INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR L-BOUNDED NORM WEAK CONVEX MAPPINGS

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Abstract. In this paper we introduce a class of functions that extends the concept of Lipschitzian function and called them L-bounded norm weak convex functions. Integral inequalities of Hermite-Hadamard type are obtained and applications for discrete inequalities of Jensen type are provided as well.

1. Introduction

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

One of the central problems in perturbation theory is to find bounds for

$$\|f(A) - f(B)\|$$

in terms of $\|A - B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [5], [34] and the references therein.

It is known that [4] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L > 0$ such that

$$\||A| - |B|\| \leq L \|A - B\|$$

for any A, $B \in \mathcal{B}(H)$.

However, as shown by Farforovskaya in [32], [33] and Kato in [39], the following inequality holds

$$\||A| - |B|\| \leq \frac{2}{\pi} \|A - B\| \left(2 + \log\left(\frac{\|A\| + \|B\|}{\|A - B\|}\right)\right) \tag{1.1}$$

for any $A, B \in \mathcal{B}(H)$ with $A \neq B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS} := (\operatorname{tr} C^*C)^{1/2}$ of an operator C, then the following inequality is true [2]

$$\||A| - |B|\|_{HS} \leq \sqrt{2} \|A - B\|_{HS} \tag{1.2}$$

for any $A, B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

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It has been shown in [4] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\| |A| - |B| \| \leq a_1 \|A - B\| + a_2 \|A - B\|^2 + O\left(\|A - B\|^3\right) \tag{1.3}$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2 .$$

In [3] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \leq f'(a) \|A - B\| \tag{1.4}$$

where f is an *operator monotone function* on $(0, \infty)$ and $A, B \geq aI_H > 0$.

Let $(X; \|\cdot\|_X)$ and $(Y; \|\cdot\|_Y)$ be two Banach spaces over the complex number field $\mathbb{C}$. Let C be a convex set in X. For any mapping $F : C \subset X \to Y$ we can consider the associated functions $\Phi_{F,x,y,\lambda}$, $\Psi_{F,x,y,\lambda} : [0,1] \to Y$, where $x, y \in C$, $\lambda \in [0,1]$, defined by [25]

$$\begin{aligned}\Phi_{F,x,y,\lambda}(t) := (1-\lambda) F[(1-t)((1-\lambda)x + \lambda y) + ty] \\ + \lambda F[(1-t)x + t((1-\lambda)x + \lambda y)]\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}\Psi_{F,x,y,\lambda}(t) := (1-\lambda) F[(1-t)((1-\lambda)x + \lambda y) + ty] \\ + \lambda F[tx + (1-t)((1-\lambda)x + \lambda y)] .\end{aligned} \tag{1.6}$$

We say that the mapping $F : B \subset X \to Y$ is *Lipschitzian* with the constant $L > 0$ on the subset B of X if

$$\|F(x) - F(y)\|_Y \leq L \|x - y\|_X \text{ for any } x, y \in B. \tag{1.7}$$

The following result holds [25]:

Theorem 1. *Let $F : C \subset X \to Y$ be a Lipschitzian mapping with the constant $L > 0$ on the convex subset C of X. If $x, y \in C$, then we have*

$$\begin{aligned}&\left\|\Lambda_{F,x,y,\lambda}(t) - \int_0^1 F[sy + (1-s)x]\, ds\right\|_Y \\ &\qquad \leq 2L\left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2\right]\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right] \|x - y\|_X\end{aligned} \tag{1.8}$$

for any $t \in [0,1]$ and $\lambda \in [0,1]$, where $\Lambda_{F,x,y,\lambda} = \Phi_{F,x,y,\lambda}$ or $\Lambda_{F,x,y,\lambda} = \Psi_{F,x,y,\lambda}$.

If we take in (1.8) $\Lambda_{F,x,y,\lambda} = \Phi_{F,x,y,\lambda}$, $\lambda = \frac{1}{2}$, then we get

$$\begin{aligned}&\left\|\frac{1}{2}\left(F\left[(1-t)\frac{x+y}{2} + ty\right] + F\left[(1-t)x + t\frac{x+y}{2}\right]\right)\right. \\ &\qquad \left. - \int_0^1 F[sy + (1-s)x]\, ds\right\| \leq \frac{1}{2}L\left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2\right] \|x - y\|_X\end{aligned} \tag{1.9}$$

for any $x, y \in C$ and $t \in [0,1]$.

If we take in (1.8) $\Lambda_{F,x,y,\lambda} = \Psi_{F,x,y,\lambda}$, $\lambda = \frac{1}{2}$, then we get

$$\begin{aligned} \text{(1.10)} \quad & \left\| \frac{1}{2}\left(F\left[(1-t)\frac{x+y}{2} + ty \right] + F\left[tx + (1-t)\frac{x+y}{2} \right] \right) \right. \\ & \left. - \int_0^1 F\left[sy + (1-s)x \right] ds \right\|_Y \le \frac{1}{2} L \left[\frac{1}{4} + \left(t - \frac{1}{2} \right)^2 \right] \|x-y\|_X \end{aligned}$$

for any $t \in [0,1]$ and $x, y \in C$.

We also have the simpler inequalities

$$\begin{aligned} \text{(1.11)} \quad & \left\| \frac{1}{2}\left[F\left(\frac{3x+y}{4} \right) + F\left(\frac{x+3y}{4} \right) \right] - \int_0^1 F\left[sy + (1-s)x \right] ds \right\|_Y \\ & \le \frac{1}{8} L \|x-y\|_X , \end{aligned}$$

$$\text{(1.12)} \quad \left\| F\left(\frac{x+y}{2} \right) - \int_0^1 F\left[sy + (1-s)x \right] ds \right\|_Y \le \frac{1}{4} L \|x-y\|_X$$

and

$$\text{(1.13)} \quad \left\| \frac{1}{2}\left[F(x) + F(y) \right] - \int_0^1 F\left[sy + (1-s)x \right] ds \right\|_Y \le \frac{1}{4} L \|x-y\|_X$$

for any $x, y \in C$. The constants $\frac{1}{8}$ and $\frac{1}{4}$ are best possible.

The inequalities (1.12) and (1.13) are the corresponding versions of Hermite-Hadamard inequalities for Lipschitzian functions. The scalar cases were obtained in [12] and [43]. For Hermite-Hadamard's type inequalities, see for instance [10], [12], [13], [35], [37], [38], [40], [42], [43], [46], [47], [48], [49], [50] and the references therein.

From (1.8) we also have the Ostrowski's inequality

$$\begin{aligned} \text{(1.14)} \quad & \left\| F\left[ty + (1-t)x \right] - \int_0^1 F\left[sy + (1-s)x \right] ds \right\|_Y \\ & \le L \left[\frac{1}{4} + \left(t - \frac{1}{2} \right)^2 \right] \|x-y\|_X \end{aligned}$$

for any $t \in [0,1]$ and x, $y \in C$. For Ostrowski's type inequalities for the Lebesgue integral, see [1], [8]-[9] and [15]-[30]. Inequalities for the Riemann-Stieltjes integral may be found in [17], [19] while the generalization for isotonic functionals was provided in [20]. For the case of functions of self-adjoint operators on complex Hilbert spaces, see the recent monograph [23].

Motivated by the above results, we introduce here a class of functions that extends the concept of Lipschitzian function and called them L-bounded norm weak convex functions. Integral inequalities of Hermite-Hadamard type are obtained and applications for discrete inequalities of Jensen type are provided as well.

2. L-Bounded Norm Weak Convex Mappings

Let $(X; \|\cdot\|_X)$ and $(Y; \|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$. Let C be a convex set in X. We consider the following class of functions:

Definition 1. *A mapping $F : C \subset X \to Y$ is called L-bounded norm weak convex, for some given $L > 0$, if it satisfies the condition*

$$\left\|(1-\lambda) F(x)+\lambda F(y)-F((1-\lambda) x+\lambda y)\right\|_Y \leq L \lambda(1-\lambda)\|x-y\|_X \tag{2.1}$$

for any x, $y \in C$ and $\lambda \in [0,1]$. For simplicity, we denote this by $F \in \mathcal{BNW}_L(C)$.

We have from (2.1) for $\lambda = \frac{1}{2}$ the Jensen's inequality

$$\left\|\frac{F(x)+F(y)}{2}-F\left(\frac{x+y}{2}\right)\right\|_Y \leq \frac{1}{4} L\|x-y\|_X \tag{2.2}$$

for any x, $y \in C$.

We observe that $\mathcal{BNW}_L(C)$ is a convex subset in the linear space of all functions defined on C and with values in Y.

The following simple result holds:

Lemma 1. *If the function $F : C \subset X \to Y$ is Lipschitzian with the constant $K > 0$, then $F \in \mathcal{BNW}_L(C)$ with $L = 2K$.*

Proof. Since F is Lipschitzian, we have

$$\|F((1-\lambda) x+\lambda y)-F(x)\|_Y \leq K \lambda\|x-y\|_X$$

and

$$\|F((1-\lambda) x+\lambda y)-F(y)\|_Y \leq K(1-\lambda)\|x-y\|_X$$

for any x, $y \in C$ and $\lambda \in [0,1]$.

If we multiply the first inequality by $1-\lambda$ and the second inequality by λ and add these inequalities, we get

$$\begin{aligned}(1-\lambda)\|F((1-\lambda) x+\lambda y)-F(x)\|_Y+\lambda\|F((1-\lambda) x+\lambda y)-F(y)\|_Y \\ \leq 2 K \lambda(1-\lambda)\|x-y\|_X\end{aligned}$$

for any x, $y \in C$ and $\lambda \in [0,1]$.

We also have

$$\begin{aligned}&(1-\lambda)\|F((1-\lambda) x+\lambda y)-F(x)\|_Y+\lambda\|F((1-\lambda) x+\lambda y)-F(y)\|_Y \\ &\geq\|(1-\lambda) F((1-\lambda) x+\lambda y)-(1-\lambda) F(x)+\lambda F((1-\lambda) x+\lambda y)-\lambda F(y)\|_Y \\ &=\|F((1-\lambda) x+\lambda y)-(1-\lambda) F(x)-\lambda F(y)\|,\end{aligned}$$

which proves that

$$\|(1-\lambda) F(x)+\lambda F(y)-F((1-\lambda) x+\lambda y)\| \leq 2 K \lambda(1-\lambda)\|x-y\|_X$$

for any x, $y \in C$ and $\lambda \in [0,1]$, namely $F \in \mathcal{BNW}_L(C)$ with $L = 2K$. □

We observe also that, by the triangle inequality, we have

$$\begin{aligned}&\|F((1-\lambda) x+\lambda y)\|_Y-\|(1-\lambda) F(x)+\lambda F(y)\|_Y \\ &\leq\|(1-\lambda) F(x)+\lambda F(y)-F((1-\lambda) x+\lambda y)\|_Y\end{aligned} \tag{2.3}$$

and by (2.1) we get

$$\|F((1-\lambda) x+\lambda y)\|_Y-\|(1-\lambda) F(x)+\lambda F(y)\|_Y \leq L \lambda(1-\lambda)\|x-y\|_X,$$

which, again, by the triangle inequality gives

$$\begin{aligned}&\|F((1-\lambda) x+\lambda y)\|_Y \\ &\leq L \lambda(1-\lambda)\|x-y\|_X+(1-\lambda)\|F(x)\|_Y+\lambda\|F(y)\|_Y\end{aligned} \tag{2.4}$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

Now, if the function $t \mapsto \|F((1-\lambda)x + \lambda y)\|_Y$, for some x, $y \in C$, is Lebesgue integrable on $[0,1]$, then by taking the integral in (2.4) we get

$$\int_0^1 \|F((1-\lambda)x+\lambda y)\|_Y \, d\lambda \le L\|x-y\|_X \int_0^1 \lambda(1-\lambda)\,d\lambda + \|F(x)\|_Y \int_0^1 (1-\lambda)\,d\lambda + \|F(y)\|_Y \int_0^1 \lambda d\lambda \tag{2.5}$$

and since

$$\int_0^1 \lambda(1-\lambda)\,d\lambda = \frac{1}{6}, \quad \int_0^1 (1-\lambda)\,d\lambda = \int_0^1 \lambda d\lambda = \frac{1}{2},$$

then we get from (2.5) that

$$\int_0^1 \|F((1-\lambda)x+\lambda y)\|_Y \, d\lambda \le \frac{1}{6}L\|x-y\|_X + \frac{1}{2}\left[\|F(x)\|_Y + \|F(y)\|_Y\right]. \tag{2.6}$$

If we assume continuity for the function F on C in the norm topology of $(X; \|\cdot\|_X)$, then the inequality (2.6) holds for any x, $y \in C$. Moreover, if we assume that $(Y; \|\cdot\|_Y)$ is a Banach space and F is continuos on C, then we have the generalized triangle inequality

$$\left\|\int_0^1 F((1-\lambda)x+\lambda y)\,d\lambda\right\|_Y \le \int_0^1 \|F((1-\lambda)x+\lambda y)\|_Y \, d\lambda,$$

and by (2.6) we get

$$\left\|\int_0^1 F((1-\lambda)x+\lambda y)\,d\lambda\right\|_Y \le \frac{1}{6}L\|x-y\|_X + \frac{1}{2}\left[\|F(x)\|_Y + \|F(y)\|_Y\right] \tag{2.7}$$

for any x, $y \in C$.

We have the following results:

Theorem 2. *Let $(X; \|\cdot\|_X)$ and $(Y; \|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$ with Y complete. Assume that the mapping $F : C \subset X \to Y$ is continuous on the convex set C in the norm topology. If $F \in \mathcal{BNW}_L(C)$ for some $L > 0$, then we have*

$$\left\|\frac{F(x)+F(y)}{2} - \int_0^1 F((1-\lambda)x+\lambda y)\,d\lambda\right\|_Y \le \frac{1}{6}L\|x-y\|_X \tag{2.8}$$

and

$$\left\|\int_0^1 F((1-\lambda)x+\lambda y)\,d\lambda - F\left(\frac{x+y}{2}\right)\right\|_Y \le \frac{1}{8}L\|x-y\|_X \tag{2.9}$$

for any x, $y \in C$.

The constants $\frac{1}{6}$ and $\frac{1}{8}$ are best possible.

Proof. From (2.1) we have successively

$$\begin{aligned}\left\|\int_0^1 \left[(1-\lambda) F(x)+\lambda F(y)-F((1-\lambda) x+\lambda y)\right] d\lambda\right\|_Y & \\ \leq \int_0^1 \|(1-\lambda) F(x)+\lambda F(y)-F((1-\lambda) x+\lambda y)\|_Y d\lambda & \\ \leq L\|x-y\|_X \int_0^1 \lambda(1-\lambda) d\lambda=\frac{1}{6} L\|x-y\|_X &\end{aligned}$$

which produces the desired result (2.8).

Utilising (2.2) we have

$$\begin{aligned}(2.10)\quad \left\|\frac{F((1-\lambda) x+\lambda y)+F(\lambda x+(1-\lambda) y)}{2}-F\left(\frac{x+y}{2}\right)\right\|_Y & \\ \leq \frac{1}{4} L\|(1-\lambda) x+\lambda y-\lambda x-(1-\lambda) y\|_X=\frac{1}{2} K\left|\lambda-\frac{1}{2}\right|\|x-y\|_X &\end{aligned}$$

for any $x,\, y \in C$ and $\lambda \in [0,1]$.

Integrating in (2.10) we get

$$\begin{aligned}(2.11)\quad \left\|\int_0^1\left[\frac{F((1-\lambda) x+\lambda y)+F(\lambda x+(1-\lambda) y)}{2}-F\left(\frac{x+y}{2}\right)\right] d\lambda\right\|_Y & \\ \leq \int_0^1\left\|\frac{F((1-\lambda) x+\lambda y)+F(\lambda x+(1-\lambda) y)}{2}-F\left(\frac{x+y}{2}\right)\right\|_Y d\lambda & \\ \leq \frac{1}{2} K\|x-y\|_X \int_0^1\left|\lambda-\frac{1}{2}\right| d\lambda=\frac{1}{8} K\|x-y\|_X &\end{aligned}$$

and since

$$\int_0^1 F((1-\lambda) x+\lambda y) d\lambda=\int_0^1 F(\lambda x+(1-\lambda) y) d\lambda,$$

then from (2.11) we get (2.9).

Now, consider the function $F_0 : H \to \mathbb{R},\ F_0(x) = \|x\|^2$ where $(H, \langle .,. \rangle)$ is a complex inner product space. If $x,\, y \in H$ and $\lambda \in [0,1]$, then

$$\begin{aligned}(1-\lambda) F_0(x)+\lambda F_0(y)-F_0((1-\lambda) x+\lambda y) & \\ =(1-\lambda)\|x\|^2+\lambda\|y\|^2-\|(1-\lambda) x+\lambda y\|^2 & \\ =(1-\lambda)\|x\|^2+\lambda\|y\|^2-(1-\lambda)^2\|x\|^2-2(1-\lambda) \lambda \operatorname{Re}\langle x, y\rangle-\lambda^2\|y\|^2 & \\ =(1-\lambda) \lambda\left[\|x\|^2-2 \operatorname{Re}\langle x, y\rangle+\|y\|^2\right]=(1-\lambda) \lambda\|x-y\|^2 . &\end{aligned}$$

Consider C_0 a convex subset of H such that $\|x-y\| \leq 1$ for any $x, y \in C$. For instance $C_0 = B\left(0, \frac{1}{2}\right)$ is the closed ball centered in 0 and with a radius $\frac{1}{2}$. Then for all $x,\, y \in B\left(0, \frac{1}{2}\right)$ we have $\|x-y\| \leq \|x\| + \|y\| \leq \frac{1}{2} + \frac{1}{2} = 1$.

Therefore, if we consider $F_0(x) = \|x\|^2$ defined on $C_0 = B\left(0, \frac{1}{2}\right)$, we have

$$0 \leq (1-\lambda) F_0(x)+\lambda F_0(y)-F_0((1-\lambda) x+\lambda y) \leq (1-\lambda) \lambda\|x-y\|$$

which shows that $F_0 \in \mathcal{BNW}_L(C_0)$ with $L = 1$.

We have

$$\begin{aligned}\int_0^1 F_0\left((1-\lambda)x+\lambda y\right)d\lambda &= \int_0^1 \left\|(1-\lambda)x+\lambda y\right\|^2 d\lambda \\ &= \int_0^1 \left[(1-\lambda)^2\|x\|^2+2(1-\lambda)\lambda\operatorname{Re}\langle x,y\rangle+\lambda^2\|y\|^2\right]d\lambda \\ &= \frac{1}{3}\left[\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right]\end{aligned}$$

for any x, $y\in H$.

Therefore

$$\begin{aligned}&\frac{F_0(x)+F_0(y)}{2}-\int_0^1 F_0\left((1-\lambda)x+\lambda y\right)d\lambda \\ &\quad=\frac{1}{2}\left[\|x\|^2+\|y\|^2\right]-\frac{1}{3}\left[\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right]=\frac{1}{6}\|x-y\|^2.\end{aligned}$$

Now, assume that the inequality (2.8) holds with a constant $A>0$, namely

$$\left\|\frac{F(x)+F(y)}{2}-\int_0^1 F\left((1-\lambda)x+\lambda y\right)d\lambda\right\|_Y\le AL\|x-y\|_X,$$

then by taking $F_0\in\mathcal{BNW}_L(C_0)$ with $L=1$ defined above, we get

$$\frac{1}{6}\|x-y\|^2\le A\|x-y\|_X$$

namely

$$\frac{1}{6}\|x-y\|\le A. \tag{2.12}$$

If $e\in H$ with $\|e\|=1$, then $x=\frac{1}{2}e$ and $y=-\frac{1}{2}e\in B\left(0,\frac{1}{2}\right)$ giving that $x-y=e$ and by (2.12) we get $A\ge\frac{1}{6}$.

Now, consider the function $F_0:X\to[0,\infty)$, $F_0(x)=\left\|x-\frac{a+b}{2}\right\|$, with $a,b\in X$ with $a\neq b$. Then

$$|F_0(x)-F_0(y)|=\left|\left\|x-\frac{a+b}{2}\right\|-\left\|y-\frac{a+b}{2}\right\|\right|\le\|x-y\|,$$

for any x, $y\in X$, which shows that F_0 is Lipschitzian with the constant $K=1$.

By utilising Lemma 1 we conclude that $F_0\in\mathcal{BNW}_L(C)$ with $L=2$.

We have

$$\int_0^1 F_0\left((1-\lambda)a+\lambda b\right)d\lambda-F_0\left(\frac{a+b}{2}\right)=\int_0^1\left\|(1-\lambda)a+\lambda b-\frac{a+b}{2}\right\|d\lambda=\frac{1}{4}\|b-a\|,$$

which shows that the inequality (2.9) holds with equality. □

3. Related Inequalities

We have the following result as well:

Theorem 3. *Let $(X;\|\cdot\|_X)$ and $(Y;\|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$ with Y complete. Assume that the mapping $F:C\subset X\to Y$*

is continuous on the convex set C in the norm topology. If $F \in \mathcal{BNW}_L(C)$ for some $L > 0$, then we have

$$\left\| \int_0^1 F(uy + (1-u)x)\,du - \frac{1}{2\lambda - 1} \int_{1-\lambda}^{\lambda} F(sx + (1-s)y)\,ds \right\|_F \leq \frac{1}{2} L\lambda(1-\lambda)\|y - x\|_X \tag{3.1}$$

for any $\lambda \in [0,1]$, $\lambda \neq \frac{1}{2}$ and $x, y \in C$.

Proof. Since $F \in \mathcal{BNW}_L(C)$ for $K > 0$, then

$$\|(1-\lambda)F(u) + \lambda F(v) - F((1-\lambda)u + \lambda v)\|_Y \leq L\lambda(1-\lambda)\|u - v\|_X \tag{3.2}$$

for any $u, v \in C$ and $\lambda \in [0,1]$.

Let $t \in [0,1]$ and for $x, y \in C$, take

$$u = (1-t)((1-\lambda)x + \lambda y) + ty, \;\; v = tx + (1-t)((1-\lambda)x + \lambda y) \in C$$

in (3.2) to get

$$\begin{aligned} &\|(1-\lambda)F((1-t)((1-\lambda)x + \lambda y) + ty) \\ &\qquad + \lambda F(tx + (1-t)((1-\lambda)x + \lambda y)) \\ &-F((1-\lambda)[(1-t)((1-\lambda)x + \lambda y) + ty] + \lambda[tx + (1-t)((1-\lambda)x + \lambda y)])\|_Y \\ &\leq L\lambda(1-\lambda)\|(1-t)((1-\lambda)x + \lambda y) + ty - [tx + (1-t)((1-\lambda)x + \lambda y)]\|_X. \end{aligned} \tag{3.3}$$

Observe that

$$\begin{aligned} &(1-\lambda)[(1-t)((1-\lambda)x + \lambda y) + ty] + \lambda[tx + (1-t)((1-\lambda)x + \lambda y)] \\ &= (1-\lambda)(1-t)((1-\lambda)x + \lambda y) + (1-\lambda)ty \\ &\quad + \lambda tx + \lambda(1-t)((1-\lambda)x + \lambda y) \\ &= (1-t)((1-\lambda)x + \lambda y) + (1-\lambda)ty + \lambda tx \\ &= [(1-t)(1-\lambda) + \lambda t]x + [(1-t)\lambda + (1-\lambda)t]y \end{aligned}$$

and

$$\begin{aligned} &(1-t)((1-\lambda)x + \lambda y) + ty - [tx + (1-t)((1-\lambda)x + \lambda y)] \\ &= (1-t)(1-\lambda)x + (1-t)\lambda y + ty - tx - (1-t)(1-\lambda)x - (1-t)\lambda y = t(y - x). \end{aligned}$$

Then by (3.3) we have

$$\begin{aligned} &\|(1-\lambda)F((1-t)((1-\lambda)x + \lambda y) + ty) \\ &\qquad + \lambda F(tx + (1-t)((1-\lambda)x + \lambda y)) \\ &\qquad -F([(1-t)(1-\lambda) + \lambda t]x + [(1-t)\lambda + (1-\lambda)t]y)\|_Y \\ &\qquad\qquad \leq L\lambda(1-\lambda)t\|y - x\|_X, \end{aligned} \tag{3.4}$$

for any $t, \lambda \in [0,1]$ and $x, y \in C$.

Integrating the inequality (3.4) over t on $[0,1]$ and using the generalized triangle inequality for norms and integrals, we get

$$\begin{aligned}(3.5)\quad & \left\|(1-\lambda)\int_0^1 F\left((1-t)\left((1-\lambda)x+\lambda y\right)+ty\right)dt\right. \\ &\quad +\lambda\int_0^1 F\left(tx+(1-t)\left((1-\lambda)x+\lambda y\right)\right)dt \\ &\quad \left.-\int_0^1 F\left(\left[(1-t)(1-\lambda)+\lambda t\right]x+\left[(1-t)\lambda+(1-\lambda)t\right]y\right)dt\right\|_Y \\ &\qquad \le \frac{1}{2}L\lambda(1-\lambda)\|y-x\|_X,\end{aligned}$$

for any $\lambda\in[0,1]$ and $x,\,y\in C$.

Observe that

$$\begin{aligned}(3.6)\quad & \int_0^1 F\left[(1-t)\left(\lambda y+(1-\lambda)x\right)+ty\right]dt \\ &\qquad = \int_0^1 F\left[\left((1-t)\lambda+t\right)y+(1-t)(1-\lambda)x\right]dt\end{aligned}$$

and

$$\begin{aligned}(3.7)\quad & \int_0^1 F\left(tx+(1-t)\left((1-\lambda)x+\lambda y\right)\right)dt \\ &= \int_0^1 F\left((1-t)x+t\left((1-\lambda)x+\lambda y\right)\right)dt=\int_0^1 F\left[t\lambda y+(1-\lambda t)x\right]dt.\end{aligned}$$

If we make the change of variable $u:=(1-t)\lambda+t$ then we have $1-u=(1-t)(1-\lambda)$ and $du=(1-\lambda)\,du$. Then

$$\int_0^1 F\left[\left((1-t)\lambda+t\right)y+(1-t)(1-\lambda)x\right]dt=\frac{1}{1-\lambda}\int_\lambda^1 F\left[uy+(1-u)x\right]du.$$

If we make the change of variable $u:=\lambda t$ then we have $du=\lambda dt$ and

$$\int_0^1 F\left[t\lambda y+(1-\lambda t)x\right]dt=\frac{1}{\lambda}\int_0^\lambda F\left[uy+(1-u)x\right]du.$$

Therefore

$$\begin{aligned}&(1-\lambda)\int_0^1 F\left[(1-t)\left(\lambda y+(1-\lambda)x\right)+ty\right]dt \\ &\qquad +\lambda\int_0^1 F\left[t\left(\lambda y+(1-\lambda)x\right)+(1-t)x\right]dt \\ &=\int_\lambda^1 F\left[uy+(1-u)x\right]du+\int_0^\lambda F\left[uy+(1-u)x\right]du=\int_0^1 F\left[uy+(1-u)x\right]du,\end{aligned}$$

and we have the simple equality

$$\begin{aligned}(3.8)\quad & (1-\lambda)\int_0^1 F\left((1-t)\left((1-\lambda)x+\lambda y\right)+ty\right)dt \\ &\quad +\lambda\int_0^1 F\left(tx+(1-t)\left((1-\lambda)x+\lambda y\right)\right)dt=\int_0^1 F\left[uy+(1-u)x\right]du\end{aligned}$$

for any $\lambda \in [0,1]$ and $x, y \in C$.

Consider now the integral

$$\int_0^1 F\left(\left[(1-t)(1-\lambda)+\lambda t\right] x+\left[(1-t)\lambda+(1-\lambda) t\right] y\right) dt.$$

Put

$$s=(1-t)(1-\lambda)+\lambda t=1-\lambda+(2\lambda-1) t.$$

Then

$$1-s=(1-t)\lambda+(1-\lambda) t.$$

If $\lambda \neq \frac{1}{2}$, then $s=1-\lambda+(2\lambda-1) t$ is a change of variable with $dt=\frac{1}{2\lambda-1}$ and we have

$$\begin{aligned}
&\int_0^1 F\left(\left[(1-t)(1-\lambda)+\lambda t\right] x+\left[(1-t)\lambda+(1-\lambda) t\right] y\right) dt \\
&\qquad\qquad=\frac{1}{2\lambda-1}\int_{1-\lambda}^{\lambda} F(sx+(1-s) y)\, ds.
\end{aligned}$$

Now, making use of (3.5) we get the desired result (3.1). □

Remark 1. *We observe that for $\lambda \to \frac{1}{2}$ we recapture from (3.1) the inequality (2.9). If we take in (3.1) $\lambda=\frac{3}{4}$, then we get*

$$\begin{aligned}
&\left\|\int_0^1 F[uy+(1-u) x]\, du-2\int_{1/4}^{3/4} F(sx+(1-s) y)\, ds\right\|_F \\
&\qquad\qquad\leq \frac{3}{32} L\|y-x\|_X . \qquad (3.9)
\end{aligned}$$

4. Applications for Gâteaux Differentiable Functions

Following [11, p. 59], let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be two normed linear spaces, Ω an open subset of X and $f:\Omega \to Y$. If $a \in \Omega$, $u \in X \setminus \{0\}$ and if the limit

$$\lim_{t \to 0} \frac{1}{t}[f(a+tu)-f(a)]$$

exists, then we denote this derivative $\partial_u f(a)$. It is called the directional derivative of f at a in the direction u. If the directional derivative is defined in all directions and there is a continuous linear mapping Φ from X into Y such that for all $u \in X$

$$\partial_u f(a)=\Phi(u),$$

then we say that f is *Gâteaux-differentiable* at a and that Φ is the *Gâteaux differential* of f at a. If a mapping f is differentiable at a point a, then clearly all its directional derivatives exist and we have

$$\partial_u f(a)=f'(a) u,\ u \in X.$$

Thus f is Gâteaux-differentiable at a. However, the Gâteaux differential may exist without the differential existing. The existence of directional derivatives at a point does not imply that the mapping is Gâteaux-differentiable. To distinguish the differential from the Gâteaux differential, the differential is often referred as the Fréchet differential.

Theorem 4. *Let $(X;\|\cdot\|_X)$ and $(Y;\|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$. Assume that the mapping $F: C \subset X \to Y$ is defined on the open convex set C and $F \in \mathcal{BNW}_L(C)$ for some $L>0$. If $x_k \in C$, $p_k \geq 0$ for $k \in \{1,...,n\}$ with $\sum_{k=1}^n p_k = 1$ and F is Gâteaux-differentiable at $\sum_{k=1}^n p_k x_k \in C$, then for any $y_j \in C$ and $q_j \geq 0$ for $j \in \{1,...,m\}$ with $\sum_{j=1}^m q_j = 1$ and $\sum_{j=1}^m q_j y_j = \sum_{k=1}^n p_k x_k$ we have*

$$\left\|\sum_{j=1}^m q_j F(y_j) - F\left(\sum_{k=1}^n p_k x_k\right)\right\|_Y \leq L \sum_{j=1}^m q_j \left\|y_j - \sum_{k=1}^n p_k x_k\right\|_X . \tag{4.1}$$

In particular, we have

$$\left\|\sum_{j=1}^n p_j F(x_j) - F\left(\sum_{k=1}^n p_k x_k\right)\right\|_Y \leq L \sum_{j=1}^n p_j \left\|x_j - \sum_{k=1}^n p_k x_k\right\|_X . \tag{4.2}$$

Proof. Since $F \in \mathcal{BNW}_L(C)$ then we have

$$\|\lambda[F(y)-F(x)] + F(x) - F((1-\lambda)x + \lambda y)\|_Y \leq L\lambda(1-\lambda)\|x-y\|_X$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

This implies that

$$\left\|F(y) - F(x) - \frac{F(x+\lambda(y-x)) - F(x)}{\lambda}\right\|_Y \leq L(1-\lambda)\|x-y\|_X \tag{4.3}$$

for any $x, y \in C$ and $\lambda \in (0,1)$.

If we assume that F is Gâteaux-differentiable at x, then by taking the limit over $\lambda \to 0+$ in (4.3) we get

$$\|F(y) - F(x) - \partial_{y-x}F(x)\|_Y \leq L\|x-y\|_X \tag{4.4}$$

for any $x, y \in C$.

Now, if F is Gâteaux-differentiable at $\sum_{k=1}^n p_k x_k \in C$, then

$$\left\|F(y) - F\left(\sum_{k=1}^n p_k x_k\right) - \partial_{y-\sum_{k=1}^n p_k x_k} F\left(\sum_{k=1}^n p_k x_k\right)\right\|_Y \leq L\left\|\sum_{k=1}^n p_k x_k - y\right\|_X \tag{4.5}$$

for any $y \in C$.

If $y_j \in C$ and $q_j \geq 0$ for $j \in \{1,...,m\}$ with $\sum_{j=1}^m q_j = 1$, then by (4.5) we have

$$\sum_{j=1}^m q_j \left\|F(y_j) - F\left(\sum_{k=1}^n p_k x_k\right) - \partial_{y_j-\sum_{k=1}^n p_k x_k} F\left(\sum_{k=1}^n p_k x_k\right)\right\|_Y \leq L\sum_{j=1}^m q_j \left\|\sum_{k=1}^n p_k x_k - y_j\right\|_X . \tag{4.6}$$

By the generalized triangle inequality we have

$$\begin{aligned}(4.7)\quad & \left\|\sum_{j=1}^{m}q_jF\left(y_j\right)-F\left(\sum_{k=1}^{n}p_kx_k\right)-\partial_{\sum_{j=1}^{m}q_jy_j-\sum_{k=1}^{n}p_kx_k}F\left(\sum_{k=1}^{n}p_kx_k\right)\right\|_Y \\ & \leq\sum_{j=1}^{m}q_j\left\|F\left(y_j\right)-F\left(\sum_{k=1}^{n}p_kx_k\right)-\partial_{y_j-\sum_{k=1}^{n}p_kx_k}F\left(\sum_{k=1}^{n}p_kx_k\right)\right\|_Y\end{aligned}$$

and by (4.6) and (4.7) we have the following inequality of interest

$$\begin{aligned}(4.8)\quad & \left\|\sum_{j=1}^{m}q_jF\left(y_j\right)-F\left(\sum_{k=1}^{n}p_kx_k\right)-\partial_{\sum_{j=1}^{m}q_jy_j-\sum_{k=1}^{n}p_kx_k}F\left(\sum_{k=1}^{n}p_kx_k\right)\right\|_Y \\ & \leq L\sum_{j=1}^{m}q_j\left\|\sum_{k=1}^{n}p_kx_k-y_j\right\|_X.\end{aligned}$$

If we take $\sum_{j=1}^{m}q_jy_j=\sum_{k=1}^{n}p_kx_k$ in (4.8), then we get the desired inequality (4.1).

The inequality (4.2) follows by (4.1) on taking $m=n$ and $q_j=p_j$, $j\in\{1,...,n\}$. □

We also have:

Theorem 5. *Let $(X;\|\cdot\|_X)$ and $(Y;\|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$. Assume that the mapping $F:C\subset X\to Y$ is defined on the open convex set C and $F\in\mathcal{BNW}_L(C)$ for some $L>0$. Let $x_k\in C$, $p_k\geq 0$ for $k\in\{1,...,n\}$ with $\sum_{k=1}^{n}p_k=1$ and F is Gâteaux-differentiable at x_k for any $k\in\{1,...,n\}$. If there exists $z\in C$ such that*

$$(4.9)\qquad \sum_{k=1}^{n}p_k\partial_zF\left(x_k\right)=\sum_{k=1}^{n}p_k\partial_{x_k}F\left(x_k\right),$$

then we have

$$(4.10)\qquad \left\|F\left(z\right)-\sum_{k=1}^{n}p_kF\left(x_k\right)\right\|_Y\leq L\sum_{k=1}^{n}p_k\left\|x_k-z\right\|_X.$$

Proof. From (4.4) we have

$$(4.11)\qquad \left\|F\left(y\right)-F\left(x_k\right)-\partial_{y-x_k}F\left(x_k\right)\right\|_Y\leq L\left\|x_k-y\right\|_X$$

for any $y\in C$ and for any $k\in\{1,...,n\}$.

If we multiply (4.11) by $p_k\geq 0$ for $k\in\{1,...,n\}$ and sum, we get

$$(4.12)\qquad \sum_{k=1}^{n}p_k\left\|F\left(y\right)-F\left(x_k\right)-\partial_{y-x_k}F\left(x_k\right)\right\|_Y\leq L\sum_{k=1}^{n}p_k\left\|x_k-y\right\|_X$$

for any $y\in C$.

By the generalized triangle inequality we get

$$\sum_{k=1}^{n} p_k \left\| F(y) - F(x_k) - \partial_{y-x_k} F(x_k) \right\|_Y \geq \left\| F(y) - \sum_{k=1}^{n} p_k F(x_k) - \sum_{k=1}^{n} p_k \partial_{y-x_k} F(x_k) \right\|_Y . \tag{4.13}$$

By the linearity of the Gâteaux differential we have

$$\sum_{k=1}^{n} p_k \partial_{y-x_k} F(x_k) = \sum_{k=1}^{n} p_k \partial_y F(x_k) - \sum_{k=1}^{n} p_k \partial_{x_k} F(x_k)$$

and by (4.12) and (4.13) we have the inequality of interest

$$\left\| F(y) - \sum_{k=1}^{n} p_k F(x_k) - \sum_{k=1}^{n} p_k \partial_y F(x_k) + \sum_{k=1}^{n} p_k \partial_{x_k} F(x_k) \right\|_Y \leq L \sum_{k=1}^{n} p_k \left\| x_k - y \right\|_X \tag{4.14}$$

for any $y \in C$.

Now, if $z \in C$ is such that (4.9) holds, then by (4.14) we get the desired result (4.10). □

Remark 2. *Let $x_k \in C$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$ and F is differentiable at x_k for any $k \in \{1, ..., n\}$. If there exists $z \in C$ such that*

$$\sum_{k=1}^{n} p_k F'(x_k) z = \sum_{k=1}^{n} p_k F(x_k) x_k , \tag{4.15}$$

then we have the inequality (4.10).

Moreover, if the operator $\sum_{k=1}^{n} p_k F'(x_k)$ is invertible and

$$z := \left(\sum_{k=1}^{n} p_k F'(x_k) \right)^{-1} \left(\sum_{k=1}^{n} p_k F(x_k) x_k \right) \in C, \tag{4.16}$$

then we have the inequality

$$\left\| F\left(\left(\sum_{k=1}^{n} p_k F'(x_k) \right)^{-1} \left(\sum_{k=1}^{n} p_k F(x_k) x_k \right) \right) - \sum_{k=1}^{n} p_k F(x_k) \right\|_Y \leq L \sum_{k=1}^{n} p_k \left\| x_k - \left(\sum_{k=1}^{n} p_k F'(x_k) \right)^{-1} \left(\sum_{k=1}^{n} p_k F(x_k) x_k \right) \right\|_X . \tag{4.17}$$

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