - \frac{2}{\ell^2(\gamma)} \operatorname{Re}\left[\overline{u} \left(\int_\gamma \left(\int_{\gamma_{u,v}} z |dz| \right) |dv| \right) \right] + |u|^2$$

and

$$\Delta_{\gamma,w,2} := \frac{1}{\ell^2(\gamma)} \int_\gamma \left(\int_{\gamma_{v,w}} |z-w|^2 |dz| \right) |dv| \tag{2.12}$$
$$= \frac{1}{\ell^2(\gamma)} \int_\gamma \left(\int_{\gamma_{v,w}} |z|^2 |dz| \right) |dv| - \frac{2}{\ell^2(\gamma)} \operatorname{Re}\left[\overline{w} \left(\int_\gamma \left(\int_{\gamma_{v,w}} z |dz| \right) |dv| \right) \right] + |w|^2.$$

Proof. If $v = z(x)$ with $x \in (a,b)$, then $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$. Using the inequality (2.7) for $p = q = 2$ we have

$$\left| f(v)(w-u) - \int_\gamma f(z)\,dz \right|$$
$$\leq \left(\int_{\gamma_{u,v}} |z-u|^2 |dz| + \int_{\gamma_{v,w}} |z-w|^2 |dz| \right)^{1/2} \|f'\|_{\gamma_{u,w};2}$$

that is equivalent to

$$\left| f(v) - \frac{1}{w-u} \int_\gamma f(z)\,dz \right|^2$$
$$\leq \left(\frac{1}{|w-u|^2} \int_{\gamma_{u,v}} |z-u|^2 |dz| + \frac{1}{|w-u|^2} \int_{\gamma_{v,w}} |z-w|^2 |dz| \right) \|f'\|^2_{\gamma_{u,w};2},$$

which implies that

$$\begin{aligned}
(2.13)\quad \mathcal{S}_{\gamma,2}(f) &= \frac{1}{\ell(\gamma)}\int_{\gamma}\left|f(v)-\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right|^{2}|dv| \\
&\leq \left(\frac{1}{|w-u|^{2}}\int_{\gamma}\left(\int_{\gamma_{u,v}}|z-u|^{2}|dz|\right)|dv|+\frac{1}{|w-u|^{2}}\int_{\gamma}\left(\int_{\gamma_{v,w}}|z-w|^{2}|dz|\right)|dv|\right) \\
&\qquad\times\frac{1}{\ell(\gamma)}\int_{\gamma_{u,w}}|f'(z)|^{2}|dz| \\
&= \epsilon^{2}(\gamma)\left(\frac{1}{\ell^{2}(\gamma)}\int_{\gamma}\left(\int_{\gamma_{u,v}}|z-u|^{2}|dz|\right)|dv|+\frac{1}{\ell^{2}(\gamma)}\int_{\gamma}\left(\int_{\gamma_{v,w}}|z-w|^{2}|dz|\right)|dv|\right) \\
&\qquad\times\frac{1}{\ell(\gamma)}\int_{\gamma_{u,w}}|f'(z)|^{2}|dz| \\
&= \epsilon^{2}(\gamma)\left(\Delta_{\gamma,u,2}+\Delta_{\gamma,w,2}\right)\frac{1}{\ell(\gamma)}\int_{\gamma_{u,w}}|f'(z)|^{2}|dz|.
\end{aligned}$$

Now, observe that

$$\begin{aligned}
\Delta_{\gamma,u,2} &= \frac{1}{\ell^{2}(\gamma)}\int_{\gamma}\left(\int_{\gamma_{u,v}}|z-u|^{2}|dz|\right)|dv| \\
&= \frac{1}{\ell^{2}(\gamma)}\int_{\gamma}\left(\int_{\gamma_{u,v}}\left(|z|^{2}-2\operatorname{Re}(z\overline{u})+|u|^{2}\right)|dz|\right)|dv| \\
&= \frac{1}{\ell^{2}(\gamma)}\int_{\gamma}\left(\int_{\gamma_{u,v}}|z|^{2}|dz|-2\operatorname{Re}\left[\overline{u}\left(\int_{\gamma_{u,v}}z\,|dz|\right)\right]+|u|^{2}\ell(\gamma)\right)|dv| \\
&= \frac{1}{\ell^{2}(\gamma)}\left[\int_{\gamma}\left(\int_{\gamma_{u,v}}|z|^{2}|dz|\right)|dv|-2\operatorname{Re}\left[\overline{u}\left(\int_{\gamma}\left(\int_{\gamma_{u,v}}z\,|dz|\right)|dv|\right)\right]+|u|^{2}\ell^{2}(\gamma)\right] \\
&= \frac{1}{\ell^{2}(\gamma)}\int_{\gamma}\left(\int_{\gamma_{u,v}}|z|^{2}|dz|\right)|dv|-\frac{2}{\ell^{2}(\gamma)}\operatorname{Re}\left[\overline{u}\left(\int_{\gamma}\left(\int_{\gamma_{u,v}}z\,|dz|\right)|dv|\right)\right]+|u|^{2}
\end{aligned}$$

and, similarly

$$\begin{aligned}
\Delta_{\gamma,w,2} &:= \frac{1}{\ell^{2}(\gamma)}\int_{\gamma}\left(\int_{\gamma_{v,w}}|z-w|^{2}|dz|\right)|dv| \\
&= \frac{1}{\ell^{2}(\gamma)}\int_{\gamma}\left(\int_{\gamma_{v,w}}|z|^{2}|dz|\right)|dv|-\frac{2}{\ell^{2}(\gamma)}\operatorname{Re}\left[\overline{w}\left(\int_{\gamma}\left(\int_{\gamma_{v,w}}z\,|dz|\right)|dv|\right)\right]+|w|^{2},
\end{aligned}$$

which proves the statement. □

Remark 1. *Similar inequalities may be obtained by taking the square in (2.5) and (2.6) and performing similar calculations. However, the details are not presented here.*

We can also consider the quantity

$$\mathcal{S}_{\gamma,1}(f) := \frac{1}{\ell(\gamma)}\int_{\gamma}\left|f(z)-\frac{1}{w-u}\int_{\gamma}f(w)\,dw\right||dz|. \tag{2.14}$$

By the use of Cauchy-Bunyakovsky-Schwarz integral inequality, we obviously have

$$\mathcal{S}_{\gamma,1}(f) \leq \left[\mathcal{S}_{\gamma,2}(f)\right]^{1/2} \tag{2.15}$$

where the path γ is piecewise smooth and $f:\gamma\to\mathbb{C}$ is continuous on γ.

Proposition 3. *Let f be holomorphic in G, an open domain and suppose $\gamma\subset G$ is a smooth path from $z(a)=u$ to $z(b)=w$. Then*

$$\mathcal{S}_{\gamma,1}(f) \leq \epsilon(\gamma)\left(\Delta_{\gamma,u,1}+\Delta_{\gamma,w,1}\right)\|f'\|_{\gamma_{u,w};\infty} \tag{2.16}$$

where

$$\Delta_{\gamma,u,1} := \frac{1}{\ell^2(\gamma)}\int_\gamma\left(\int_{\gamma_{u,v}}|z-u|\,|dz|\right)|dv| \tag{2.17}$$

and

$$\Delta_{\gamma,w,1} := \frac{1}{\ell^2(\gamma)}\int_\gamma\left(\int_{\gamma_{v,w}}|z-w|\,|dz|\right)|dv|. \tag{2.18}$$

Proof. From (2.5) we have

$$\begin{aligned}
&\left|f(v)-\frac{1}{w-u}\int_\gamma f(z)\,dz\right| \\
&\quad\leq\left[\frac{1}{|w-u|}\int_{\gamma_{u,v}}|z-u|\,|dz|+\frac{1}{|w-u|}\int_{\gamma_{v,w}}|z-w|\,|dz|\right]\|f'\|_{\gamma_{u,w};\infty},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{1}{\ell(\gamma)}\int_\gamma\left|f(v)-\frac{1}{w-u}\int_\gamma f(z)\,dz\right||dv| \\
&\quad\leq\left[\frac{1}{|w-u|}\frac{1}{\ell(\gamma)}\int_\gamma\left(\int_{\gamma_{u,v}}|z-u|\,|dz|\right)|dv|\right. \\
&\qquad\left.+\frac{1}{|w-u|}\frac{1}{\ell(\gamma)}\int_\gamma\left(\int_{\gamma_{v,w}}|z-w|\,|dz|\right)|dv|\right]\|f'\|_{\gamma_{u,w};\infty} \\
&\quad=\frac{\ell(\gamma)}{|w-u|}\left[\frac{1}{\ell^2(\gamma)}\int_\gamma\left(\int_{\gamma_{u,v}}|z-u|\,|dz|\right)|dv|\right. \\
&\qquad\left.+\frac{1}{\ell^2(\gamma)}\int_\gamma\left(\int_{\gamma_{v,w}}|z-w|\,|dz|\right)|dv|\right]\|f'\|_{\gamma_{u,w};\infty} \\
&\qquad\qquad=\epsilon(\gamma)\left(\Delta_{\gamma,u,1}+\Delta_{\gamma,w,1}\right)\|f'\|_{\gamma_{u,w};\infty}
\end{aligned}$$

and the inequality (2.16) is proved. □

3. Some Grüss Type Inequalities

Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t\in\gamma$ from $z(a)=u$ to $z(b)=w$. Now, for ϕ, $\Phi\in\mathbb{C}$, define the sets of complex-valued functions

$$\bar{U}_\gamma(\phi,\Phi) := \left\{f:\gamma\to\mathbb{C}\,|\,\operatorname{Re}\left[(\Phi-f(z))\left(\overline{f(z)}-\overline{\phi}\right)\right]\geq 0\ \text{ for each }\ z\in\gamma\right\}$$

and

$$\bar{\Delta}_{\gamma}(\phi,\Phi) := \left\{ f : \gamma \to \mathbb{C} \,|\, \left| f(z) - \frac{\phi+\Phi}{2} \right| \le \frac{1}{2} |\Phi - \phi| \text{ for each } z \in \gamma \right\}.$$

The following representation result may be stated.

Proposition 4. *For any ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that $\bar{U}_{\gamma}(\phi,\Phi)$ and $\bar{\Delta}_{\gamma}(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_{\gamma}(\phi,\Phi) = \bar{\Delta}_{\gamma}(\phi,\Phi). \tag{3.1}$$

Proof. We observe that for any $w \in \mathbb{C}$ we have the equivalence

$$\left| w - \frac{\phi+\Phi}{2} \right| \le \frac{1}{2} |\Phi - \phi|$$

if and only if

$$\operatorname{Re}\left[(\Phi - w)\left(\overline{w} - \overline{\phi} \right) \right] \ge 0.$$

This follows by the equality

$$\frac{1}{4} |\Phi - \phi|^2 - \left| w - \frac{\phi+\Phi}{2} \right|^2 = \operatorname{Re}\left[(\Phi - w)\left(\overline{w} - \overline{\phi} \right) \right]$$

that holds for any $w \in \mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 1. *For any ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that*

$$\begin{aligned} \bar{U}_{\gamma}(\phi,\Phi) = \{ f : \gamma \to \mathbb{C} \mid & (\operatorname{Re}\Phi - \operatorname{Re} f(z))(\operatorname{Re} f(z) - \operatorname{Re}\phi) \\ & + (\operatorname{Im}\Phi - \operatorname{Im} f(z))(\operatorname{Im} f(z) - \operatorname{Im}\phi) \ge 0 \textit{ for each } z \in \gamma \}. \end{aligned} \tag{3.2}$$

Now, if we assume that $\operatorname{Re}(\Phi) \ge \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \ge \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{\gamma}(\phi,\Phi) := \{ f : \gamma \to \mathbb{C} \mid & \operatorname{Re}(\Phi) \ge \operatorname{Re} f(z) \ge \operatorname{Re}(\phi) \\ & \text{and } \operatorname{Im}(\Phi) \ge \operatorname{Im} f(z) \ge \operatorname{Im}(\phi) \text{ for each } z \in \gamma \}. \end{aligned} \tag{3.3}$$

One can easily observe that $\bar{S}_{\gamma}(\phi,\Phi)$ is closed, convex and

$$\emptyset \neq \bar{S}_{\gamma}(\phi,\Phi) \subseteq \bar{U}_{\gamma}(\phi,\Phi). \tag{3.4}$$

We have the following result:

Theorem 1. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ, then*

$$|\mathcal{D}_{\gamma}(f,g)| \le \epsilon(\gamma) \max_{z\in\gamma} |g(z) - \lambda| \, \mathcal{S}_{\gamma,1}(f) \le \epsilon(\gamma) \max_{z\in\gamma} |g(z) - \lambda| \left[\mathcal{S}_{\gamma,2}(f) \right]^{1/2} \tag{3.5}$$

for any $\lambda \in \mathbb{C}$.

Proof. For any $\lambda \in \mathbb{C}$ we have

$$\begin{aligned}
&\frac{1}{w-u}\int_{\gamma}\left(f(z)-\frac{1}{w-u}\int_{\gamma}f(w)\,dw\right)(g(z)-\lambda)\,dz\\
&=\frac{1}{w-u}\int_{\gamma}f(z)\,g(z)\,dz-\frac{1}{w-u}\int_{\gamma}f(w)\,dw\frac{1}{w-u}\int_{\gamma}g(z)\,dz\\
&-\lambda\frac{1}{w-u}\int_{\gamma}\left(f(z)-\frac{1}{w-u}\int_{\gamma}f(w)\,dw\right)dz\\
&=\frac{1}{w-u}\int_{\gamma}f(z)\,g(z)\,dz-\frac{1}{w-u}\int_{\gamma}f(w)\,dw\frac{1}{w-u}\int_{\gamma}g(z)\,dz,
\end{aligned}$$

therefore, we have the following identity of interest, which is a generalization of Sonin's identity for functions of real variables

$$\mathcal{D}_{\gamma}(f,g)=\frac{1}{w-u}\int_{\gamma}\left(f(z)-\frac{1}{w-u}\int_{\gamma}f(w)\,dw\right)(g(z)-\lambda)\,dz \tag{3.6}$$

for any $\lambda \in \mathbb{C}$.

Taking the modulus in (3.6), we get

$$\begin{aligned}
|\mathcal{D}_{\gamma}(f,g)| &= \left|\frac{1}{w-u}\int_{\gamma}\left(f(z)-\frac{1}{w-u}\int_{\gamma}f(w)\,dw\right)(g(z)-\lambda)\,dz\right|\\
&\le \frac{1}{|w-u|}\int_{\gamma}\left|f(z)-\frac{1}{w-u}\int_{\gamma}f(w)\,dw\right||g(z)-\lambda|\,|dz|\\
&\le \frac{\ell(\gamma)}{|w-u|}\max_{z\in\gamma}|g(z)-\lambda|\frac{1}{\ell(\gamma)}\int_{\gamma}\left|f(z)-\frac{1}{w-u}\int_{\gamma}f(w)\,dw\right||dz|\\
&= \epsilon(\gamma)\max_{z\in\gamma}|g(z)-\lambda|\,\mathcal{S}_{\gamma,1}(f)\le \epsilon(\gamma)\max_{z\in\gamma}|g(z)-\lambda|\,[\mathcal{S}_{\gamma,2}(f)]^{1/2}
\end{aligned} \tag{3.7}$$

for any $\lambda \in \mathbb{C}$. □

Remark 2. *We observe that, by using the upper bounds for* $\mathcal{S}_{\gamma,2}(f)$ *we can get further upper bounds such as*

$$\begin{aligned}
|\mathcal{D}_{\gamma}(f,g)| &\le \epsilon(\gamma)\max_{z\in\gamma}|g(z)-\lambda|\,\mathcal{S}_{\gamma,1}(f)\\
&\le \epsilon(\gamma)\max_{z\in\gamma}|g(z)-\lambda|\,[\mathcal{S}_{\gamma,2}(f)]^{1/2}\le\sqrt{2}\epsilon^{2}(\gamma)\max_{z\in\gamma}|g(z)-\lambda|\left[\mathcal{P}_{\gamma}\left(f,\overline{f}\right)\right]^{1/2}
\end{aligned} \tag{3.8}$$

for any $\lambda \in \mathbb{C}$.

By using (2.10) we get

$$\begin{aligned}
|\mathcal{D}_{\gamma}(f,g)| &\le \epsilon(\gamma)\max_{z\in\gamma}|g(z)-\lambda|\,\mathcal{S}_{\gamma,1}(f)\\
&\le \epsilon(\gamma)\max_{z\in\gamma}|g(z)-\lambda|\,[\mathcal{S}_{\gamma,2}(f)]^{1/2}\le\\
&\le \max_{z\in\gamma}|g(z)-\lambda|\,\epsilon^{2}(\gamma)\left(\Delta_{\gamma,u,2}+\Delta_{\gamma,w,2}\right)^{1/2}\left(\frac{1}{\ell(\gamma)}\int_{\gamma_{u,w}}|f'(z)|^{2}\,|dz|\right)^{1/2},
\end{aligned} \tag{3.9}$$

while by (2.16) we get

$$
\begin{aligned}
\text{(3.10)} \quad |\mathcal{D}_\gamma(f,g)| &\le \epsilon(\gamma)\max_{z\in\gamma}|g(z)-\lambda|\,\mathcal{S}_{\gamma,1}(f)\\
&\le \epsilon^2(\gamma)\max_{z\in\gamma}|g(z)-\lambda|\left(\Delta_{\gamma,u,1}+\Delta_{\gamma,w,1}\right)\|f'\|_{\gamma_{u,w};\infty},
\end{aligned}
$$

provided f is holomorphic in G, an open domain and $\gamma\subset G$ is a smooth path from $z(a)=u$ to $z(b)=w$.

Corollary 2. *With the assumptions of Theorem 1 and, in addition, if $g\in\bar{\Delta}_\gamma(\phi,\Phi)$ for some ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$, then*

$$
\text{(3.11)} \quad |\mathcal{D}_\gamma(f,g)| \le \frac{1}{2}|\Phi-\phi|\,\epsilon(\gamma)\,\mathcal{S}_{\gamma,1}(f) \le \frac{1}{2}|\Phi-\phi|\,\epsilon(\gamma)\left[\mathcal{S}_{\gamma,2}(f)\right]^{1/2}.
$$

Remark 3. *For $g=f$ we have*

$$
\text{(3.12)} \quad \mathcal{D}_\gamma(f,f) = \frac{1}{w-u}\int_\gamma f^2(z)\,dz - \left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)^2
$$

and by (3.8) we get

$$
\text{(3.13)} \quad |\mathcal{D}_\gamma(f,f)| \le \frac{1}{2}|\Phi-\phi|\,\epsilon(\gamma)\,\mathcal{S}_{\gamma,1}(f) \le \frac{1}{2}|\Phi-\phi|\,\epsilon(\gamma)\left[\mathcal{S}_{\gamma,2}(f)\right]^{1/2},
$$

provided $f\in\bar{\Delta}_\gamma(\phi,\Phi)$ for some ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$.

For $g=\bar{f}$ we have

$$
\text{(3.14)} \quad \mathcal{D}_\gamma\left(f,\bar{f}\right) = \frac{1}{w-u}\int_\gamma |f(z)|^2\,dz - \frac{1}{w-u}\int_\gamma f(z)\,dz\,\frac{1}{w-u}\int_\gamma \overline{f(z)}dz
$$

and by (3.8) we get

$$
\text{(3.15)} \quad \left|\mathcal{D}_\gamma\left(f,\bar{f}\right)\right| \le \frac{1}{2}|\Phi-\phi|\,\epsilon(\gamma)\,\mathcal{S}_{\gamma,1}(f) \le \frac{1}{2}|\Phi-\phi|\,\epsilon(\gamma)\left[\mathcal{S}_{\gamma,2}(f)\right]^{1/2},
$$

provided $f\in\bar{\Delta}_\gamma(\phi,\Phi)$.

Corollary 3. *With the assumptions of Theorem 1 and, in addition, if f is holomorphic in G, an open domain and $\gamma\subset G$ is a smooth path from $z(a)=u$ to $z(b)=w$ and $g\in\bar{\Delta}_\gamma(\phi,\Phi)$ for some ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$, then*

$$
\begin{aligned}
\text{(3.16)} \quad |\mathcal{D}_\gamma(f,g)| &\le \frac{1}{2}|\Phi-\phi|\,\epsilon(\gamma)\,\mathcal{S}_{\gamma,1}(f)\\
&\le \frac{1}{2}|\Phi-\phi|\,\epsilon^2(\gamma)\left(\Delta_{\gamma,u,1}+\Delta_{\gamma,w,1}\right)\|f'\|_{\gamma_{u,w};\infty}
\end{aligned}
$$

and

$$
\begin{aligned}
\text{(3.17)} \quad |\mathcal{D}_\gamma(f,g)| &\le \frac{1}{2}|\Phi-\phi|\,\epsilon(\gamma)\left[\mathcal{S}_{\gamma,2}(f)\right]^{1/2}\\
&\le \frac{1}{2}|\Phi-\phi|\,\epsilon^2(\gamma)\left(\Delta_{\gamma,u,2}+\Delta_{\gamma,w,2}\right)^{1/2}\left(\frac{1}{\ell(\gamma)}\int_{\gamma_{u,w}}|f'(z)|^2\,|dz|\right)^{1/2}.
\end{aligned}
$$

4. Examples for Circular Paths

Let $[a,b] \subseteq [0,2\pi]$ and the circular path $\gamma_{[a,b],R}$ centered in 0 and with radius $R>0$

$$z(t) = R\exp(it) = R(\cos t + i\sin t), \ t\in [a,b].$$

If $[a,b]=[0,\pi]$ then we get a half circle while for $[a,b]=[0,2\pi]$ we get the full circle.

Since

$$\begin{aligned}\left|e^{is}-e^{it}\right|^2 &= \left|e^{is}\right|^2 - 2\operatorname{Re}\left(e^{i(s-t)}\right) + \left|e^{it}\right|^2 \\ &= 2-2\cos(s-t) = 4\sin^2\left(\frac{s-t}{2}\right)\end{aligned}$$

for any $t,\ s\in\mathbb{R}$, then

$$\left|e^{is}-e^{it}\right|^r = 2^r\left|\sin\left(\frac{s-t}{2}\right)\right|^r \tag{4.1}$$

for any $t,\ s\in\mathbb{R}$ and $r>0$. In particular,

$$\left|e^{is}-e^{it}\right| = 2\left|\sin\left(\frac{s-t}{2}\right)\right|$$

for any $t,\ s\in\mathbb{R}$.

If $u=R\exp(ia)$ and $w=R\exp(ib)$ then

$$\begin{aligned}w-u &= R[\exp(ib)-\exp(ia)] = R[\cos b + i\sin b - \cos a - i\sin a] \\ &= R[\cos b - \cos a + i(\sin b - \sin a)].\end{aligned}$$

Since

$$\cos b - \cos a = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

and

$$\sin b - \sin a = 2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right),$$

hence

$$\begin{aligned}w-u &= R\left[-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right) + 2i\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right)\right] \\ &= 2R\sin\left(\frac{b-a}{2}\right)\left[-\sin\left(\frac{a+b}{2}\right) + i\cos\left(\frac{a+b}{2}\right)\right] \\ &= 2Ri\sin\left(\frac{b-a}{2}\right)\left[\cos\left(\frac{a+b}{2}\right) + i\sin\left(\frac{a+b}{2}\right)\right] \\ &= 2Ri\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right].\end{aligned}$$

If $\gamma = \gamma_{[a,b],R}$, then the *circular complex Čebyšev functional* is defined by

$$
\begin{aligned}
(4.2)\quad \mathcal{C}_{[a,b],R}(f,g) &:= \mathcal{D}_{\gamma_{[a,b],R}}(f,g) \\
&= \frac{1}{2\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right]}\int_a^b f(R\exp(it))\, g(R\exp(it))\exp(it)\,dt \\
&\quad - \frac{1}{4\sin^2\left(\frac{b-a}{2}\right)\exp\left[2\left(\frac{a+b}{2}\right)i\right]} \\
&\quad \times \int_a^b f(R\exp(it))\exp(it)\,dt \int_a^b g(R\exp(it))\exp(it)\,dt.
\end{aligned}
$$

For $\gamma = \gamma_{[a,b],R}$ and $v = Re^{it}$ with $t \in [a,b]$ we have

$$
\begin{aligned}
(4.3)\quad \Delta_{\gamma,u,1} &:= \frac{1}{\ell^2\left(\gamma_{[a,b],R}\right)}\int_{\gamma_{[a,b],R}}\left(\int_{\gamma_{[a,t],R}}|z-u|\,|dz|\right)|dv| \\
&= \frac{R^3}{R^2(b-a)^2}\int_a^b\left(\int_a^t\left|e^{is}-e^{ia}\right|ds\right)dt \\
&= \frac{2R^3}{R^2(b-a)^2}\int_a^b\left(\int_a^t\left|\sin\left(\frac{s-a}{2}\right)\right|ds\right)dt \\
&= \frac{2R}{(b-a)^2}\int_a^b\left(\int_a^t\sin\left(\frac{s-a}{2}\right)ds\right)dt \\
&= \frac{4R}{(b-a)^2}\int_a^b\left[1-\cos\left(\frac{t-a}{2}\right)\right]dt = \frac{4R}{(b-a)^2}\left[b-a-2\sin\left(\frac{b-a}{2}\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(4.4)\quad \Delta_{\gamma,w,1} &:= \frac{1}{\ell^2\left(\gamma_{[a,b],R}\right)}\int_{\gamma_{[a,b],R}}\left(\int_{\gamma_{[t,b],R}}|z-w|\,|dz|\right)|dv| \\
&= \frac{R^3}{R^2(b-a)^2}\int_a^b\left(\int_t^b\left|e^{ib}-e^{is}\right|ds\right)dt \\
&= \frac{2R^3}{R^2(b-a)^2}\int_a^b\left(\int_t^b\left|\sin\left(\frac{b-s}{2}\right)\right|ds\right)dt \\
&= \frac{2R}{(b-a)^2}\int_a^b\left(\int_t^b\sin\left(\frac{b-s}{2}\right)ds\right)dt \\
&= \frac{4R}{(b-a)^2}\int_a^b\left(1-\cos\left(\frac{b-t}{2}\right)\right)dt = \frac{4R}{(b-a)^2}\left[b-a-2\sin\left(\frac{b-a}{2}\right)\right].
\end{aligned}
$$

We have the following result:

Proposition 5. *Let $\gamma_{[a,b],R}$ be a circular path centered in 0 and with radius $R>0$ and $[a,b]\subset[0,2\pi]$. If f is holomorphic in G, an open domain and $\gamma_{[a,b],R}\subset G$ while g is continuous on $\gamma_{[a,b],R}$ and $g\in\bar{\Delta}_{\gamma_{[a,b],R}}(\phi,\Phi)$ for some ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$,* *then*

$$\left|\mathcal{C}_{[a,b],R}\left(f,g\right)\right| \leq \left|\Phi-\phi\right| \frac{2R}{\sin^{2}\left(\frac{b-a}{2}\right)} \left[\frac{b-a}{2}-\sin\left(\frac{b-a}{2}\right)\right] \left\|f'\right\|_{\gamma_{u,w};\infty}. \tag{4.5}$$

The proof follows by the inequality (3.16).

References

[1] M. W. Alomari, A companion of Grüss type inequality for Riemann-Stieltjes integral and applications. *Mat. Vesnik* **66** (2014), no. 2, 202–212.

[2] D. Andrica and C. Badea, Grüss' inequality for positive linear functionals. *Period. Math. Hungar.* **19** (1988), no. 2, 155–167.

[3] D. Baleanu, S. D. Purohit and F. Uçar, On Grüss type integral inequality involving the Saigo's fractional integral operators. *J. Comput. Anal. Appl.* **19** (2015), no. 3, 480–489

[4] P. L. Chebyshev, Sur les expressions approximatives des intègrals dèfinis par les outres prises entre les même limites, *Proc. Math. Soc. Charkov,* **2** (1882), 93-98.

[5] P. Cerone, On a Čebyšev-type functional and Grüss-like bounds. *Math. Inequal. Appl.* **9** (2006), no. 1, 87–102.

[6] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications, *Tamkang J. Math.*, **38**(1) (2007), 37-49. Preprint *RGMIA Res. Rep. Coll.,* **5**(2) (2002), Article 14. [ONLINE: `http://rgmia.vu.edu.au/v5n2.html`].

[7] P. Cerone and S. S. Dragomir, Some new Ostrowski-type bounds for the Čebyšev functional and applications. *J. Math. Inequal.* **8** (2014), no. 1, 159–170.

[8] P. Cerone, S. S. Dragomir and J. Roumeliotis, Grüss inequality in terms of Δ-seminorms and applications. *Integral Transforms Spec. Funct.* **14** (2003), no. 3, 205–216.

[9] X. L. Cheng and J. Sun, A note on the perturbed trapezoid inequality, *J. Ineq. Pure and Appl. Math.,* **3**(2) Art. 29, (2002).

[10] S. S. Dragomir, A generalization of Grüss's inequality in inner product spaces and applications. *J. Math. Anal. Appl.* **237** (1999), no. 1, 74–82.

[11] S. S. Dragomir, A Grüss' type integral inequality for mappings of r-Hölder's type and applications for trapezoid formula. *Tamkang J. Math.* **31** (2000), no. 1, 43–47.

[12] S. S. Dragomir, Some integral inequalities of Grüss type. *Indian J. Pure Appl. Math.* **31** (2000), no. 4, 397–415.

[13] S. S. Dragomir, Integral Grüss inequality for mappings with values in Hilbert spaces and applications. *J. Korean Math. Soc.* **38** (2001), no. 6, 1261–1273.

[14] S. S. Dragomir and I. A. Fedotov, An inequality of Grüss' type for Riemann-Stieltjes integral and applications for special means. *Tamkang J. Math.* **29** (1998), no. 4, 287–292.

[15] S. S. Dragomir and I. Gomm, Some integral and discrete versions of the Grüss inequality for real and complex functions and sequences. *Tamsui Oxf. J. Math. Sci.* **19** (2003), no. 1, 67–77.

[16] A. M. Fink, A treatise on Grüss' inequality. *Analytic and Geometric Inequalities and Applications*, 93–113, Math. Appl., 478, Kluwer Acad. Publ., Dordrecht, 1999.

[17] G. Grüss, Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$, *Math. Z.* , **39**(1935), 215-226.

[18] D. Jankov Maširević and T. K. Pogány, Bounds on Čebyšev functional for $C_\varphi[0,1]$ function class. *J. Anal.* **22** (2014), 107–117.

[19] Z. Liu, Refinement of an inequality of Grüss type for Riemann-Stieltjes integral. *Soochow J. Math.* **30** (2004), no. 4, 483–489.

[20] Z. Liu, Notes on a Grüss type inequality and its application. *Vietnam J. Math.* **35** (2007), no. 2, 121–127.

[21] A. Lupaş, The best constant in an integral inequality, *Mathematica* (Cluj, Romania), **15**(**38**)(2) (1973), 219-222.

[22] A. Mc.D. Mercer and P. R. Mercer, New proofs of the Grüss inequality. *Aust. J. Math. Anal. Appl.* **1** (2004), no. 2, Art. 12, 6 pp.

[23] N. Minculete and L. Ciurdariu, A generalized form of Grüss type inequality and other integral inequalities. *J. Inequal. Appl.* **2014**, 2014:119, 18 pp.

[24] A. M. Ostrowski, On an integral inequality, *Aequat. Math.,* **4** (1970), 358-373.

[25] B. G. Pachpatte, A note on some inequalities analogous to Grüss inequality. *Octogon Math. Mag.* **5** (1997), no. 2, 62–66
[26] J. Pečarić and Š. Ungar, On a inequality of Grüss type. *Math. Commun.* **11** (2006), no. 2, 137–141.
[27] M. Z. Sarikaya and H. Budak, An inequality of Grüss like via variant of Pompeiu's mean value theorem. *Konuralp J. Math.* **3** (2015), no. 1, 29–35.
[28] N. Ujević, A generalization of the pre-Grüss inequality and applications to some quadrature formulae. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 1, Article 13, 9 pp.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa