SOME WEIGHTED INEQUALITIES FOR THE COMPLEX INTEGRAL (I)

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Abstract. In this paper we provide some upper bounds for the magnitude of the error in approximating the weighted integral

$$\int_{\gamma} f(z)\, g(z)\, dz$$

with the simple quantity

$$f(w)[G(w)-\beta]+f(u)[\alpha-G(u)]+(\beta-\alpha) f(v)$$

under the assumptions that f and g are holomorphic functions in D, an open domain, $\gamma \subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x \in (a,b)$ while G is a primitive for the function g on γ. Some particular results for certain selections of the complex parameters α and β are also given.

1. Introduction

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a)=u$ and $z(b)=w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w}=\gamma$ as

$$\int_{\gamma} f(z)\, dz=\int_{\gamma_{u,w}} f(z)\, dz:=\int_{a}^{b} f(z(t))\, z'(t)\, dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\, dz:=\int_{\gamma_{u,v}} f(z)\, dz+\int_{\gamma_{v,w}} f(z)\, dz$$

where $v:=zz$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\, |dz|:=\int_{a}^{b} f(z(t))\, |z'(t)|\, dt$$

and the length of the curve γ is then

$$\ell(\gamma)=\int_{\gamma_{u,w}} |dz|=\int_{a}^{b} |z'(t)|\, dt.$$

1991 *Mathematics Subject Classification.* 26D15, 26D10, 30A10, 30A86.

Key words and phrases. Complex integral, Continuous functions, Holomorphic functions, Three points inequalities, Ostrowski inequality, Trapezoid inequality.

RGMIA Res. Rep. Coll. 21 (2018), Art. 127, 22 pp. Received 23/10/18

Let f and g be holomorphic in D, and open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z) g'(z) dz = f(w) g(w) - f(u) g(u) - \int_{\gamma_{u,w}} f'(z) g(z) dz. \tag{1.1}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left| \int_\gamma f(z) dz \right| \le \int_\gamma |f(z)| |dz| \le \|f\|_{\gamma,\infty} \ell(\gamma) \tag{1.2}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p |dz| \right)^{1/p}.$$

For $p=1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)| |dz|.$$

If p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q} \|f\|_{\gamma,p}.$$

Suppose that a continuous function g on γ has a *primitive* on γ, namely a function G analytic on γ such that $G'(z)=g(z)$ for all $z\in\gamma$. Suppose γ is a smooth path parametrized by $z(t)$, $t\in[a,b]$. Put $z(a)=u$ and $z(b)=w$ with u, $w\in\mathbb{C}$. Then

$$\int_\gamma g(z) dz = \int_a^b g(z(t)) z'(t) dt = \int_a^b (G(z(t)))' dt = G(w) - G(u).$$

In this paper we provide some upper bounds for the magnitude of the error in approximating the weighted integral

$$\int_\gamma f(z) g(z) dz$$

with the simple quantity

$$f(w)[G(w)-\beta] + f(u)[\alpha - G(u)] + (\beta-\alpha) f(v)$$

under the assumptions that f and g are holomorphic functions in D, an open domain, $\gamma\subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x\in(a,b)$ while G is a primitive for the function g on γ. Some particular results for certain selections of the complex parameters α and β are also given.

For several previous results concerning three points inequalities, see [1], [2] and [8]-[14]. For some trapezoid, Ostrowski, Grüss and quasi-Grüss type inequalities for complex functions defined on the unit circle centered in zero, see [3]-[7].

2. Some Preliminary Facts

We have:

Lemma 1. *Let f and g be holomorphic in D, an open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v = z(x)$ with $x \in (a,b)$. If G is a primitive for the function g on γ, then for any complex numbers α, β we have*

$$f(w)[G(w)-\beta] + f(u)[\alpha - G(u)] + (\beta-\alpha) f(v) - \int_\gamma f(z) g(z)\, dz = \int_{\gamma_{u,v}} f'(z)[G(z)-\alpha]\, dz + \int_{\gamma_{v,w}} f'(z)[G(z)-\beta]\, dz. \tag{2.1}$$

In particular, for $\beta = \alpha$, we get

$$\begin{aligned} f(w)[G(w)-\alpha] + f(u)[\alpha - G(u)] - \int_\gamma f(z) g(z)\, dz &= \int_{\gamma_{u,v}} f'(z)[G(z)-\alpha]\, dz + \int_{\gamma_{v,w}} f'(z)[G(z)-\alpha]\, dz \\ &= \int_\gamma f'(z)[G(z)-\alpha]\, dz. \end{aligned} \tag{2.2}$$

Proof. Using the integration by parts formula, we have

$$\begin{aligned} \int_{\gamma_{u,v}} f'(z)(G(z)-\alpha)\, dz &= f(z)(G(z)-\alpha)\big|_u^v - \int_{\gamma_{u,v}} f(z)(G(z)-\alpha)'\, dz \\ &= f(v)(G(v)-\alpha) - f(u)(G(u)-\alpha) - \int_{\gamma_{u,v}} f(z) g(z)\, dz \end{aligned}$$

and

$$\begin{aligned} \int_{\gamma_{v,w}} f'(z)(G(z)-\beta)\, dz &= f(z)(G(z)-\beta)\big|_v^w - \int_{\gamma_{v,w}} f(z)(G(z)-\beta)'\, dz \\ &= f(w)(G(w)-\beta) - f(v)(G(v)-\beta) - \int_{\gamma_{v,w}} f(z) g(z)\, dz. \end{aligned}$$

If we add these two equalities, we get

$$\begin{aligned} &\int_{\gamma_{u,v}} f'(z)(G(z)-\alpha)\, dz + \int_{\gamma_{v,w}} f'(z)(G(z)-\beta)\, dz \\ &= f(v)(G(v)-\alpha) - f(u)(G(u)-\alpha) - \int_{\gamma_{u,v}} f(z) g(z)\, dz \\ &+ f(w)(G(w)-\beta) - f(v)(G(v)-\beta) - \int_{\gamma_{v,w}} f(z) g(z)\, dz \\ &= f(w)(G(w)-\beta) + f(u)(\alpha - G(u)) + (\beta-\alpha) f(v) - \int_\gamma f(z) g(z)\, dz \end{aligned}$$

which proves the desired result (2.1). □

Corollary 1. *With the assumptions of Lemma 1 and if* $\beta \neq \alpha$ *and* $w \neq u$*, then*

$$\begin{aligned}(2.3)\quad & \frac{f(w)[G(w)-\beta]+f(u)[\alpha-G(u)]}{w-u}+\left(\frac{\beta-\alpha}{w-u}\right)\frac{1}{w-u}\int_\gamma f(v)\,dv \\ & -\frac{1}{w-u}\int_\gamma f(z)\,g(z)\,dz \\ &= \frac{1}{(w-u)^2}\int_\gamma\left(\int_{\gamma_{u,v}} f'(z)[G(z)-\alpha]\,dz\right)dv \\ &+\frac{1}{(w-u)^2}\int_\gamma\left(\int_{\gamma_{v,w}} f'(z)[G(z)-\beta]\,dz\right)dv.\end{aligned}$$

Proof. Taking the integral on γ over v we have

$$\begin{aligned}&(w-u)\{f(w)[G(w)-\beta]+f(u)[\alpha-G(u)]\}+(\beta-\alpha)\int_\gamma f(v)\,dv \\ &-(w-u)\int_\gamma f(z)\,g(z)\,dz \\ &= \int_\gamma\left(\int_{\gamma_{u,v}} f'(z)[G(z)-\alpha]\,dz\right)dv+\int_\gamma\left(\int_{\gamma_{v,w}} f'(z)[G(z)-\beta]\,dz\right)dv,\end{aligned}$$

which is equivalent to (2.3). □

From the equality (2.2) we have for $\alpha = G(v)$ with $v \in \gamma$

$$\begin{aligned}(2.4)\quad & f(w)[G(w)-G(v)]+f(u)[G(v)-G(u)]-\int_\gamma f(z)\,g(z)\,dz \\ &= \int_{\gamma_{u,v}} f'(z)[G(z)-G(v)]\,dz+\int_{\gamma_{v,w}} f'(z)[G(z)-G(v)]\,dz \\ &= \int_\gamma f'(z)[G(z)-G(v)]\,dz.\end{aligned}$$

If $m = z\left(\frac{a+b}{2}\right)$, then by (2.4) we get

$$\begin{aligned}(2.5)\quad & f(w)[G(w)-G(m)]+f(u)[G(m)-G(u)]-\int_\gamma f(z)\,g(z)\,dz \\ &= \int_{\gamma_{u,m}} f'(z)[G(z)-G(m)]\,dz+\int_{\gamma_{m,w}} f'(z)[G(z)-G(m)]\,dz \\ &= \int_\gamma f'(z)[G(z)-G(m)]\,dz.\end{aligned}$$

If $p \in \gamma$ is such that $G(p) = \frac{G(u)+G(w)}{2}$, then by (2.4) we get

$$\frac{f(w)+f(u)}{2}[G(w)-G(u)] - \int_\gamma f(z) g(z) dz = \int_{\gamma_{u,p}} f'(z)[G(z)-G(p)] dz + \int_{\gamma_{p,w}} f'(z)[G(z)-G(p)] dz = \int_\gamma f'(z)[G(z)-G(p)] dz. \tag{2.6}$$

Now, if we take $\alpha = (1-s)G(u) + sG(w)$ with $s \in [0,1]$ in (2.2), then we get

$$[(1-s)f(w) + sf(u)][G(w)-G(u)] - \int_\gamma f(z) g(z) dz = \int_\gamma f'(z)[G(z)-(1-s)G(u)-sG(w)] dz. \tag{2.7}$$

and, in particular

$$\frac{f(w)+f(u)}{2}[G(w)-G(u)] - \int_\gamma f(z) g(z) dz = \int_\gamma f'(z)\left[G(z) - \frac{G(u)+G(w)}{2}\right] dz. \tag{2.8}$$

If we take $\alpha = \frac{1}{w-u}\int_\gamma G(v) dv$ in (2.2), then we get

$$f(w)\left[G(w) - \frac{1}{w-u}\int_\gamma G(v) dv\right] + f(u)\left[\frac{1}{w-u}\int_\gamma G(v) dv - G(u)\right] - \int_\gamma f(z) g(z) dz = \int_\gamma f'(z)\left[G(z) - \frac{1}{w-u}\int_\gamma G(v) dv\right] dz. \tag{2.9}$$

If we take $\alpha = \frac{1}{w-u}\int_\gamma g(v) dv = \frac{G(w)-G(u)}{w-u}$, then we get

$$\frac{G(w)(w-u-1)+G(u)}{w-u} f(w) + \frac{G(w)+(u-w-1)G(u)}{w-u} f(u) - \int_\gamma f(z) g(z) dz = \int_\gamma f'(z)\left[G(z) - \frac{1}{w-u}\int_\gamma g(v) dv\right] dz. \tag{2.10}$$

If we take in (2.1) $\alpha = sG(u) + (1-s)G(v)$ and $\beta = (1-s)G(v) + sG(w)$, then we get

$$(2.11)\quad (1-s)\{f(w)[G(w)-G(v)] + f(u)[G(v)-G(u)]\} + s[G(w)-G(u)]f(v) - \int_{\gamma} f(z)g(z)\,dz = \int_{\gamma_{u,v}} f'(z)[G(z) - sG(u) - (1-s)G(v)]\,dz + \int_{\gamma_{v,w}} f'(z)[G(z) - (1-s)G(v) - sG(w)]\,dz.$$

for $s \in [0,1]$.

If we take in (2.11) $s = 1$, then we get the Montgomery type identity

$$(2.12)\quad [G(w)-G(u)]f(v) - \int_{\gamma} f(z)g(z)\,dz = \int_{\gamma_{u,v}} f'(z)[G(z)-G(u)]\,dz + \int_{\gamma_{v,w}} f'(z)[G(z)-G(w)]\,dz.$$

If in (2.11) we take $s = \frac{1}{2}$, then we get

$$(2.13)\quad \frac{1}{2}\{f(w)[G(w)-G(v)] + f(u)[G(v)-G(u)] + [G(w)-G(u)]f(v)\} - \int_{\gamma} f(z)g(z)\,dz = \int_{\gamma_{u,v}} f'(z)\left[G(z) - \frac{G(u)+G(v)}{2}\right]dz + \int_{\gamma_{v,w}} f'(z)\left[G(z) - \frac{G(v)+G(w)}{2}\right]dz.$$

If in (2.11) we take $s = 0$, we recapture the equality (2.4).

If in (2.1) we take

$$\alpha = \frac{1}{v-u}\int_{\gamma_{u,v}} G(q)\,dq \text{ and } \beta = \frac{1}{w-v}\int_{\gamma_{v,w}} G(q)\,dq,$$

then we get

$$
\begin{aligned}
(2.14)\quad & f(w)\left[G(w)-\frac{1}{w-v}\int_{\gamma_{v,w}}G(q)\,dq\right] \\
&+f(u)\left[\frac{1}{v-u}\int_{\gamma_{u,v}}G(q)\,dq-G(u)\right] \\
&+\left(\frac{1}{w-v}\int_{\gamma_{v,w}}G(q)\,dq-\frac{1}{v-u}\int_{\gamma_{u,v}}G(q)\,dq\right)f(v)-\int_{\gamma}f(z)\,g(z)\,dz \\
&=\int_{\gamma_{u,v}}f'(z)\left[G(z)-\frac{1}{v-u}\int_{\gamma_{u,v}}G(q)\,dq\right]dz \\
&+\int_{\gamma_{v,w}}f'(z)\left[G(z)-\frac{1}{w-v}\int_{\gamma_{v,w}}G(q)\,dq\right]dz.
\end{aligned}
$$

Moreover, if we take the integral mean over $v\in\gamma$ in (2.11), we get

$$
\begin{aligned}
(2.15)\quad & (1-s)\left\{f(w)\left[G(w)-\frac{1}{w-u}\int_{\gamma}G(v)\,dv\right]\right. \\
&\left.+f(u)\left[\frac{1}{w-u}\int_{\gamma}G(v)\,dv-G(u)\right]\right\} \\
&+s\frac{G(w)-G(u)}{w-u}\int_{\gamma}f(v)\,dv-\int_{\gamma}f(z)\,g(z)\,dz \\
&=\frac{1}{w-u}\int_{\gamma}\left(\int_{\gamma_{u,v}}f'(z)\left[G(z)-sG(u)-(1-s)\,G(v)\right]dz\right)dv \\
&+\frac{1}{w-u}\int_{\gamma}\left(\int_{\gamma_{v,w}}f'(z)\left[G(z)-(1-s)\,G(v)-sG(w)\right]dz\right)dv.
\end{aligned}
$$

for all $s\in[0,1]$.

In particular, for $s=1$ we get

$$
\begin{aligned}
(2.16)\quad & [G(w)-G(u)]\,\frac{1}{w-u}\int_{\gamma}f(v)\,dv-\int_{\gamma}f(z)\,g(z)\,dz \\
&=\frac{1}{w-u}\int_{\gamma}\left(\int_{\gamma_{u,v}}f'(z)\left[G(z)-G(u)\right]dz\right)dv \\
&+\frac{1}{w-u}\int_{\gamma}\left(\int_{\gamma_{v,w}}f'(z)\left[G(z)-G(w)\right]dz\right)dv.
\end{aligned}
$$

Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t\in\gamma$ from $z(a)=u$ to $z(b)=w$ with $w\neq u$. If f and g are continuous on γ, we consider the *complex Čebyšev functional* defined by

$$
\mathcal{D}_{\gamma}(f,g):=\frac{1}{w-u}\int_{\gamma}f(z)\,g(z)\,dz-\frac{1}{w-u}\int_{\gamma}f(z)\,dz\frac{1}{w-u}\int_{\gamma}g(z)\,dz.
$$

Now, if we use (2.16), then we get the representation

$$\mathcal{D}_{\gamma}(f,g)=\frac{1}{(w-u)^{2}}\int_{\gamma}\left(\int_{\gamma_{u,v}}f'(z)\left[G(z)-G(u)\right]dz\right)dv+\frac{1}{(w-u)^{2}}\int_{\gamma}\left(\int_{\gamma_{v,w}}f'(z)\left[G(z)-G(w)\right]dz\right)dv. \tag{2.17}$$

For $s=\frac{1}{2}$ we get from (2.15) the following equality as well:

$$\begin{aligned}&\frac{1}{2}\left\{f(w)\left[G(w)-\frac{1}{w-u}\int_{\gamma}G(v)\,dv\right]\right.\\&\qquad\left.+f(u)\left[\frac{1}{w-u}\int_{\gamma}G(v)\,dv-G(u)\right]\right\}\\&\quad+\frac{1}{2}\frac{G(w)-G(u)}{w-u}\int_{\gamma}f(v)\,dv-\int_{\gamma}f(z)\,g(z)\,dz\\&=\frac{1}{w-u}\int_{\gamma}\left(\int_{\gamma_{u,v}}f'(z)\left[G(z)-\frac{G(u)+G(v)}{2}\right]dz\right)dv\\&\quad+\frac{1}{w-u}\int_{\gamma}\left(\int_{\gamma_{v,w}}f'(z)\left[G(z)-\frac{G(v)+G(w)}{2}\right]dz\right)dv.\end{aligned} \tag{2.18}$$

3. Inequalities for p-Norms of Primitives

We consider the norms sup-norm or ∞-norm defined by

$$\|f\|_{\gamma,\infty}:=\sup_{z\in\gamma}|f(z)|.$$

We also define the p-norm with $p\geq 1$ by

$$\|f\|_{\gamma,p}:=\left(\int_{\gamma}|f(z)|^{p}\,|dz|\right)^{1/p}.$$

For $p=1$ we have

$$\|f\|_{\gamma,1}:=\int_{\gamma}|f(z)|\,|dz|.$$

We have the following result:

Theorem 1. *Let f and g be holomorphic in D, an open domain and suppose $\gamma\subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x\in(a,b)$. If G is a primitive for the function g on γ, then for any complex numbers α, β we* *have*

$$\left|f(w)[G(w)-\beta]+f(u)[\alpha-G(u)]+(\beta-\alpha)f(v)-\int_{\gamma}f(z)g(z)\,dz\right| \tag{3.1}$$
$$\leq\int_{\gamma_{u,v}}|f'(z)||G(z)-\alpha||dz|+\int_{\gamma_{v,w}}|f'(z)||G(z)-\beta||dz|$$
$$\leq\begin{cases}\|f'\|_{\gamma_{u,v},\infty}\|G-\alpha\|_{\gamma_{u,v},1}\,,\\[2mm] \|f'\|_{\gamma_{u,v},p}\|G-\alpha\|_{\gamma_{u,v},q}\ \ \textit{if } p,q>1 \textit{ with } \frac{1}{p}+\frac{1}{q}=1,\\[2mm] \|f'\|_{\gamma_{u,v},1}\|G-\alpha\|_{\gamma_{u,v},\infty}\,,\end{cases}$$
$$+\begin{cases}\|f'\|_{\gamma_{v,w},\infty}\|G-\beta\|_{\gamma_{v,w},1}\,,\\[2mm] \|f'\|_{\gamma_{v,w},p}\|G-\beta\|_{\gamma_{v,w},q}\ \ \textit{if } p,q>1 \textit{ with } \frac{1}{p}+\frac{1}{q}=1,\\[2mm] \|f'\|_{\gamma_{v,w},1}\|G-\beta\|_{\gamma_{v,w},\infty}\,.\end{cases}$$

Proof. Taking the modulus in (2.1) we get

$$\left|f(w)[G(w)-\beta]+f(u)[\alpha-G(u)]+(\beta-\alpha)f(v)-\int_{\gamma}f(z)g(z)\,dz\right| \tag{3.2}$$
$$\leq\left|\int_{\gamma_{u,v}}f'(z)[G(z)-\alpha]\,dz\right|+\left|\int_{\gamma_{v,w}}f'(z)[G(z)-\beta]\,dz\right|$$
$$\leq\int_{\gamma_{u,v}}|f'(z)[G(z)-\alpha]||dz|+\int_{\gamma_{v,w}}|f'(z)[G(z)-\beta]||dz|$$
$$=\int_{\gamma_{u,v}}|f'(z)||G(z)-\alpha||dz|+\int_{\gamma_{v,w}}|f'(z)||G(z)-\beta||dz|=:B(v)\,,$$

which proves the first inequality in (3.1).

By making use of Hölder's inequality, we have

$$\int_{\gamma_{u,v}}|f'(z)||G(z)-\alpha||dz|\leq\begin{cases}\|f'\|_{\gamma_{u,v},\infty}\|G-\alpha\|_{\gamma_{u,v},1}\,,\\[2mm] \|f'\|_{\gamma_{u,v},p}\|G-\alpha\|_{\gamma_{u,v},q}\\ \text{if } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\[2mm] \|f'\|_{\gamma_{u,v},1}\|G-\alpha\|_{\gamma_{u,v},\infty}\end{cases}$$

and

$$\int_{\gamma_{v,w}}|f'(z)||G(z)-\beta||dz|\leq\begin{cases}\|f'\|_{\gamma_{v,w},\infty}\|G-\beta\|_{\gamma_{v,w},1}\,,\\[2mm] \|f'\|_{\gamma_{v,w},p}\|G-\beta\|_{\gamma_{v,w},q}\\ \text{if } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\[2mm] \|f'\|_{\gamma_{v,w},1}\|G-\beta\|_{\gamma_{v,w},\infty}\,.\end{cases}$$

□

We have the following generalized trapezoid inequality:

Corollary 2. *With the assumptions of Theorem 1 we have*

$$\left|f(w)[G(w)-\alpha]+f(u)[\alpha-G(u)]-\int_{\gamma}f(z)g(z)\,dz\right| \tag{3.3}$$

$$\leq\int_{\gamma_{u,v}}|f'(z)||G(z)-\alpha||dz|+\int_{\gamma_{v,w}}|f'(z)||G(z)-\alpha||dz|$$

$$\leq\begin{cases}\|f'\|_{\gamma_{u,v},\infty}\|G-\alpha\|_{\gamma_{u,v},1}, \\ \\ \|f'\|_{\gamma_{u,v},p}\|G-\alpha\|_{\gamma_{u,v},q} \ \text{if } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \|f'\|_{\gamma_{u,v},1}\|G-\alpha\|_{\gamma_{u,v},\infty},\end{cases}$$

$$+\begin{cases}\|f'\|_{\gamma_{v,w},\infty}\|G-\alpha\|_{\gamma_{v,w},1}, \\ \\ \|f'\|_{\gamma_{v,w},p}\|G-\alpha\|_{\gamma_{v,w},q} \ \text{if } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \|f'\|_{\gamma_{v,w},1}\|G-\alpha\|_{\gamma_{v,w},\infty}.\end{cases}$$

$$\leq\begin{cases}\|f'\|_{\gamma_{u,w},\infty}\|G-\alpha\|_{\gamma_{u,w},1}, \\ \\ \|f'\|_{\gamma_{u,w},p}\|G-\alpha\|_{\gamma_{u,w},q} \ \text{if } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \|f'\|_{\gamma_{u,w},1}\|G-\alpha\|_{\gamma_{u,w},\infty}.\end{cases}$$

4. Inequalities for Bounded Primitives

Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t\in\gamma$ from $z(a)=u$ to $z(b)=w$. Now, for ϕ, $\Phi\in\mathbb{C}$ and γ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{\gamma}(\phi,\Phi):=\left\{h:\gamma\to\mathbb{C}|\operatorname{Re}\left[(\Phi-h(z))\left(\overline{h(z)}-\overline{\phi}\right)\right]\geq0 \ \text{ for each } \ z\in\gamma\right\}$$

and

$$\bar{\Delta}_{\gamma}(\phi,\Phi):=\left\{h:\gamma\to\mathbb{C}|\ \left|h(z)-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi| \ \text{ for each } \ z\in\gamma\right\}.$$

The following representation result may be stated.

Proposition 1. *For any ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that $\bar{U}_{\gamma}(\phi,\Phi)$ and $\bar{\Delta}_{\gamma}(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_{\gamma}(\phi,\Phi)=\bar{\Delta}_{\gamma}(\phi,\Phi). \tag{4.1}$$

Proof. We observe that for any $w\in\mathbb{C}$ we have the equivalence

$$\left|w-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi|$$

if and only if

$$\operatorname{Re}\left[(\Phi-w)\left(\overline{w}-\overline{\phi}\right)\right]\geq0.$$

This follows by the equality

$$\frac{1}{4}\left|\Phi-\phi\right|^{2}-\left|w-\frac{\phi+\Phi}{2}\right|^{2}=\operatorname{Re}\left[\left(\Phi-w\right)\left(\overline{w}-\overline{\phi}\right)\right]$$

that holds for any $w\in\mathbb{C}$.

The equality (4.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 3. *For any $\phi,\ \Phi\in\mathbb{C},\ \phi\neq\Phi$, we have that*

$$\bar{U}_{\gamma}\left(\phi,\Phi\right)=\{h:\gamma\rightarrow\mathbb{C}\ |\ \left(\operatorname{Re}\Phi-\operatorname{Re}h\left(z\right)\right)\left(\operatorname{Re}h\left(z\right)-\operatorname{Re}\phi\right)+\left(\operatorname{Im}\Phi-\operatorname{Im}h\left(z\right)\right)\left(\operatorname{Im}h\left(z\right)-\operatorname{Im}\phi\right)\geq 0\ \textit{for each}\ z\in\gamma\}. \tag{4.2}$$

Now, if we assume that $\operatorname{Re}\left(\Phi\right)\geq\operatorname{Re}\left(\phi\right)$ and $\operatorname{Im}\left(\Phi\right)\geq\operatorname{Im}\left(\phi\right)$, then we can define the following set of functions as well:

$$\bar{S}_{\gamma}\left(\phi,\Phi\right):=\{h:\gamma\rightarrow\mathbb{C}\ |\ \operatorname{Re}\left(\Phi\right)\geq\operatorname{Re}h\left(z\right)\geq\operatorname{Re}\left(\phi\right)\ \text{and}\ \operatorname{Im}\left(\Phi\right)\geq\operatorname{Im}h\left(z\right)\geq\operatorname{Im}\left(\phi\right)\ \text{for each}\ \ z\in\gamma\}. \tag{4.3}$$

One can easily observe that $\bar{S}_{\gamma}\left(\phi,\Phi\right)$ is closed, convex and

$$\emptyset\neq\bar{S}_{\gamma}\left(\phi,\Phi\right)\subseteq\bar{U}_{\gamma}\left(\phi,\Phi\right). \tag{4.4}$$

Proposition 2. *Let f and g be holomorphic in D, an open domain and suppose $\gamma\subset D$ is a piecewise smooth path from $z\left(a\right)=u$ to $z\left(b\right)=w$ and $v=z\left(x\right)$ with $x\in\left(a,b\right)$. If G is a primitive for the function g on γ and there exists the constants $\phi_i,\ \Phi_i\in\mathbb{C},\ \phi_i\neq\Phi_i,\ i\in\{1,2\}$ with $G\in\bar{\Delta}_{\gamma_{u,v}}\left(\phi_1,\Phi_1\right)\cap\bar{\Delta}_{\gamma_{v,w}}\left(\phi_2,\Phi_2\right)$, then*

$$\begin{aligned}&\left|f\left(w\right)\left[G\left(w\right)-\frac{\phi_2+\Phi_2}{2}\right]+f\left(u\right)\left[\frac{\phi_1+\Phi_1}{2}-G\left(u\right)\right]\right.\\&\qquad\left.+\left(\frac{\phi_2+\Phi_2}{2}-\frac{\phi_1+\Phi_1}{2}\right)f\left(v\right)-\int_{\gamma}f\left(z\right)g\left(z\right)dz\right|\\&\leq\int_{\gamma_{u,v}}\left|f'\left(z\right)\right|\left|G\left(z\right)-\frac{\phi_1+\Phi_1}{2}\right|\left|dz\right|+\int_{\gamma_{v,w}}\left|f'\left(z\right)\right|\left|G\left(z\right)-\frac{\phi_2+\Phi_2}{2}\right|\left|dz\right|\\&\leq\frac{1}{2}\left|\Phi_1-\phi_1\right|\int_{\gamma_{u,v}}\left|f'\left(z\right)\right|\left|dz\right|+\frac{1}{2}\left|\Phi_2-\phi_2\right|\int_{\gamma_{v,w}}\left|f'\left(z\right)\right|\left|dz\right|\\&\leq\frac{1}{2}\max\left\{\left|\Phi_1-\phi_1\right|,\left|\Phi_2-\phi_2\right|\right\}\int_{\gamma_{u,w}}\left|f'\left(z\right)\right|\left|dz\right|.\end{aligned} \tag{4.5}$$

The proof follows by the first inequality in (3.1) by taking $\alpha=\frac{\phi_1+\Phi_1}{2}$ and $\beta=\frac{\phi_2+\Phi_2}{2}$.

Remark 1. *If we take $\phi_1=\phi_2=\phi$ and $\Phi_1=\Phi_2=\Phi$ then we get, by (4.6) for $G\in\bar{\Delta}_{\gamma_{u,w}}\left(\phi,\Phi\right)$, the following trapezoid type inequality*

$$\begin{aligned}&\left|f\left(w\right)\left[G\left(w\right)-\frac{\phi+\Phi}{2}\right]+f\left(u\right)\left[\frac{\phi+\Phi}{2}-G\left(u\right)\right]-\int_{\gamma}f\left(z\right)g\left(z\right)dz\right|\\&\leq\int_{\gamma_{u,w}}\left|f'\left(z\right)\right|\left|G\left(z\right)-\frac{\phi+\Phi}{2}\right|\left|dz\right|\leq\frac{1}{2}\left|\Phi-\phi\right|\int_{\gamma_{u,w}}\left|f'\left(z\right)\right|\left|dz\right|.\end{aligned} \tag{4.6}$$

5. Trapezoid Type Inequalities

We have

Theorem 2. *Let f and g be holomorphic in D, an open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v = z(x)$ with $x \in (a,b)$. If G is a primitive for the function g on γ, then*

$$\left| f(w) \int_{\gamma_{v,w}} g(z)\,dz + f(u) \int_{\gamma_{u,v}} g(z)\,dz - \int_{\gamma} f(z)\,g(z)\,dz \right| \leq B(v) \tag{5.1}$$

where

$$B(v) := \int_{\gamma_{u,v}} \left|f'(z)\right| \left| \int_{\gamma_{z,v}} g(q)\,dq \right| |dz| + \int_{\gamma_{v,w}} \left|f'(z)\right| \left| \int_{\gamma_{v,z}} g(q)\,dq \right| |dz|. \tag{5.2}$$

We also have the following bounds for $B(v)$,

$$B(v) \leq \begin{cases} \max_{z\in\gamma_{u,v}} \left|f'(z)\right| \int_{\gamma_{u,v}} \left| \int_{\gamma_{z,v}} g(q)\,dq \right| |dz|, \\ \\ \left(\int_{\gamma_{u,v}} \left|f'(z)\right|^p |dz| \right)^{1/p} \left(\int_{\gamma_{u,v}} \left| \int_{\gamma_{z,v}} g(q)\,dq \right|^q |dz| \right)^{1/q} \\ p,q>1 \ \text{with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max_{z\in\gamma_{u,v}} \left| \int_{\gamma_{z,v}} g(q)\,dq \right| \int_{\gamma_{u,v}} \left|f'(z)\right| |dz| \end{cases} \tag{5.3}$$

$$+ \begin{cases} \max_{z\in\gamma_{v,w}} \left|f'(z)\right| \int_{\gamma_{v,w}} \left| \int_{\gamma_{v,z}} g(q)\,dq \right| |dz|, \\ \\ \left(\int_{\gamma_{v,w}} \left|f'(z)\right|^p |dz| \right)^{1/p} \left(\int_{\gamma_{v,w}} \left| \int_{\gamma_{v,z}} g(q)\,dq \right|^q |dz| \right)^{1/q} \\ p,q>1 \ \text{with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max_{z\in\gamma_{v,w}} \left| \int_{\gamma_{v,z}} g(q)\,dq \right| \int_{\gamma_{v,w}} \left|f'(z)\right| |dz|. \end{cases}$$

$$\leq \begin{cases} \max_{z\in\gamma_{u,w}} \left|f'(z)\right| \int_{\gamma_{u,w}} \left| \int_{\gamma_{z,v}} g(q)\,dq \right| |dz| \\ \\ \left(\int_{\gamma_{u,w}} \left|f'(z)\right|^p |dz| \right)^{1/p} \left(\int_{\gamma_{u,w}} \left| \int_{\gamma_{z,v}} g(q)\,dq \right|^q |dz| \right)^{1/q} \\ p,q>1 \ \text{with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max_{z\in\gamma_{u,w}} \left| \int_{\gamma_{z,v}} g(q)\,dq \right| \int_{\gamma_{u,w}} \left|f'(z)\right| |dz| \end{cases}$$

Proof. We have $\gamma = \gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$ where u, v, w are as above.

Then by (2.4) we have

$$
\begin{aligned}
(5.4)\quad & \left| f(w)\int_{\gamma_{v,w}} g(z)\,dz + f(u)\int_{\gamma_{u,v}} g(z)\,dz - \int_{\gamma} f(z)\,g(z)\,dz \right| \\
&= \left| f(w)\left[G(w)-G(v)\right] + f(u)\left[G(v)-G(u)\right] - \int_{\gamma} f(z)\,g(z)\,dz \right| \\
&\le \left| \int_{\gamma_{u,v}} f'(z)\left[G(z)-G(v)\right] dz \right| + \left| \int_{\gamma_{v,w}} f'(z)\left[G(z)-G(v)\right] dz \right| \\
&\le \int_{\gamma_{u,v}} \left| f'(z)\left[G(z)-G(v)\right]\right| |dz| + \int_{\gamma_{v,w}} \left| f'(z)\left[G(z)-G(v)\right]\right| |dz| \\
&= \int_{\gamma_{u,v}} \left| f'(z)\right| \left|G(v)-G(z)\right| |dz| + \int_{\gamma_{v,w}} \left| f'(z)\right| \left|G(z)-G(v)\right| |dz| \\
&= \int_{\gamma_{u,v}} \left| f'(z)\right| \left| \int_{\gamma_{z,v}} g(q)\,dq \right| |dz| + \int_{\gamma_{v,w}} \left| f'(z)\right| \left| \int_{\gamma_{v,z}} g(q)\,dq \right| |dz| = B(v),
\end{aligned}
$$

which proves the inequality (5.1).

Using the Hölder's inequality we have

$$
\int_{\gamma_{u,v}} \left| f'(z)\right| \left| \int_{\gamma_{z,v}} g(q)\,dq \right| |dz| \le
\begin{cases}
\max_{z\in\gamma_{u,v}} \left| f'(z)\right| \int_{\gamma_{u,v}} \left| \int_{\gamma_{z,v}} g(q)\,dq \right| |dz|, \\[2ex]
\left(\int_{\gamma_{u,v}} \left| f'(z)\right|^{p} |dz| \right)^{1/p} \left(\int_{\gamma_{u,v}} \left| \int_{\gamma_{z,v}} g(q)\,dq \right|^{q} |dz| \right)^{1/q} \\
p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\[2ex]
\max_{z\in\gamma_{u,v}} \left| \int_{\gamma_{z,v}} g(q)\,dq \right| \int_{\gamma_{u,v}} \left| f'(z)\right| |dz|
\end{cases}
$$

and

$$
\int_{\gamma_{v,w}} \left| f'(z)\right| \left| \int_{\gamma_{v,z}} g(q)\,dq \right| |dz| \le
\begin{cases}
\max_{z\in\gamma_{v,w}} \left| f'(z)\right| \int_{\gamma_{v,w}} \left| \int_{\gamma_{v,z}} g(q)\,dq \right| |dz|, \\[2ex]
\left(\int_{\gamma_{v,w}} \left| f'(z)\right|^{p} |dz| \right)^{1/p} \left(\int_{\gamma_{v,w}} \left| \int_{\gamma_{v,z}} g(q)\,dq \right|^{q} |dz| \right)^{1/q} \\
p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\[2ex]
\max_{z\in\gamma_{v,w}} \left| \int_{\gamma_{v,z}} g(q)\,dq \right| \int_{\gamma_{v,w}} \left| f'(z)\right| |dz|.
\end{cases}
$$

This implies that

$$B(v) \leq \begin{cases} \max_{z\in\gamma_{u,v}} |f'(z)| \int_{\gamma_{u,v}} \left| \int_{\gamma_{z,v}} g(q)\,dq \right| |dz|, \\ \\ \left(\int_{\gamma_{u,v}} |f'(z)|^p |dz| \right)^{1/p} \left(\int_{\gamma_{u,v}} \left| \int_{\gamma_{z,v}} g(q)\,dq \right|^q |dz| \right)^{1/q} \\ p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max_{z\in\gamma_{u,v}} \left| \int_{\gamma_{z,v}} g(q)\,dq \right| \int_{\gamma_{u,v}} |f'(z)| |dz| \end{cases}$$

$$+ \begin{cases} \max_{z\in\gamma_{v,w}} |f'(z)| \int_{\gamma_{v,w}} \left| \int_{\gamma_{v,z}} g(q)\,dq \right| |dz|, \\ \\ \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz| \right)^{1/p} \left(\int_{\gamma_{v,w}} \left| \int_{\gamma_{v,z}} g(q)\,dq \right|^q |dz| \right)^{1/q} \\ p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max_{z\in\gamma_{v,w}} \left| \int_{\gamma_{v,z}} g(q)\,dq \right| \int_{\gamma_{v,w}} |f'(z)| |dz|, \end{cases}$$

which proves the first inequality in (5.3).

Since

$$\begin{aligned} &\max_{z\in\gamma_{u,v}} |f'(z)| \int_{\gamma_{u,v}} \left| \int_{\gamma_{z,v}} g(q)\,dq \right| |dz| + \max_{z\in\gamma_{v,w}} |f'(z)| \int_{\gamma_{v,w}} \left| \int_{\gamma_{v,z}} g(q)\,dq \right| |dz|, \\ &\leq \max\left\{ \max_{z\in\gamma_{u,v}} |f'(z)|, \max_{z\in\gamma_{v,w}} |f'(z)| \right\} \\ &\times \left[\int_{\gamma_{u,v}} \left| \int_{\gamma_{z,v}} g(q)\,dq \right| |dz| + \int_{\gamma_{v,w}} \left| \int_{\gamma_{v,z}} g(q)\,dq \right| |dz| \right] \\ &= \max_{z\in\gamma_{u,w}} |f'(z)| \int_{\gamma_{u,w}} \left| \int_{\gamma_{z,v}} g(q)\,dq \right| |dz|, \end{aligned}$$

$$\begin{aligned} &\left(\int_{\gamma_{u,v}} |f'(z)|^p |dz| \right)^{1/p} \left(\int_{\gamma_{u,v}} \left| \int_{\gamma_{z,v}} g(q)\,dq \right|^q |dz| \right)^{1/q} \\ &+ \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz| \right)^{1/p} \left(\int_{\gamma_{v,w}} \left| \int_{\gamma_{v,z}} g(q)\,dq \right|^q |dz| \right)^{1/q} \\ &\leq \left[\left(\left(\int_{\gamma_{u,v}} |f'(z)|^p |dz| \right)^{1/p} \right)^p + \left(\left(\int_{\gamma_{v,w}} |f'(z)|^p |dz| \right)^{1/p} \right)^p \right]^{1/p} \\ &\times \left[\left(\left(\int_{\gamma_{u,v}} \left| \int_{\gamma_{z,v}} g(q)\,dq \right|^q |dz| \right)^{1/q} \right)^q + \left(\left(\int_{\gamma_{v,w}} \left| \int_{\gamma_{v,z}} g(q)\,dq \right|^q |dz| \right)^{1/q} \right)^q \right]^{1/q} \end{aligned}$$

$$= \left[\int_{\gamma_{u,v}} |f'(z)|^p |dz| + \int_{\gamma_{v,w}} |f'(z)|^p |dz|\right]^{1/p}$$

$$\times \left[\int_{\gamma_{u,v}} \left|\int_{\gamma_{z,v}} g(q)\,dq\right|^q |dz| + \int_{\gamma_{v,w}} \left|\int_{\gamma_{v,z}} g(q)\,dq\right|^q |dz|\right]^{1/q}$$

$$= \left(\int_{\gamma_{u,w}} |f'(z)|^p |dz|\right)^{1/p} \left(\int_{\gamma_{u,w}} \left|\int_{\gamma_{z,v}} g(q)\,dq\right|^q |dz|\right)^{1/q}$$

and

$$\max_{z\in\gamma_{u,v}} \left|\int_{\gamma_{z,v}} g(q)\,dq\right| \int_{\gamma_{u,v}} |f'(z)|\,|dz| + \max_{z\in\gamma_{v,w}} \left|\int_{\gamma_{v,z}} g(q)\,dq\right| \int_{\gamma_{v,w}} |f'(z)|\,|dz|$$

$$\leq \max\left\{\max_{z\in\gamma_{u,v}} \left|\int_{\gamma_{z,v}} g(q)\,dq\right|, \max_{z\in\gamma_{v,w}} \left|\int_{\gamma_{v,z}} g(q)\,dq\right|\right\}$$

$$\times \left[\int_{\gamma_{u,v}} |f'(z)|\,|dz| + \int_{\gamma_{v,w}} |f'(z)|\,|dz|\right]$$

$$= \max_{z\in\gamma_{u,w}} \left|\int_{\gamma_{z,v}} g(q)\,dq\right| \int_{\gamma_{u,w}} |f'(z)|\,|dz|,$$

which proves the last part of (5.3). □

Corollary 4. *With the assumptions of Theorem 2 we also have the following inequalities*

(5.5) $B(v)$

$$\leq \int_{\gamma_{u,v}} |f'(z)| \left(\int_{\gamma_{z,v}} |g(q)|\,|dq|\right) |dz| + \int_{\gamma_{v,w}} |f'(z)| \left(\int_{\gamma_{v,z}} |g(q)|\,|dq|\right) |dz|$$

$$\leq \begin{cases} \max_{z\in\gamma_{u,v}} |f'(z)| \int_{\gamma_{u,v}} \left(\int_{\gamma_{z,v}} |g(q)|\,|dq|\right) |dz|, \\ \\ \left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} \left(\int_{\gamma_{u,v}} \left(\int_{\gamma_{z,v}} |g(q)|\,|dq|\right)^q |dz|\right)^{1/q} \\ p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max_{z\in\gamma_{u,v}} \left(\left(\int_{\gamma_{z,v}} |g(q)|\,|dq|\right)\right) \int_{\gamma_{u,v}} |f'(z)|\,|dz| \end{cases}$$

$$+ \begin{cases} \max_{z\in\gamma_{v,w}} |f'(z)| \int_{\gamma_{v,w}} \left(\int_{\gamma_{v,z}} |g(q)|\,|dq|\right) |dz|, \\ \\ \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p} \left(\int_{\gamma_{v,w}} \left(\int_{\gamma_{v,z}} |g(q)|\,|dq|\right)^q |dz|\right)^{1/q} \\ p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max_{z\in\gamma_{v,w}} \left(\int_{\gamma_{v,z}} |g(q)|\,|dq|\right) \int_{\gamma_{v,w}} |f'(z)|\,|dz|. \end{cases}$$

$$\leq \begin{cases} \max_{z\in\gamma_{u,w}} |f'(z)| \int_{\gamma_{u,w}} \left| \int_{\gamma_{z,v}} |g(q)| \, |dq| \right| |dz| \\ \\ \left(\int_{\gamma_{u,w}} |f'(z)|^p \, |dz| \right)^{1/p} \left(\int_{\gamma_{u,w}} \left| \int_{\gamma_{z,v}} |g(q)| \, |dq| \right|^q |dz| \right)^{1/q} \\ p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \max_{z\in\gamma_{u,w}} \left(\left| \int_{\gamma_{z,v}} g(q) \, dq \right| \right) \int_{\gamma_{u,w}} |f'(z)| \, |dz| . \end{cases}$$

The proof is similar to the one from Theorem 2 and we omit the details.

6. Further Trapezoid Inequalities

We have the following result as well:

Theorem 3. *Let f and g be holomorphic in D, an open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. If G is a primitive for the function g on γ and $\alpha \in [0,1]$, then*

$$(6.1) \quad \left| [(1-s) f(w) + s f(u)] \int_\gamma g(z) \, dz - \int_\gamma f(z) g(z) \, dz \right|$$
$$\leq (1-s) \int_\gamma |f'(z)| \left| \int_{\gamma_{u,z}} g(q) \, dq \right| |dz| + s \int_\gamma |f'(z)| \left| \int_{\gamma_{z,w}} g(q) \, dq \right| |dz| =: C(s) .$$

We also have

$$(6.2) \quad C(s)$$
$$\leq \begin{cases} \max_{z\in\gamma} |f'(z)| \left[(1-s) \int_\gamma \left| \int_{\gamma_{u,z}} g(q) \, dq \right| |dz| + s \int_\gamma \left| \int_{\gamma_{z,w}} g(q) \, dq \right| |dz| \right] \\ \\ \left(\int_\gamma |f'(z)|^p \, |dz| \right)^{1/p} \\ \times \left[(1-s) \left(\int_\gamma \left| \int_{\gamma_{u,z}} g(q) \, dq \right|^q |dz| \right)^{1/q} + s \left(\int_\gamma \left| \int_{\gamma_{z,w}} g(q) \, dq \right|^q |dz| \right)^{1/q} \right], \\ p, q > 1 \text{ where } \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \int_\gamma |f'(z)| \, |dz| \left[(1-s) \max_{z\in\gamma} \left| \int_{\gamma_{u,z}} g(q) \, dq \right| + s \max_{z\in\gamma} \left| \int_{\gamma_{z,w}} g(q) \, dq \right| \right] \end{cases}$$

In particular,

$$(6.3) \quad \left| \frac{f(w) + f(u)}{2} \int_\gamma g(z) \, dz - \int_\gamma f(z) g(z) \, dz \right|$$
$$\leq \frac{1}{2} \left[\int_\gamma |f'(z)| \left| \int_{\gamma_{u,z}} g(q) \, dq \right| |dz| + \int_\gamma |f'(z)| \left| \int_{\gamma_{z,w}} g(q) \, dq \right| |dz| \right] =: C_{1/2}$$

and

(6.4) $C_{1/2}$

$$\leq \frac{1}{2}\begin{cases} \max_{z\in\gamma}\left|f'(z)\right|\left[\int_\gamma\left[\left|\int_{\gamma_{u,z}} g(q)\,dq\right| + \left|\int_{\gamma_{z,w}} g(q)\,dq\right|\right]|dz|\right] \\ \left(\int_\gamma\left|f'(z)\right|^p|dz|\right)^{1/p}\left[\left(\int_\gamma\left|\int_{\gamma_{u,z}} g(q)\,dq\right|^q|dz|\right)^{1/q} + \left(\int_\gamma\left|\int_{\gamma_{z,w}} g(q)\,dq\right|^q|dz|\right)^{1/q}\right], \\ p,q>1 \text{ where } \frac{1}{p}+\frac{1}{q}=1; \\ \int_\gamma\left|f'(z)\right||dz|\left[\max_{z\in\gamma}\left|\int_{\gamma_{u,z}} g(q)\,dq\right| + \max_{z\in\gamma}\left|\int_{\gamma_{z,w}} g(q)\,dq\right|\right]. \end{cases}$$

Proof. Using the identity (2.7) we have

$$\begin{aligned} &[(1-s)f(w)+sf(u)][G(w)-G(u)] - \int_\gamma f(z)g(z)\,dz \quad (6.5)\\ &= \int_\gamma f'(z)[G(z)-(1-s)G(u)-sG(w)]\,dz \\ &= \int_\gamma f'(z)\{(1-s)[G(z)-G(u)] + s[G(z)-G(w)]\}\,dz \end{aligned}$$

for $\alpha\in[0,1]$.

Taking the modulus in (6.5), we get

$$\begin{aligned} &\left|[(1-s)f(w)+sf(u)][G(w)-G(u)] - \int_\gamma f(z)g(z)\,dz\right| \quad (6.6)\\ &\leq \int_\gamma f'(z)\left|(1-s)[G(z)-G(u)] + s[G(z)-G(w)]\right||dz| \\ &\leq (1-s)\int_\gamma\left|f'(z)\right||G(z)-G(u)||dz| + s\int_\gamma\left|f'(z)\right||G(w)-G(z)||dz| \\ &= (1-s)\int_\gamma\left|f'(z)\right|\left|\int_{\gamma_{u,z}} g(q)\,dq\right||dz| + s\int_\gamma\left|f'(z)\right|\left|\int_{\gamma_{z,w}} g(q)\,dq\right||dz| = C(s) \end{aligned}$$

for $\alpha\in[0,1]$, which proves the inequality (6.1).

Using the Hölder's inequality, we have

$$\int_\gamma\left|f'(z)\right|\left|\int_{\gamma_{u,z}} g(q)\,dq\right||dz| \leq \begin{cases} \max_{z\in\gamma}\left|f'(z)\right|\int_\gamma\left|\int_{\gamma_{u,z}} g(q)\,dq\right||dz|; \\ \left(\int_\gamma\left|f'(z)\right|^p|dz|\right)^{1/p}\left(\int_\gamma\left|\int_{\gamma_{u,z}} g(q)\,dq\right|^q|dz|\right)^{1/q}, \\ p,q>1 \text{ where } \frac{1}{p}+\frac{1}{q}=1; \\ \max_{z\in\gamma}\left|\int_{\gamma_{u,z}} g(q)\,dq\right|\int_\gamma\left|f'(z)\right||dz| \end{cases}$$

and

$$\int_{\gamma}|f'(z)|\left|\int_{\gamma_{z,w}}g(q)\,dq\right||dz|\leq\begin{cases}\max_{z\in\gamma}|f'(z)|\int_{\gamma}\left|\int_{\gamma_{z,w}}g(q)\,dq\right||dz|;\\[2ex]\left(\int_{\gamma}|f'(z)|^{p}|dz|\right)^{1/p}\left(\int_{\gamma}\left|\int_{\gamma_{z,w}}g(q)\,dq\right|^{q}|dz|\right)^{1/q},\\ p,q>1\text{ where }\frac{1}{p}+\frac{1}{q}=1;\\[2ex]\max_{z\in\gamma}\left|\int_{\gamma_{z,w}}g(q)\,dq\right|\int_{\gamma}|f'(z)||dz|,\end{cases}$$

which provides the inequality (6.2). □

We also have the result

Theorem 4. *Let f and g be holomorphic in D, an open domain and suppose $\gamma\subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$. If G is a primitive for the function g on γ, then*

$$\begin{aligned}(6.7)\quad&\left|f(w)\left[G(w)-\frac{1}{w-u}\int_{\gamma}G(v)\,dv\right]+f(u)\left[\frac{1}{w-u}\int_{\gamma}G(v)\,dv-G(u)\right]\right.\\&\qquad\left.-\int_{\gamma}f(z)\,g(z)\,dz\right|\\&\leq\int_{\gamma}|f'(z)|\left|G(z)-\frac{1}{w-u}\int_{\gamma}G(v)\,dv\right||dz|\\&\leq\begin{cases}\max_{z\in\gamma}|f'(z)|\int_{\gamma}\left|G(z)-\frac{1}{w-u}\int_{\gamma}G(v)\,dv\right||dz|,\\[2ex]\left(\int_{\gamma}|f'(z)|^{p}|dz|\right)^{1/p}\left(\int_{\gamma}\left|G(z)-\frac{1}{w-u}\int_{\gamma}G(v)\,dv\right|^{q}|dz|\right)^{1/q}\\ p,q>1\ \text{and}\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]\max_{z\in\gamma}\left|G(z)-\frac{1}{w-u}\int_{\gamma}G(v)\,dv\right|\int_{\gamma}|f'(z)||dz|.\end{cases}\end{aligned}$$

The proof follows as above by employing the identity (2.9). We omit the details.

7. Some Unweighted Inequalities

The case $g(z)=1$, $z\in\mathbb{C}$ in the inequality (3.1) gives simple unweighted inequalities as follows:

$$\begin{aligned}(7.1)\quad&\left|f(w)(w-\beta)+f(u)(\alpha-u)+(\beta-\alpha)f(v)-\int_{\gamma}f(z)\,dz\right|\\&\qquad\leq\int_{\gamma_{u,v}}|f'(z)||z-\alpha||dz|+\int_{\gamma_{v,w}}|f'(z)||z-\beta||dz|\end{aligned}$$

$$\leq \begin{cases} \|f'\|_{\gamma_{u,v},\infty} \|\ell-\alpha\|_{\gamma_{u,v},1}, \\ \\ \|f'\|_{\gamma_{u,v},p} \|\ell-\alpha\|_{\gamma_{u,v},q} \text{ if } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \|f'\|_{\gamma_{u,v},1} \|\ell-\alpha\|_{\gamma_{u,v},\infty}, \end{cases}$$

$$+\begin{cases} \|f'\|_{\gamma_{v,w},\infty} \|\ell-\beta\|_{\gamma_{v,w},1}, \\ \\ \|f'\|_{\gamma_{v,w},p} \|\ell-\beta\|_{\gamma_{v,w},q} \text{ if } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \|f'\|_{\gamma_{v,w},1} \|\ell-\beta\|_{\gamma_{v,w},\infty}, \end{cases}$$

for α, $\beta \in \mathbb{C}$, where $\ell(z)=z$ is the identity complex function, f is holomorphic in D, an open domain and $\gamma \subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$.

In particular, for $\beta=\alpha$, we get

$$\left|f(w)(w-\alpha)+f(u)(\alpha-u)-\int_{\gamma} f(z)\,dz\right| \leq \int_{\gamma_{u,v}} |f'(z)|\,|z-\alpha|\,|dz| + \int_{\gamma_{v,w}} |f'(z)|\,|z-\alpha|\,|dz| \tag{7.2}$$

$$\leq \begin{cases} \|f'\|_{\gamma_{u,v},\infty} \|\ell-\alpha\|_{\gamma_{u,v},1}, \\ \\ \|f'\|_{\gamma_{u,v},p} \|\ell-\alpha\|_{\gamma_{u,v},q} \text{ if } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \|f'\|_{\gamma_{u,v},1} \|\ell-\alpha\|_{\gamma_{u,v},\infty}, \end{cases}$$

$$+\begin{cases} \|f'\|_{\gamma_{v,w},\infty} \|\ell-\alpha\|_{\gamma_{v,w},1}, \\ \\ \|f'\|_{\gamma_{v,w},p} \|\ell-\alpha\|_{\gamma_{v,w},q} \text{ if } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \|f'\|_{\gamma_{v,w},1} \|\ell-\alpha\|_{\gamma_{v,w},\infty}. \end{cases}$$

$$\leq \begin{cases} \|f'\|_{\gamma_{u,w},\infty} \|\ell-\alpha\|_{\gamma_{u,w},1}, \\ \\ \|f'\|_{\gamma_{u,w},p} \|\ell-\alpha\|_{\gamma_{u,w},q} \text{ if } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \|f'\|_{\gamma_{u,w},1} \|\ell-\alpha\|_{\gamma_{u,w},\infty}. \end{cases}$$

We assume that the path $\gamma \subset D$ belongs to the class $\bar{\Delta}(\phi,\Phi)$ for ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, if

$$\left|w-\frac{\phi+\Phi}{2}\right| \leq \frac{1}{2}|\Phi-\phi| \text{ for any } w \in \gamma$$

that is equivalent to

$$\operatorname{Re}\left[(\Phi-w)\left(\overline{w}-\overline{\phi}\right)\right] \geq 0 \text{ for any } w \in \gamma.$$

Under this assumption for γ and f is holomorphic in D, then by (4.6) we get

$$\left|f(w)\left(w-\frac{\phi+\Phi}{2}\right)+f(u)\left(\frac{\phi+\Phi}{2}-u\right)-\int_{\gamma}f(z)\,dz\right| \le \int_{\gamma_{u,w}}|f'(z)|\left|z-\frac{\phi+\Phi}{2}\right||dz| \le \frac{1}{2}|\Phi-\phi|\int_{\gamma_{u,w}}|f'(z)||dz|. \tag{7.3}$$

From Theorem 2 we have for $v\in\gamma$ that

$$\begin{aligned}
&\left|f(w)(w-v)+f(u)(v-u)-\int_{\gamma}f(z)\,dz\right| \\
&\le \int_{\gamma_{u,v}}|f'(z)||v-z||dz|+\int_{\gamma_{v,w}}|f'(z)||z-v||dz| \\
&\le \begin{cases} \max_{z\in\gamma_{u,v}}|f'(z)|\int_{\gamma_{u,v}}|v-z||dz|, \\ \left(\int_{\gamma_{u,v}}|f'(z)|^p|dz|\right)^{1/p}\left(\int_{\gamma_{u,v}}|v-z|^q|dz|\right)^{1/q} \\ p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \max_{z\in\gamma_{u,v}}|v-z|\int_{\gamma_{u,v}}|f'(z)||dz| \end{cases} \\
&+\begin{cases} \max_{z\in\gamma_{v,w}}|f'(z)|\int_{\gamma_{v,w}}|z-v||dz|, \\ \left(\int_{\gamma_{v,w}}|f'(z)|^p|dz|\right)^{1/p}\left(\int_{\gamma_{v,w}}|z-v|^q|dz|\right)^{1/q} \\ p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \max_{z\in\gamma_{v,w}}|z-v|\int_{\gamma_{v,w}}|f'(z)||dz|. \end{cases} \\
&\le \begin{cases} \max_{z\in\gamma_{u,w}}|f'(z)|\int_{\gamma_{u,w}}|z-v||dz| \\ \left(\int_{\gamma_{u,w}}|f'(z)|^p|dz|\right)^{1/p}\left(\int_{\gamma_{u,w}}|z-v|^q|dz|\right)^{1/q} \\ p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \max_{z\in\gamma_{u,w}}|z-v|\int_{\gamma_{u,w}}|f'(z)||dz| \end{cases}
\end{aligned} \tag{7.4}$$

provided that f is holomorphic in D.

From Theorem 3 we have

$$\left|[(1-s)f(w)+sf(u)](w-u)-\int_{\gamma}f(z)\,dz\right| \le (1-s)\int_{\gamma}|f'(z)||z-u||dz|+s\int_{\gamma}|f'(z)||w-z||dz| \tag{7.5}$$

$$\leq \begin{cases} \max_{z\in\gamma} |f'(z)| \left[(1-s)\int_\gamma |z-u|\,|dz| + s\int_\gamma |w-z|\,|dz|\right] \\ \left(\int_\gamma |f'(z)|^p\,|dz|\right)^{1/p} \left[(1-s)\left(\int_\gamma |z-u|^q\,|dz|\right)^{1/q} + s\left(\int_\gamma |w-z|^q\,|dz|\right)^{1/q}\right], \\ p,q>1 \text{ where } \frac{1}{p}+\frac{1}{q}=1; \\ \int_\gamma |f'(z)|\,|dz| \left[(1-s)\max_{z\in\gamma}|z-u| + s\max_{z\in\gamma}|w-z|\right] \end{cases}$$

and, in particular,

$$\left|\frac{f(w)+f(u)}{2}(w-u) - \int_\gamma f(z)\,dz\right| \tag{7.6}$$

$$\leq \frac{1}{2}\left[\int_\gamma |f'(z)|\,|z-u|\,|dz| + \int_\gamma |f'(z)|\,|w-z|\,|dz|\right]$$

$$\leq \frac{1}{2}\begin{cases} \max_{z\in\gamma} |f'(z)| \left[\int_\gamma [|z-u|+|w-z|]\,|dz|\right] \\ \left(\int_\gamma |f'(z)|^p\,|dz|\right)^{1/p} \left[\left(\int_\gamma |z-u|^q\,|dz|\right)^{1/q} + \left(\int_\gamma |w-z|^q\,|dz|\right)^{1/q}\right], \\ p,q>1 \text{ where } \frac{1}{p}+\frac{1}{q}=1; \\ \int_\gamma |f'(z)|\,|dz| \left[\max_{z\in\gamma}|z-u| + \max_{z\in\gamma}|w-z|\right]. \end{cases}$$

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