A REFINEMENT OF GRÜSS INEQUALITY FOR THE COMPLEX INTEGRAL

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Abstract. Assume that f and g are continuous on γ, $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$ and the *complex Čebyšev functional* is defined by

$$\mathcal{D}_\gamma(f,g) := \frac{1}{w-u}\int_\gamma f(z)g(z)\,dz - \frac{1}{w-u}\int_\gamma f(z)\,dz \frac{1}{w-u}\int_\gamma g(z)\,dz.$$

In this paper we establish some Grüss type inequalities for $\mathcal{D}_\gamma(f,g)$ under some complex boundedness conditiond for the functions f and g.

1. Introduction

For two Lebesgue integrable functions $f,\ g : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \frac{1}{b-a}\int_a^b g(t)\,dt.$$

In 1934, G. Grüss [17] showed that

$$|C(f,g)| \leq \frac{1}{4}(M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on} \quad [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.6) in the sense that it cannot be replaced by a smaller one.

In [6], P. Cerone and S. S. Dragomir proved the following inequalities:

$$|C(f,g)| \leq \begin{cases} \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt, \\ \\ \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_q \frac{1}{b-a}\left(\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right|^p dt \right)^{\frac{1}{p}}, \\ \qquad\qquad \text{where } p > 1,\ 1/p + 1/q = 1. \end{cases} \tag{1.3}$$

For $\gamma = 0$, we get from the first inequality in (1.3)

$$|C(f,g)| \leq \|g\|_\infty \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt \tag{1.4}$$

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for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq g \leq M$ for a.e. $x \in [a,b]$, then $\left\| g - \frac{m+M}{2} \right\|_{\infty} \leq \frac{1}{2}(M-m)$ and by the first inequality in (1.3) we can deduce the following result obtained by Cheng and Sun [9]

$$|C(f,g)| \leq \frac{1}{2}(M-m)\frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt. \tag{1.5}$$

The constant $\frac{1}{2}$ is best in (1.5) as shown by Cerone and Dragomir in [7].

For other inequality of Grüss' type see [1]-[5], [7]-[16], [18]-[23] and [25]-[28].

In order to extend Grüss' inequality to complex integral we need the following preparations.

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, and open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.6}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left| \int_{\gamma} f(z)\,dz \right| \leq \int_{\gamma} |f(z)|\,|dz| \leq \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.7}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \geq 1$ by

$$\|f\|_{\gamma,p} := \left(\int_{\gamma} |f(z)|^p\,|dz| \right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)| |dz| .$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q} \|f\|_{\gamma,p} .$$

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $f : \gamma \to \mathbb{C}$ a continuous function on γ. Define the quantity:

$$\begin{aligned} \mathcal{P}_\gamma\left(f, \overline{f}\right) &= \frac{1}{\ell(\gamma)} \int_\gamma |f(z)|^2 |dz| - \left| \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right|^2 \\ &= \frac{1}{\ell(\gamma)} \int_\gamma \left| f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right|^2 |dv| \ge 0. \end{aligned} \tag{1.8}$$

If f and g are continuous on γ, we consider the *complex Čebyšev functional* defined by

$$\mathcal{D}_\gamma(f,g) := \frac{1}{w-u} \int_\gamma f(z) g(z) dz - \frac{1}{w-u} \int_\gamma f(z) dz \frac{1}{w-u} \int_\gamma g(z) dz.$$

In this paper we establish some Grüss type inequalities for $\mathcal{D}_\gamma(f,g)$ under some complex boundedness conditions for the functions f and g.

2. Some Preliminary Results

We have the following equalities:

Lemma 1. *Assume that the path γ is piecewise smooth and $f : \gamma \to \mathbb{C}$ is continuous on γ. Then for ϕ, $\Phi \in \mathbb{C}$ with $\phi \ne \Phi$ we have*

$$\begin{aligned} \mathcal{P}_\gamma\left(f, \overline{f}\right) = \operatorname{Re}&\left[\left(\Phi - \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right) \left(\frac{1}{\ell(\gamma)} \int_\gamma \overline{f(z)} |dz| - \overline{\phi} \right) \right] \\ &- \frac{1}{\ell(\gamma)} \int_\gamma \operatorname{Re}\left[(\Phi - f(z)) \left(\overline{f(z)} - \overline{\phi} \right) \right] |dz| \end{aligned} \tag{2.1}$$

and, equivalently,

$$\begin{aligned} \mathcal{P}_\gamma\left(f, \overline{f}\right) = \frac{1}{4} |\Phi - \phi|^2 &- \left| \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| - \frac{\phi + \Phi}{2} \right|^2 \\ &- \frac{1}{\ell(\gamma)} \int_\gamma \operatorname{Re}\left[(\Phi - f(z)) \left(\overline{f(z)} - \overline{\phi} \right) \right] |dz| . \end{aligned} \tag{2.2}$$

Proof. We have

$$
\begin{aligned}
I_1 &:= \operatorname{Re}\left[\left(\Phi-\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right)\left(\frac{1}{\ell(\gamma)}\int_\gamma \overline{f(z)}\,|dz|-\overline{\phi}\right)\right] \\
&= \operatorname{Re}\left[\frac{\Phi}{\ell(\gamma)}\int_\gamma \overline{f(z)}\,|dz|+\frac{\overline{\phi}}{\ell(\gamma)}\int_\gamma f(z)\,|dz|-\Phi\overline{\phi}-\left|\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|^2\right] \\
&= \operatorname{Re}\left(\frac{\Phi}{\ell(\gamma)}\int_\gamma \overline{f(z)}\,|dz|\right)+\operatorname{Re}\left(\frac{\overline{\phi}}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right) \\
&\qquad -\operatorname{Re}\left(\Phi\overline{\phi}\right)-\left|\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|^2
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &:= \frac{1}{\ell(\gamma)}\int_\gamma \operatorname{Re}\left[(\Phi-f(z))\left(\overline{f(z)}-\overline{\phi}\right)\right]|dz| \\
&= \frac{1}{\ell(\gamma)}\int_\gamma \operatorname{Re}\left[\Phi\overline{f(z)}+\overline{\phi}f(z)-\Phi\overline{\phi}-|f(z)|^2\right]|dz| \\
&= \frac{1}{\ell(\gamma)}\int_\gamma \left[\operatorname{Re}\left(\Phi\overline{f(z)}\right)+\operatorname{Re}\left(\overline{\phi}f(z)\right)-\operatorname{Re}\left(\Phi\overline{\phi}\right)-|f(z)|^2\right]|dz| \\
&= \operatorname{Re}\left(\frac{\Phi}{\ell(\gamma)}\int_\gamma \overline{f(z)}\,|dz|\right)+\operatorname{Re}\left(\frac{\overline{\phi}}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right) \\
&\qquad -\operatorname{Re}\left(\Phi\overline{\phi}\right)-\frac{1}{\ell(\gamma)}\int_\gamma |f(z)|^2\,|dz|.
\end{aligned}
$$

Therefore

$$
I_1-I_2=\frac{1}{\ell(\gamma)}\int_\gamma |f(z)|^2\,|dz|-\left|\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|^2
$$

proving the identity (2.1).

We have the equality for complex numbers

$$
\operatorname{Re}\left[(\Phi-w)\left(\overline{w}-\overline{\phi}\right)\right]=\frac{1}{4}|\Phi-\phi|^2-\left|w-\frac{\phi+\Phi}{2}\right|^2
$$

for $w\in\mathbb{C}$, then by taking

$$
w=\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|
$$

in this equality, we get

$$
\begin{aligned}
&\operatorname{Re}\left[\left(\Phi-\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right)\left(\frac{1}{\ell(\gamma)}\int_\gamma \overline{f(z)}\,|dz|-\overline{\phi}\right)\right] \\
&\qquad =\frac{1}{4}|\Phi-\phi|^2-\left|\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|-\frac{\phi+\Phi}{2}\right|^2
\end{aligned}
$$

and by (2.1) we obtain the desired result (2.2). □

Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t\in\gamma$ from $z(a)=u$ to $z(b)=w$. Now, for ϕ, $\Phi\in\mathbb{C}$, define the sets of complex-valued functions

$$\bar{U}_{\gamma}(\phi,\Phi) := \left\{ f:\gamma\to\mathbb{C} | \operatorname{Re}\left[(\Phi - f(z)) \left(\overline{f(z)} - \overline{\phi} \right) \right] \geq 0 \text{ for each } z\in\gamma \right\}$$

and

$$\bar{\Delta}_{\gamma}(\phi,\Phi) := \left\{ f:\gamma\to\mathbb{C} | \left| f(z) - \frac{\phi+\Phi}{2} \right| \leq \frac{1}{2}|\Phi-\phi| \text{ for each } z\in\gamma \right\}.$$

The following representation result may be stated.

Proposition 1. *For any ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that $\bar{U}_{\gamma}(\phi,\Phi)$ and $\bar{\Delta}_{\gamma}(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_{\gamma}(\phi,\Phi) = \bar{\Delta}_{\gamma}(\phi,\Phi). \tag{2.3}$$

Proof. We observe that for any $w\in\mathbb{C}$ we have the equivalence

$$\left| w - \frac{\phi+\Phi}{2} \right| \leq \frac{1}{2}|\Phi-\phi|$$

if and only if

$$\operatorname{Re}\left[(\Phi - w)\left(\overline{w} - \overline{\phi} \right) \right] \geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Phi-\phi|^2 - \left| w - \frac{\phi+\Phi}{2} \right|^2 = \operatorname{Re}\left[(\Phi - w)\left(\overline{w} - \overline{\phi} \right) \right]$$

that holds for any $w\in\mathbb{C}$.

The equality (2.3) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 1. *For any ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that*

$$\begin{aligned} \bar{U}_{\gamma}(\phi,\Phi) = \{ f:\gamma\to\mathbb{C} \mid & (\operatorname{Re}\Phi - \operatorname{Re} f(z))(\operatorname{Re} f(z) - \operatorname{Re}\phi) \\ & + (\operatorname{Im}\Phi - \operatorname{Im} f(z))(\operatorname{Im} f(z) - \operatorname{Im}\phi) \geq 0 \textit{ for each } z\in\gamma \}. \end{aligned} \tag{2.4}$$

Now, if we assume that $\operatorname{Re}(\Phi) \geq \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \geq \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{\gamma}(\phi,\Phi) := \{ f:\gamma\to\mathbb{C} \mid & \operatorname{Re}(\Phi) \geq \operatorname{Re} f(z) \geq \operatorname{Re}(\phi) \\ & \text{and } \operatorname{Im}(\Phi) \geq \operatorname{Im} f(z) \geq \operatorname{Im}(\phi) \text{ for each } z\in\gamma \}. \end{aligned} \tag{2.5}$$

One can easily observe that $\bar{S}_{\gamma}(\phi,\Phi)$ is closed, convex and

$$\emptyset \neq \bar{S}_{\gamma}(\phi,\Phi) \subseteq \bar{U}_{\gamma}(\phi,\Phi). \tag{2.6}$$

Theorem 1. *Assume that the path γ is piecewise smooth and $f:\gamma\to\mathbb{C}$ is continuous on γ and there exist ϕ, $\Phi\in\mathbb{C}$ with $\phi\neq\Phi$ such that $f\in\bar{\Delta}_{\gamma}(\phi,\Phi)$. Then*

$$\mathcal{P}_{\gamma}\left(f,\overline{f}\right) \leq \operatorname{Re}\left[\left(\Phi - \frac{1}{\ell(\gamma)}\int_{\gamma} f(z)|dz| \right)\left(\frac{1}{\ell(\gamma)}\int_{\gamma}\overline{f(z)}|dz| - \overline{\phi} \right) \right] \leq \frac{1}{4}|\Phi-\phi|^2 \tag{2.7}$$

or, equivalently,

$$\mathcal{P}_{\gamma}\left(f,\overline{f}\right) \leq \frac{1}{4}|\Phi-\phi|^2 - \left| \frac{1}{\ell(\gamma)}\int_{\gamma} f(z)|dz| - \frac{\phi+\Phi}{2} \right|^2 \leq \frac{1}{4}|\Phi-\phi|^2. \tag{2.8}$$

Proof. Since $f \in \bar{\Delta}_{\gamma}(\phi, \Phi)$, hence

$$\operatorname{Re}\left[(\Phi - f(z))\left(\overline{f(z)} - \overline{\phi}\right)\right] \geq 0 \text{ for each } \ z \in \gamma.$$

Therefore

$$\frac{1}{\ell(\gamma)} \int_{\gamma} \operatorname{Re}\left[(\Phi - f(z))\left(\overline{f(z)} - \overline{\phi}\right)\right] |dz| \geq 0$$

and by Lemma 1 we deduce the desired result. □

We have:

Lemma 2. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$. If f is continuous on γ, then for all $\lambda \in \mathbb{C}$ we have*

$$\mathcal{P}_{\gamma}\left(f, \overline{f}\right) = \frac{1}{\ell(\gamma)} \int_{\gamma} \left(f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz|\right)\left(\overline{f(v)} - \lambda\right) |dv| . \tag{2.9}$$

Proof. We observe that

$$\begin{aligned}
&\frac{1}{\ell(\gamma)} \int_{\gamma} \left(f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz|\right)\left(\overline{f(v)} - \lambda\right) |dv| \\
&= \frac{1}{\ell(\gamma)} \int_{\gamma} |f(v)|^2 |dv| - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \frac{1}{\ell(\gamma)} \int_{\gamma} \overline{f(v)} |dv| \\
&- \lambda \frac{1}{\ell(\gamma)} \int_{\gamma} \left(f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz|\right) |dv| \\
&= \frac{1}{\ell(\gamma)} \int_{\gamma} |f(v)|^2 |dv| - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \frac{1}{\ell(\gamma)} \overline{\left(\int_{\gamma} f(v) |dv|\right)} \\
&= \frac{1}{\ell(\gamma)} \int_{\gamma} |f(v)|^2 |dv| - \left|\frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz|\right|^2
\end{aligned}$$

for any $\lambda \in \mathbb{C}$, which proves (2.9). □

We have:

Lemma 3. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$. If f is continuous on γ and there exists $c \in \mathbb{C}$ and $\rho > 0$ such that*

$$f \in \overline{D}(c, \rho) := \{z \in \mathbb{C} | \ |z - c| \leq \rho\}, \tag{2.10}$$

then

$$0 \leq \mathcal{P}_{\gamma}\left(f, \overline{f}\right) \leq \rho \frac{1}{\ell(\gamma)} \left|\int_{\gamma} f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz|\right| |dv| \tag{2.11}$$

and

$$\begin{aligned}
0 \leq \mathcal{P}_{\gamma}^2\left(f, \overline{f}\right) &\leq \rho^2 \left(\frac{1}{\ell(\gamma)} \left|\int_{\gamma} f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz|\right| |dv|\right)^2 \\
&\leq \rho^2 \mathcal{P}_{\gamma}\left(f, \overline{f}\right).
\end{aligned} \tag{2.12}$$

Proof. For the equality (2.9) for $\lambda = \overline{c}$ we have

$$\begin{aligned}\mathcal{P}_\gamma\left(f,\overline{f}\right) &= \left|\frac{1}{\ell(\gamma)}\int_\gamma\left(f(v)-\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right)\left(\overline{f(v)}-\overline{c}\right)|dv|\right| \\ &\le \frac{1}{\ell(\gamma)}\left|\int_\gamma f(v)-\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|\left|\overline{f(v)}-\overline{c}\right||dv| \\ &= \frac{1}{\ell(\gamma)}\left|\int_\gamma f(v)-\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|\left|f(v)-c\right||dv| \\ &\le \rho\frac{1}{\ell(\gamma)}\left|\int_\gamma f(v)-\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right||dv|,\end{aligned}$$

which proves (2.11).

Using Cauchy-Bunyakovsky-Schwarz integral inequality, we have

$$\begin{aligned}&\frac{1}{\ell(\gamma)}\left|\int_\gamma f(v)-\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right||dv| \\ &\le\left(\frac{1}{\ell(\gamma)}\int_\gamma\left|f(v)-\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|^2|dv|\right)^{1/2} \\ &=\left(\frac{1}{\ell(\gamma)}\int_\gamma |f(z)|^2\,|dz|-\left|\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|^2\right)^{1/2},\end{aligned}\tag{2.13}$$

where for the last equality we used (1.8).

From (2.11) and (2.13) we have

$$0\le\mathcal{P}_\gamma\left(f,\overline{f}\right)\le\rho\frac{1}{\ell(\gamma)}\left|\int_\gamma f(v)-\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right||dv|\le\rho\left[\mathcal{P}_\gamma\left(f,\overline{f}\right)\right]^{1/2},$$

which implies (2.12). □

Corollary 2. *Assume that the path γ is piecewise smooth and $f:\gamma\to\mathbb{C}$ is continuous on γ and there exist $\phi,\ \Phi\in\mathbb{C}$ with $\phi\ne\Phi$ such that $f\in\bar{\Delta}_\gamma(\phi,\Phi)$. Then*

$$0\le\mathcal{P}_\gamma\left(f,\overline{f}\right)\le\frac{1}{2}|\Phi-\phi|\frac{1}{\ell(\gamma)}\left|\int_\gamma f(v)-\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right||dv|\tag{2.14}$$

and

$$\begin{aligned}0&\le\mathcal{P}_\gamma^2\left(f,\overline{f}\right)\le\frac{1}{4}|\Phi-\phi|^2\left(\frac{1}{\ell(\gamma)}\int_\gamma\left|f(v)-\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right||dv|\right)^2 \\ &\le\frac{1}{4}|\Phi-\phi|^2\operatorname{Re}\left[\left(\Phi-\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right)\left(\frac{1}{\ell(\gamma)}\int_\gamma\overline{f(z)}\,|dz|-\overline{\phi}\right)\right] \\ &=\frac{1}{4}|\Phi-\phi|^2\left[\frac{1}{4}|\Phi-\phi|^2-\left|\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|-\frac{\phi+\Phi}{2}\right|^2\right] \\ &\le\frac{1}{16}|\Phi-\phi|^4.\end{aligned}\tag{2.15}$$

3. Refinements of Grüss' Inequality

We start with the following identity of interest:

Lemma 4. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ, then*

$$\mathcal{D}_\gamma(f,g) = \frac{1}{2(w-u)^2}\int_\gamma \left(\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dw\right)dz \tag{3.1}$$
$$= \frac{1}{2(w-u)^2}\int_\gamma \left(\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dz\right)dw$$
$$= \frac{1}{2(w-u)^2}\int_\gamma \int_\gamma (f(z)-f(w))(g(z)-g(w))\,dz dw.$$

Proof. For any $z \in \gamma$ the integral $\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dw$ exists and

$$I(z) := \int_\gamma (f(z)-f(w))(g(z)-g(w))\,dw$$
$$= \int_\gamma (f(z)g(z)+f(w)g(w)-g(z)f(w)-f(z)g(w))\,dw$$
$$= f(z)g(z)\int_\gamma dw + \int_\gamma f(w)g(w)\,dw - g(z)\int_\gamma f(w)\,dw - f(z)\int_\gamma g(w)\,dw$$
$$= (w-u)f(z)g(z) + \int_\gamma f(w)g(w)\,dw - g(z)\int_\gamma f(w)\,dw - f(z)\int_\gamma g(w)\,dw.$$

The function $I(z)$ is also continuous on γ, then the integral $\int_\gamma I(z)\,dz$ exists and

$$\int_\gamma I(z)\,dz = \int_\gamma \left[(w-u)f(z)g(z) + \int_\gamma f(w)g(w)\,dw \right.$$
$$\left. -g(z)\int_\gamma f(w)\,dw - f(z)\int_\gamma g(w)\,dw\right]dz$$
$$= (w-u)\int_\gamma f(z)g(z)\,dz + (w-u)\int_\gamma f(w)g(w)\,dw$$
$$- \int_\gamma f(w)\,dw \int_\gamma g(z)\,dz - \int_\gamma g(w)\,dw\int_\gamma f(z)\,dz$$
$$= 2(w-u)\int_\gamma f(z)g(z)\,dz - 2\int_\gamma f(z)\,dz\int_\gamma g(z)\,dz = 2(w-u)^2\mathcal{P}_\gamma(f,g),$$

which proves the first equality in (3.1).

The rest follows in a similar manner and we omit the details. □

We have:

Lemma 5. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ, then*

$$|\mathcal{D}_\gamma(f,g)| \le \epsilon^2(\gamma)\left[\mathcal{P}_\gamma\left(f,\overline{f}\right)\right]^{1/2}\left[\mathcal{P}_\gamma(g,\overline{g})\right]^{1/2}, \tag{3.2}$$

where $\epsilon(\gamma) := \frac{\ell(\gamma)}{|w-u|}$ can be interpreted as the deviation of the path γ from the segment joining the points u and w in $\mathbb{C}$.

Proof. Taking the modulus in the first equality in (3.1), we get

$$\begin{aligned}\left|\mathcal{D}_{\gamma}\left(f,g\right)\right| &= \frac{1}{2\left|w-u\right|^{2}}\left|\int_{\gamma}\left(\int_{\gamma}\left(f\left(z\right)-f\left(w\right)\right)\left(g\left(z\right)-g\left(w\right)\right)dw\right)dz\right| \\ &\leq \frac{1}{2\left|w-u\right|^{2}}\int_{\gamma}\left|\int_{\gamma}\left(f\left(z\right)-f\left(w\right)\right)\left(g\left(z\right)-g\left(w\right)\right)dw\right|\left|dz\right| =: A\end{aligned}$$

Using the Cauchy-Bunyakovsky-Schwarz integral inequality, we have

$$\begin{aligned}&\left|\int_{\gamma}\left(f\left(z\right)-f\left(w\right)\right)\left(g\left(z\right)-g\left(w\right)\right)dw\right| \\ &\quad\leq\left(\int_{\gamma}\left|f\left(z\right)-f\left(w\right)\right|^{2}\left|dw\right|\right)^{1/2}\left(\int_{\gamma}\left|g\left(z\right)-g\left(w\right)\right|^{2}\left|dw\right|\right)^{1/2},\end{aligned}$$

which implies that

$$\begin{aligned}&A \\ &\leq\frac{1}{2\left|w-u\right|^{2}}\int_{\gamma}\left(\int_{\gamma}\left|f\left(z\right)-f\left(w\right)\right|^{2}\left|dw\right|\right)^{1/2}\left(\int_{\gamma}\left|g\left(z\right)-g\left(w\right)\right|^{2}\left|dw\right|\right)^{1/2}\left|dz\right| \\ &=: B.\end{aligned}$$

By the Cauchy-Bunyakovsky-Schwarz integral inequality, we also have

$$\begin{aligned}&\int_{\gamma}\left(\int_{\gamma}\left|f\left(z\right)-f\left(w\right)\right|^{2}\left|dw\right|\right)^{1/2}\left(\int_{\gamma}\left|g\left(z\right)-g\left(w\right)\right|^{2}\left|dw\right|\right)^{1/2}\left|dz\right| \\ &\quad\leq\left(\int_{\gamma}\left[\left(\int_{\gamma}\left|f\left(z\right)-f\left(w\right)\right|^{2}\left|dw\right|\right)^{1/2}\right]^{2}\left|dz\right|\right)^{1/2} \\ &\quad\times\left(\int_{\gamma}\left[\left(\int_{\gamma}\left|g\left(z\right)-g\left(w\right)\right|^{2}\left|dw\right|\right)^{1/2}\right]^{2}\left|dz\right|\right)^{1/2} \\ &=\left(\int_{\gamma}\left(\int_{\gamma}\left|f\left(z\right)-f\left(w\right)\right|^{2}\left|dw\right|\right)\left|dz\right|\right)^{1/2}\left(\int_{\gamma}\left(\int_{\gamma}\left|g\left(z\right)-g\left(w\right)\right|^{2}\left|dw\right|\right)\left|dz\right|\right)^{1/2},\end{aligned}$$

which implies that

$$\begin{aligned}B &\leq\frac{1}{2\left|w-u\right|^{2}}\left(\int_{\gamma}\left(\int_{\gamma}\left|f\left(z\right)-f\left(w\right)\right|^{2}\left|dw\right|\right)\left|dz\right|\right)^{1/2} \\ &\quad\times\left(\int_{\gamma}\left(\int_{\gamma}\left|g\left(z\right)-g\left(w\right)\right|^{2}\left|dw\right|\right)\left|dz\right|\right)^{1/2}. \qquad (3.3)\end{aligned}$$

Now, observe that

$$\begin{aligned}
(3.4)\quad & \int_{\gamma}\left(\int_{\gamma}|f(z)-f(w)|^{2}|dw|\right)|dz| \\
&= \int_{\gamma}\left(\int_{\gamma}\left(|f(z)|^{2}-2\operatorname{Re}\left(f(z)\overline{f(w)}\right)+|f(w)|^{2}\right)|dw|\right)|dz| \\
&= \int_{\gamma}\left(\ell(\gamma)|f(z)|^{2}-2\operatorname{Re}\left(f(z)\int_{\gamma}\overline{f(w)}|dw|\right)+\int_{\gamma}|f(w)|^{2}|dw|\right)|dz| \\
&= \ell(\gamma)\int_{\gamma}|f(z)|^{2}|dz|-2\operatorname{Re}\left(\int_{\gamma}f(z)|dz|\int_{\gamma}\overline{f(w)}|dw|\right)+\ell(\gamma)\int_{\gamma}|f(w)|^{2}|dw| \\
&= 2\ell(\gamma)\int_{\gamma}|f(z)|^{2}|dz|-2\operatorname{Re}\left(\int_{\gamma}f(z)|dz|\overline{\left(\int_{\gamma}f(w)|dw|\right)}\right) \\
&= 2\left[\ell(\gamma)\int_{\gamma}|f(z)|^{2}|dz|-\left|\int_{\gamma}f(z)|dz|\right|^{2}\right]=2\ell^{2}(\gamma)\mathcal{P}_{\gamma}\left(f,\overline{f}\right)
\end{aligned}$$

and, similarly

$$\int_{\gamma}\left(\int_{\gamma}|g(z)-g(w)|^{2}|dw|\right)|dz|=2\ell^{2}(\gamma)\mathcal{P}_{\gamma}(g,\overline{g}). \tag{3.5}$$

Making use of (3.4) and (3.5), we get

$$\begin{aligned}
B &\leq \frac{1}{2|w-u|^{2}}\left[2\ell^{2}(\gamma)\mathcal{P}_{\gamma}\left(f,\overline{f}\right)\right]^{1/2}\left[2\ell^{2}(\gamma)\mathcal{P}_{\gamma}(g,\overline{g})\right]^{1/2} \\
&= \frac{\ell^{2}(\gamma)}{|w-u|^{2}}\left[\mathcal{P}_{\gamma}\left(f,\overline{f}\right)\right]^{1/2}\left[\mathcal{P}_{\gamma}(g,\overline{g})\right]^{1/2},
\end{aligned}$$

which proves the desired result (3.2). □

Remark 1. *For $g=f$ we have*

$$\mathcal{D}_{\gamma}(f,f)=\frac{1}{w-u}\int_{\gamma}f^{2}(z)\,dz-\left(\frac{1}{w-u}\int_{\gamma}f(z)\,dz\right)^{2} \tag{3.6}$$

and by (3.2) we get

$$|\mathcal{D}_{\gamma}(f,f)|\leq\epsilon^{2}(\gamma)\mathcal{P}_{\gamma}\left(f,\overline{f}\right). \tag{3.7}$$

For $g=\bar{f}$ we have

$$\mathcal{D}_{\gamma}\left(f,\bar{f}\right)=\frac{1}{w-u}\int_{\gamma}|f(z)|^{2}dz-\frac{1}{w-u}\int_{\gamma}f(z)\,dz\frac{1}{w-u}\int_{\gamma}\overline{f(z)}dz \tag{3.8}$$

and by (3.2) we get

$$\left|\mathcal{D}_{\gamma}\left(f,\bar{f}\right)\right|\leq\epsilon^{2}(\gamma)\mathcal{P}_{\gamma}\left(f,\overline{f}\right). \tag{3.9}$$

Theorem 2. *Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t\in\gamma$ from $z(a)=u$ to $z(b)=w$ with $w\neq u$. If f and g are continuous on γ and there* *exists* ϕ, Φ, ψ, $\Psi\in\mathbb{C}$, $\phi\neq\Phi$, $\psi\neq\Psi$ *such that* $f\in\bar{\Delta}_{\gamma}(\phi,\Phi)$ *and* $g\in\bar{\Delta}_{\gamma}(\psi,\Psi)$ *then*

$$\begin{aligned}(3.10)\quad |\mathcal{D}_{\gamma}(f,g)| &\leq \epsilon^{2}(\gamma)\left(\frac{1}{4}|\Phi-\phi|^{2}-\left|\frac{1}{\ell(\gamma)}\int_{\gamma}f(z)|dz|-\frac{\phi+\Phi}{2}\right|^{2}\right)^{1/2}\\ &\quad\times\left(\frac{1}{4}|\Psi-\psi|^{2}-\left|\frac{1}{\ell(\gamma)}\int_{\gamma}g(z)|dz|-\frac{\psi+\Psi}{2}\right|^{2}\right)^{1/2}\\ &\leq \epsilon^{2}(\gamma)\left[\frac{1}{4}|\Phi-\phi||\Psi-\psi|\right.\\ &\quad\left.-\left|\frac{1}{\ell(\gamma)}\int_{\gamma}f(z)|dz|-\frac{\phi+\Phi}{2}\right|\left|\frac{1}{\ell(\gamma)}\int_{\gamma}g(z)|dz|-\frac{\psi+\Psi}{2}\right|\right]\\ &\leq \frac{1}{4}\epsilon^{2}(\gamma)|\Phi-\phi||\Psi-\psi|.\end{aligned}$$

Proof. The first inequality in (3.10) follows by Corollary 2 and Lemma 5.

Using the elementary inequality

$$\left(m^{2}-n^{2}\right)^{1/2}\left(p^{2}-q^{2}\right)^{1/2}\leq mp-nq$$

that holds for $m\geq n\geq 0$ and $p\geq q\geq 0$, for the choices

$$m=\frac{1}{2}|\Phi-\phi|,\ n=\left|\frac{1}{\ell(\gamma)}\int_{\gamma}f(z)|dz|-\frac{\phi+\Phi}{2}\right|$$

and

$$p=\frac{1}{2}|\Psi-\psi|,\ q=\left|\frac{1}{\ell(\gamma)}\int_{\gamma}g(z)|dz|-\frac{\psi+\Psi}{2}\right|$$

we get the second inequality in (3.10).

The last part is obvious. □

Remark 2. *If there is information on the boundedness of only one function, namely* $f\in\bar{\Delta}_{\gamma}(\phi,\Phi)$, *where* $g:\gamma\to\mathbb{C}$ *is continuous, then we have the "premature" Grüss inequality*

$$(3.11)\quad |\mathcal{D}_{\gamma}(f,g)|\leq \epsilon^{2}(\gamma)\left(\frac{1}{4}|\Phi-\phi|^{2}-\left|\frac{1}{\ell(\gamma)}\int_{\gamma}f(z)|dz|-\frac{\phi+\Phi}{2}\right|^{2}\right)^{1/2}\times\left[\mathcal{P}_{\gamma}(g,\overline{g})\right]^{1/2}.$$

4. Applications for Trapezoid Inequality

Suppose γ is a smooth path parametrized by $z(t)$, $t\in[a,b]$ and f holomorphic in G, an open domain and suppose $\gamma\subset G$. Put $z(a)=u$ and $z(b)=w$ with u, $w\in\mathbb{C}$. Using the integration by parts formula (1.6) twice we have

$$\int_{\gamma_{u,v}}(z-v)f'(z)\,dz=(v-u)f(u)-\int_{\gamma_{u,v}}f(z)\,dz$$

and

$$\int_{\gamma_{v,w}}(z-v)f'(z)\,dz=(w-v)f(w)-\int_{\gamma_{v,w}}f(z)\,dz,$$

for any $v\in\gamma$.

If we add these two equalities, we get the *generalized trapezoid equality*

$$(v-u) f(u)+(w-v) f(w)-\int_{\gamma} f(z) dz = \int_{\gamma_{u,v}}(z-v) f'(z) dz+\int_{\gamma_{v,w}}(z-v) f'(z) dz=\int_{\gamma}(z-v) f'(z) dz \tag{4.1}$$

with the above assumptions for u, v and w on γ.

If we take $v=\frac{u+w}{2}$, then we get the *trapezoid equality*

$$(w-u) \frac{f(u)+f(w)}{2}-\int_{\gamma} f(z) dz=\int_{\gamma}\left(z-\frac{u+w}{2}\right) f'(z) dz, \tag{4.2}$$

which also can be written as

$$\frac{f(u)+f(w)}{2}-\frac{1}{w-u} \int_{\gamma} f(z) dz=\mathcal{D}_{\gamma}(f', h)$$

where $h(z)=z-\frac{u+w}{2}$, $z \in \mathbb{C}$, since $\int_{\gamma}\left(z-\frac{u+w}{2}\right) dz=0$.

If $f' \in \bar{\Delta}_{\gamma}(\theta, \Theta)$ for some θ, $\Theta \in \mathbb{C}$ with $\theta \neq \Theta$, then by (3.11) we get the following trapezoid inequality of interest:

$$\begin{aligned}
&\left|\frac{f(u)+f(w)}{2}-\frac{1}{w-u} \int_{\gamma} f(z) dz\right|=\mathcal{D}_{\gamma}(f', h) \\
&\leq \epsilon^{2}(\gamma)\left(\frac{1}{4}|\Theta-\theta|^{2}-\left|\frac{1}{\ell(\gamma)} \int_{\gamma} f'(z)|dz|-\frac{\theta+\Theta}{2}\right|^{2}\right)^{1/2}\left[\mathcal{P}_{\gamma}\left(h, \overline{h}\right)\right]^{1/2} \\
&\leq \frac{1}{2}|\Theta-\theta| \epsilon^{2}(\gamma)\left[\mathcal{P}_{\gamma}\left(h, \overline{h}\right)\right]^{1/2},
\end{aligned}$$

where

$$\mathcal{P}_{\gamma}\left(h, \overline{h}\right)=\frac{1}{\ell(\gamma)} \int_{\gamma}\left|z-\frac{u+w}{2}\right|^{2}|dz|-\left|\frac{1}{\ell(\gamma)} \int_{\gamma}\left(z-\frac{u+w}{2}\right)|dz|\right|^{2}.$$

If the path γ is a segment $[u, w] \subset G$ connecting two distinct points u and w in G then we write $\int_{\gamma} f(z) dz$ as $\int_{u}^{w} f(z) dz$. We then have

$$\epsilon(\gamma)=\frac{\ell(\gamma)}{|w-u|}=\frac{|w-u|}{|w-u|}=1,$$

$$\frac{1}{\ell([u, w])} \int_{u}^{w} f'(z)|dz|=\frac{|w-u|}{|w-u|} \int_{0}^{1} f'((1-t) u+t w) dt=\frac{f(w)-f(u)}{w-u}$$

and

$$\begin{aligned}
\mathcal{P}_{[u, w]}\left(h, \overline{h}\right)= & \frac{|w-u|}{|w-u|} \int_{0}^{1}\left|(1-t) u+t w-\frac{u+w}{2}\right|^{2} dt \\
& -\left|\frac{|w-u|}{|w-u|} \int_{0}^{1}\left((1-t) u+t w-\frac{u+w}{2}\right) dt\right|^{2} \\
= & |w-u|^{2} \int_{0}^{1}\left(t-\frac{1}{2}\right)^{2} dt-|w-u|^{2}\left|\int_{0}^{1}\left(t-\frac{1}{2}\right) dt\right|^{2}=\frac{1}{12}|w-u|^{2}.
\end{aligned}$$

We can state the following result:

Proposition 2. *Assume that f is holomorphic in G, an open domain and suppose $[u,w]\subset G$. If $f'\in\bar{\Delta}_{[u,w]}(\theta,\Theta)$ for some θ, $\Theta\in\mathbb{C}$ with $\theta\neq\Theta$, then*

$$\left|\frac{f(u)+f(w)}{2}-\frac{1}{w-u}\int_u^w f(z)\,dz\right| \le \frac{\sqrt{3}}{6}|w-u|\left(\frac{1}{4}|\Theta-\theta|^2-\left|\frac{f(w)-f(u)}{w-u}-\frac{\theta+\Theta}{2}\right|^2\right)^{1/2} \le \frac{\sqrt{3}}{12}|w-u||\Theta-\theta|. \tag{4.3}$$

Since

$$\operatorname{Re}\left[\left(\Phi-\frac{f(w)-f(u)}{w-u}\right)\left(\overline{\frac{f(w)-f(u)}{w-u}}-\overline{\phi}\right)\right] = \frac{1}{4}|\Phi-\phi|^2-\left|\frac{f(w)-f(u)}{w-u}-\frac{\phi+\Phi}{2}\right|^2,$$

then the inequality can also be written as

$$\left|\frac{f(u)+f(w)}{2}-\frac{1}{w-u}\int_u^w f(z)\,dz\right| \le \frac{\sqrt{3}}{6}|w-u|\left(\operatorname{Re}\left[\left(\Phi-\frac{f(w)-f(u)}{w-u}\right)\left(\overline{\frac{f(w)-f(u)}{w-u}}-\overline{\phi}\right)\right]\right)^{1/2} \le \frac{\sqrt{3}}{12}|w-u||\Theta-\theta|. \tag{4.4}$$

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