SOME WEIGHTED INEQUALITIES FOR THE COMPLEX INTEGRAL (II)

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Abstract. In this paper that is a continuation of the one with (I) in the title, we provide some upper bounds for the magnitude of the error in approximating the weighted integral

$$\int_{\gamma} f(z) g(z) dz$$

with the simple quantity

$$(1-s)\{f(w)[G(w)-G(v)]+f(u)[G(v)-G(u)]\} + s[G(w)-G(u)] f(v), \ s \in [0,1]$$

under the assumptions that f and g are holomorphic functions in D, an open domain, $\gamma \subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x \in (a,b)$ while G is a primitive for the function g on γ. Some particular results for certain selections of $s \in [0,1]$ are also given.

1. Introduction

Suppose γ is a smooth path from $\mathbb{C}$ parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a)=u$ and $z(b)=w$ with u, $w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w}=\gamma$ as

$$\int_{\gamma} f(z) dz = \int_{\gamma_{u,w}} f(z) dz := \int_a^b f(z(t)) z'(t) dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z) dz := \int_{\gamma_{u,v}} f(z) dz + \int_{\gamma_{v,w}} f(z) dz$$

where $v := zz$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z) |dz| := \int_a^b f(z(t)) |z'(t)| dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)| dt.$$

1991 *Mathematics Subject Classification.* 26D15, 26D10, 30A10, 30A86.

Key words and phrases. Complex integral, Continuous functions, Holomorphic functions, Three points inequalities, Ostrowski inequality, Trapezoid inequality.

RGMIA Res. Rep. Coll. 21 (2018), Art. 129, 21 pp. Received 29/10/18

Let f and g be holomorphic in D, and open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z) g'(z) dz = f(w) g(w) - f(u) g(u) - \int_{\gamma_{u,w}} f'(z) g(z) dz. \tag{1.1}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left| \int_{\gamma} f(z) dz \right| \leq \int_{\gamma} |f(z)| |dz| \leq \|f\|_{\gamma,\infty} \ell(\gamma) \tag{1.2}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \geq 1$ by

$$\|f\|_{\gamma,p} := \left(\int_{\gamma} |f(z)|^p |dz| \right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_{\gamma} |f(z)| |dz|.$$

If $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \leq [\ell(\gamma)]^{1/q} \|f\|_{\gamma,p}.$$

Suppose that a continuous function g on γ has a *primitive* on γ, namely a function G analytic on γ such that $G'(z) = g(z)$ for all $z \in \gamma$. Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a, b]$. Put $z(a) = u$ and $z(b) = w$ with $u,\ w \in \mathbb{C}$. Then

$$\int_{\gamma} g(z) dz = \int_a^b g(z(t)) z'(t) dt = \int_a^b (G(z(t)))' dt = G(w) - G(u).$$

In this paper we provide some upper bounds for the magnitude of the error in approximating the weighted integral

$$\int_{\gamma} f(z) g(z) dz$$

with the simple quantity

$$\begin{aligned}(1-s)\{f(w)[G(w)-G(v)] + f(u)[G(v)-G(u)]\} \\ + s[G(w)-G(u)] f(v),\ s \in [0,1]\end{aligned}$$

under the assumptions that f and g are holomorphic functions in D, an open domain, $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v = z(x)$ with $x \in (a, b)$ while G is a primitive for the function g on γ.

For several previous results concerning three points inequalities, see [1], [2] and [9]-[15]. For some trapezoid, Ostrowski, Grüss and quasi-Grüss type inequalities for complex functions defined on the unit circle centered in zero, see [3]-[7].

2. Some Preliminary Facts

We have the following general equality of interest [8]:

Lemma 1. *Let f and g be holomorphic in D, an open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v = z(x)$ with $x \in (a,b)$. If G is a primitive for the function g on γ, then for any complex numbers α, β we have*

$$\begin{aligned} &f(w)[G(w)-\beta] + f(u)[\alpha - G(u)] + (\beta-\alpha) f(v) - \int_{\gamma} f(z) g(z)\,dz \\ &\quad = \int_{\gamma_{u,v}} f'(z)[G(z)-\alpha]\,dz + \int_{\gamma_{v,w}} f'(z)[G(z)-\beta]\,dz. \end{aligned} \tag{2.1}$$

Proof. Using the integration by parts formula, we have

$$\begin{aligned} \int_{\gamma_{u,v}} f'(z)(G(z)-\alpha)\,dz &= f(z)(G(z)-\alpha)\Big|_u^v - \int_{\gamma_{u,v}} f(z)(G(z)-\alpha)'\,dz \\ &= f(v)(G(v)-\alpha) - f(u)(G(u)-\alpha) - \int_{\gamma_{u,v}} f(z) g(z)\,dz \end{aligned}$$

and

$$\begin{aligned} \int_{\gamma_{v,w}} f'(z)(G(z)-\beta)\,dz &= f(z)(G(z)-\beta)\Big|_v^w - \int_{\gamma_{v,w}} f(z)(G(z)-\beta)'\,dz \\ &= f(w)(G(w)-\beta) - f(v)(G(v)-\beta) - \int_{\gamma_{v,w}} f(z) g(z)\,dz. \end{aligned}$$

If we add these two equalities, we get

$$\begin{aligned} &\int_{\gamma_{u,v}} f'(z)(G(z)-\alpha)\,dz + \int_{\gamma_{v,w}} f'(z)(G(z)-\beta)\,dz \\ &= f(v)(G(v)-\alpha) - f(u)(G(u)-\alpha) - \int_{\gamma_{u,v}} f(z) g(z)\,dz \\ &+ f(w)(G(w)-\beta) - f(v)(G(v)-\beta) - \int_{\gamma_{v,w}} f(z) g(z)\,dz \\ &= f(w)(G(w)-\beta) + f(u)(\alpha - G(u)) + (\beta-\alpha) f(v) - \int_{\gamma} f(z) g(z)\,dz, \end{aligned}$$

which proves the desired result (2.1). $\square$

Corollary 1. *With the assumptions of Lemma 1 and if $\beta \neq \alpha$ and $w \neq u$, then*

$$\begin{aligned}(2.2)\quad &\frac{f(w)[G(w)-\beta]+f(u)[\alpha-G(u)]}{w-u}+\left(\frac{\beta-\alpha}{w-u}\right)\frac{1}{w-u}\int_{\gamma}f(v)\,dv \\ &-\frac{1}{w-u}\int_{\gamma}f(z)\,g(z)\,dz \\ &=\frac{1}{(w-u)^{2}}\int_{\gamma}\left(\int_{\gamma_{u,v}}f'(z)[G(z)-\alpha]\,dz\right)dv \\ &+\frac{1}{(w-u)^{2}}\int_{\gamma}\left(\int_{\gamma_{v,w}}f'(z)[G(z)-\beta]\,dz\right)dv.\end{aligned}$$

Proof. Taking the integral on γ over v we have

$$\begin{aligned}&(w-u)\{f(w)[G(w)-\beta]+f(u)[\alpha-G(u)]\}+(\beta-\alpha)\int_{\gamma}f(v)\,dv \\ &-(w-u)\int_{\gamma}f(z)\,g(z)\,dz \\ &=\int_{\gamma}\left(\int_{\gamma_{u,v}}f'(z)[G(z)-\alpha]\,dz\right)dv+\int_{\gamma}\left(\int_{\gamma_{v,w}}f'(z)[G(z)-\beta]\,dz\right)dv,\end{aligned}$$

which is equivalent to (2.2). □

If we take in (2.1) $\alpha = sG(u)+(1-s)G(v)$ and $\beta=(1-s)G(v)+sG(w)$, then we get

$$\begin{aligned}(2.3)\quad &(1-s)\{f(w)[G(w)-G(v)]+f(u)[G(v)-G(u)]\} \\ &+s[G(w)-G(u)]f(v)-\int_{\gamma}f(z)\,g(z)\,dz \\ &=\int_{\gamma_{u,v}}f'(z)[G(z)-sG(u)-(1-s)G(v)]\,dz \\ &+\int_{\gamma_{v,w}}f'(z)[G(z)-(1-s)G(v)-sG(w)]\,dz\end{aligned}$$

for $s\in[0,1]$.

If we take in (2.3) $s=1$, then we get the *Montgomery type identity*

$$\begin{aligned}(2.4)\quad &[G(w)-G(u)]f(v)-\int_{\gamma}f(z)\,g(z)\,dz \\ &=\int_{\gamma_{u,v}}f'(z)[G(z)-G(u)]\,dz+\int_{\gamma_{v,w}}f'(z)[G(z)-G(w)]\,dz.\end{aligned}$$

If in (2.3) we take $s=\frac{1}{2}$, then we get

$$\begin{aligned}(2.5)\quad &\frac{1}{2}\left\{f(w)\left[G(w)-G(v)\right]+f(u)\left[G(v)-G(u)\right]+\left[G(w)-G(u)\right]f(v)\right\}\\ &\qquad -\int_{\gamma} f(z)\,g(z)\,dz\\ &=\int_{\gamma_{u,v}} f'(z)\left[G(z)-\frac{G(u)+G(v)}{2}\right]dz\\ &\qquad+\int_{\gamma_{v,w}} f'(z)\left[G(z)-\frac{G(v)+G(w)}{2}\right]dz.\end{aligned}$$

If in (2.3) we take $s=0$, we get

$$\begin{aligned}(2.6)\quad &f(w)\left[G(w)-G(v)\right]+f(u)\left[G(v)-G(u)\right]\\ &=\int_{\gamma_{u,v}} f'(z)\left[G(z)-G(v)\right]dz+\int_{\gamma_{v,w}} f'(z)\left[G(z)-G(v)\right]dz.\end{aligned}$$

If in (2.1) we take

$$\alpha=\frac{1}{v-u}\int_{\gamma_{u,v}} G(y)\,dy \text{ and } \beta=\frac{1}{w-v}\int_{\gamma_{v,w}} G(y)\,dy,$$

then we get

$$\begin{aligned}(2.7)\quad &f(w)\left[G(w)-\frac{1}{w-v}\int_{\gamma_{v,w}} G(y)\,dy\right]\\ &\qquad+f(u)\left[\frac{1}{v-u}\int_{\gamma_{u,v}} G(y)\,dy-G(u)\right]\\ &+\left(\frac{1}{w-v}\int_{\gamma_{v,w}} G(y)\,dy-\frac{1}{v-u}\int_{\gamma_{u,v}} G(y)\,dy\right)f(v)-\int_{\gamma} f(z)\,g(z)\,dz\\ &=\int_{\gamma_{u,v}} f'(z)\left[G(z)-\frac{1}{v-u}\int_{\gamma_{u,v}} G(y)\,dy\right]dz\\ &\qquad+\int_{\gamma_{v,w}} f'(z)\left[G(z)-\frac{1}{w-v}\int_{\gamma_{v,w}} G(y)\,dy\right]dz.\end{aligned}$$

Moreover, if we take the integral mean over $v \in \gamma$ in (2.3), we get

$$(1-s)\left\{f(w)\left[G(w)-\frac{1}{w-u}\int_{\gamma}G(v)\,dv\right]+f(u)\left[\frac{1}{w-u}\int_{\gamma}G(v)\,dv-G(u)\right]\right\}+s\frac{G(w)-G(u)}{w-u}\int_{\gamma}f(v)\,dv-\int_{\gamma}f(z)\,g(z)\,dz$$
$$=\frac{1}{w-u}\int_{\gamma}\left(\int_{\gamma_{u,v}}f'(z)\left[G(z)-sG(u)-(1-s)\,G(v)\right]dz\right)dv+\frac{1}{w-u}\int_{\gamma}\left(\int_{\gamma_{v,w}}f'(z)\left[G(z)-(1-s)\,G(v)-sG(w)\right]dz\right)dv. \tag{2.8}$$

for all $s \in [0,1]$.

In particular, for $s = 1$ we get

$$[G(w)-G(u)]\frac{1}{w-u}\int_{\gamma}f(v)\,dv-\int_{\gamma}f(z)\,g(z)\,dz=\frac{1}{w-u}\int_{\gamma}\left(\int_{\gamma_{u,v}}f'(z)\left[G(z)-G(u)\right]dz\right)dv+\frac{1}{w-u}\int_{\gamma}\left(\int_{\gamma_{v,w}}f'(z)\left[G(z)-G(w)\right]dz\right)dv. \tag{2.9}$$

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ, we consider the *complex Čebyšev functional* defined by

$$\mathcal{D}_{\gamma}(f,g) := \frac{1}{w-u}\int_{\gamma}f(z)\,g(z)\,dz-\frac{1}{w-u}\int_{\gamma}f(z)\,dz\frac{1}{w-u}\int_{\gamma}g(z)\,dz.$$

Now, if we use (2.9), then we get the representation

$$\mathcal{D}_{\gamma}(f,g)=\frac{1}{(w-u)^2}\int_{\gamma}\left(\int_{\gamma_{u,v}}f'(z)\left[G(z)-G(u)\right]dz\right)dv+\frac{1}{(w-u)^2}\int_{\gamma}\left(\int_{\gamma_{v,w}}f'(z)\left[G(z)-G(w)\right]dz\right)dv. \tag{2.10}$$

For $s=\frac{1}{2}$ we get from (2.8) the following equality as well:

$$\begin{aligned}(2.11)\quad & \frac{1}{2}\left\{f(w)\left[G(w)-\frac{1}{w-u}\int_{\gamma}G(v)\,dv\right]\right.\\ & \left.\qquad +f(u)\left[\frac{1}{w-u}\int_{\gamma}G(v)\,dv-G(u)\right]\right\}\\ & +\frac{1}{2}\frac{G(w)-G(u)}{w-u}\int_{\gamma}f(v)\,dv-\int_{\gamma}f(z)\,g(z)\,dz\\ & =\frac{1}{w-u}\int_{\gamma}\left(\int_{\gamma_{u,v}}f'(z)\left[G(z)-\frac{G(u)+G(v)}{2}\right]dz\right)dv\\ & \qquad +\frac{1}{w-u}\int_{\gamma}\left(\int_{\gamma_{v,w}}f'(z)\left[G(z)-\frac{G(v)+G(w)}{2}\right]dz\right)dv.\end{aligned}$$

3. Three Point Inequalities In Terms of p-Norms

We consider the norms sup-norm or ∞-norm defined by

$$\|f\|_{\gamma,\infty}:=\sup_{z\in\gamma}|f(z)|.$$

We also define the p-norm with $p\geq 1$ by

$$\|f\|_{\gamma,p}:=\left(\int_{\gamma}|f(z)|^{p}\,|dz|\right)^{1/p}.$$

For $p=1$ we have

$$\|f\|_{\gamma,1}:=\int_{\gamma}|f(z)|\,|dz|.$$

We have the following result:

Theorem 1. *Let f and g be holomorphic in D, an open domain and suppose $\gamma\subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x\in(a,b)$. If G is a primitive for the function g on γ, then for $s\in[0,1]$ we have*

$$\begin{aligned}(3.1)\quad & \left|(1-s)\left\{f(w)\int_{\gamma_{v,w}}g(z)\,dz+f(u)\int_{\gamma_{u,v}}g(z)\,dz\right\}+sf(v)\int_{\gamma_{u,w}}g(z)\,dz\right.\\ & \qquad\left.-\int_{\gamma}f(z)\,g(z)\,dz\right|\leq B(s,v)\end{aligned}$$

where

$$\begin{aligned}(3.2)\quad B(s,v):=&\int_{\gamma_{u,v}}|f'(z)|\left|s\int_{\gamma_{u,z}}g(y)\,dy-(1-s)\int_{\gamma_{z,v}}g(y)\,dy\right||dz|\\ & +\int_{\gamma_{v,w}}|f'(z)|\left|(1-s)\int_{\gamma_{v,z}}g(y)\,dy-s\int_{\gamma_{z,w}}g(y)\,dy\right||dz|.\end{aligned}$$

Moreover, we have

$$
\begin{aligned}
(3.3)\quad & B(s,v) \\
&\leq s\left\{\int_{\gamma_{u,v}}|f'(z)|\left|\int_{\gamma_{u,z}} g(y)\,dy\right||dz| + \int_{\gamma_{v,w}}|f'(z)|\left|\int_{\gamma_{z,w}} g(y)\,dy\right||dz|\right\} \\
&\qquad + (1-s)\int_{\gamma_{u,w}}|f'(z)|\left|\int_{\gamma_{z,v}} g(y)\,dy\right||dz| \\
&\leq s\begin{cases}
\max_{z\in\gamma_{u,v}}|f'(z)|\int_{\gamma_{u,v}}\left|\int_{\gamma_{u,z}} g(y)\,dy\right||dz| \\
\left(\int_{\gamma_{u,v}}|f'(z)|^p\right)^{1/p}\left(\int_{\gamma_{u,v}}\left|\int_{\gamma_{u,z}} g(y)\,dy\right|^q|dz|\right)^{1/q} \\
p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1; \\
\max_{z\in\gamma_{u,v}}\left|\int_{\gamma_{u,z}} g(y)\,dy\right|\int_{\gamma_{u,v}}|f'(z)||dz|,
\end{cases} \\
&\quad + s\begin{cases}
\max_{z\in\gamma_{v,w}}|f'(z)|\int_{\gamma_{v,w}}\left|\int_{\gamma_{z,w}} g(y)\,dy\right||dz| \\
\left(\int_{\gamma_{v,w}}|f'(z)|^p\right)^{1/p}\left(\int_{\gamma_{v,w}}\left|\int_{\gamma_{z,w}} g(y)\,dy\right|^q|dz|\right)^{1/q} \\
p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1; \\
\max_{z\in\gamma_{v,w}}\left|\int_{\gamma_{z,w}} g(y)\,dy\right|\int_{\gamma_{v,w}}|f'(z)||dz|,
\end{cases} \\
&\quad + (1-s)\begin{cases}
\max_{z\in\gamma_{u,w}}|f'(z)|\int_{\gamma_{u,w}}\left|\int_{\gamma_{z,v}} g(y)\,dy\right||dz| \\
\left(\int_{\gamma_{u,w}}|f'(z)|^p\right)^{1/p}\left(\int_{\gamma_{u,w}}\left|\int_{\gamma_{z,v}} g(y)\,dy\right|^q|dz|\right)^{1/q} \\
p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1; \\
\max_{z\in\gamma_{u,w}}\left|\int_{\gamma_{z,v}} g(y)\,dy\right|\int_{\gamma_{u,w}}|f'(z)||dz|,
\end{cases}
\end{aligned}
$$

for any $v\in\gamma$ *and* $s\in[0,1]$.

Proof. From the identity (2.3) we get

$$
\begin{aligned}
&(1-s)\{f(w)[G(w)-G(v)]+f(u)[G(v)-G(u)]\} \\
&\qquad + s[G(w)-G(u)]f(v) - \int_\gamma f(z)g(z)\,dz \\
&= \int_{\gamma_{u,v}} f'(z)[s(G(z)-G(u))+(1-s)(G(z)-G(v))]\,dz \\
&\qquad + \int_{\gamma_{v,w}} f'(z)[(1-s)(G(z)-G(v))+s(G(z)-G(w))]\,dz,
\end{aligned}
$$

that can be written in terms of integrals as

$$
\begin{aligned}
(3.4)\quad & (1-s)\left\{f(w)\int_{\gamma_{v,w}} g(z)\,dz+f(u)\int_{\gamma_{u,v}} g(z)\,dz\right\}+sf(v)\int_{\gamma_{u,w}} g(z)\,dz \\
& \qquad -\int_{\gamma} f(z)\,g(z)\,dz \\
&= \int_{\gamma_{u,v}} f'(z)\left[s\int_{\gamma_{u,z}} g(y)\,dy-(1-s)\int_{\gamma_{z,v}} g(y)\,dy\right]dz \\
&\quad +\int_{\gamma_{v,w}} f'(z)\left[(1-s)\int_{\gamma_{v,z}} g(y)\,dy-s\int_{\gamma_{z,w}} g(y)\,dy\right]dz,
\end{aligned}
$$

for $s\in[0,1]$. This is an equality of interest in itself.

Taking the modulus, we have

$$
\begin{aligned}
& \left|(1-s)\left\{f(w)\int_{\gamma_{v,w}} g(z)\,dz+f(u)\int_{\gamma_{u,v}} g(z)\,dz\right\}+sf(v)\int_{\gamma_{u,w}} g(z)\,dz\right. \\
& \qquad \left. -\int_{\gamma} f(z)\,g(z)\,dz\right| \\
&\le \left|\int_{\gamma_{u,v}} f'(z)\left[s\int_{\gamma_{u,z}} g(y)\,dy-(1-s)\int_{\gamma_{z,v}} g(y)\,dy\right]dz\right| \\
&\quad +\left|\int_{\gamma_{v,w}} f'(z)\left[(1-s)\int_{\gamma_{v,z}} g(y)\,dy-s\int_{\gamma_{z,w}} g(y)\,dy\right]dz\right| \\
&\le \int_{\gamma_{u,v}} |f'(z)|\left|s\int_{\gamma_{u,z}} g(y)\,dy-(1-s)\int_{\gamma_{z,v}} g(y)\,dy\right||dz| \\
&\quad +\int_{\gamma_{v,w}} |f'(z)|\left|(1-s)\int_{\gamma_{v,z}} g(y)\,dy-s\int_{\gamma_{z,w}} g(y)\,dy\right||dz| = B(s,v)
\end{aligned}
$$

for $s\in[0,1]$, which proves (3.1).

Further, we have

$$
\begin{aligned}
B(s,v) &\le \int_{\gamma_{u,v}} |f'(z)|\left[s\left|\int_{\gamma_{u,z}} g(y)\,dy\right|+(1-s)\left|\int_{\gamma_{z,v}} g(y)\,dy\right|\right]|dz| \\
&\quad +\int_{\gamma_{v,w}} |f'(z)|\left[(1-s)\left|\int_{\gamma_{v,z}} g(y)\,dy\right|+s\left|\int_{\gamma_{z,w}} g(y)\,dy\right|\right]|dz| \\
&= s\int_{\gamma_{u,v}} |f'(z)|\left|\int_{\gamma_{u,z}} g(y)\,dy\right||dz|+(1-s)\int_{\gamma_{u,v}} |f'(z)|\left|\int_{\gamma_{z,v}} g(y)\,dy\right||dz| \\
&\quad +(1-s)\int_{\gamma_{v,w}} |f'(z)|\left|\int_{\gamma_{v,z}} g(y)\,dy\right||dz|+s\int_{\gamma_{v,w}} |f'(z)|\left|\int_{\gamma_{z,w}} g(y)\,dy\right||dz|
\end{aligned}
$$

$$= s\left\{\int_{\gamma_{u,v}}|f'(z)|\left|\int_{\gamma_{u,z}}g(y)\,dy\right||dz|+\int_{\gamma_{v,w}}|f'(z)|\left|\int_{\gamma_{z,w}}g(y)\,dy\right||dz|\right\}$$
$$+(1-s)\left\{\int_{\gamma_{u,v}}|f'(z)|\left|\int_{\gamma_{z,v}}g(y)\,dy\right||dz|+\int_{\gamma_{v,w}}|f'(z)|\left|\int_{\gamma_{v,z}}g(y)\,dy\right||dz|\right\}$$
$$= s\left\{\int_{\gamma_{u,v}}|f'(z)|\left|\int_{\gamma_{u,z}}g(y)\,dy\right||dz|+\int_{\gamma_{v,w}}|f'(z)|\left|\int_{\gamma_{z,w}}g(y)\,dy\right||dz|\right\}$$
$$+(1-s)\int_{\gamma_{u,w}}|f'(z)|\left|\int_{\gamma_{z,v}}g(y)\,dy\right||dz|,$$

which proves the first inequality in (3.3).

Using Hölder's integral inequality, we have

$$\int_{\gamma_{u,v}}|f'(z)|\left|\int_{\gamma_{u,z}}g(y)\,dy\right||dz|\le\begin{cases}\max_{z\in\gamma_{u,v}}|f'(z)|\int_{\gamma_{u,v}}\left|\int_{\gamma_{u,z}}g(y)\,dy\right||dz|\\[2ex]\left(\int_{\gamma_{u,v}}|f'(z)|^p\right)^{1/p}\left(\int_{\gamma_{u,v}}\left|\int_{\gamma_{u,z}}g(y)\,dy\right|^q|dz|\right)^{1/q}\\ p,q>1\text{ and }\frac{1}{p}+\frac{1}{q}=1;\\[2ex]\max_{z\in\gamma_{u,v}}\left|\int_{\gamma_{u,z}}g(y)\,dy\right|\int_{\gamma_{u,v}}|f'(z)||dz|,\end{cases}$$

$$\int_{\gamma_{v,w}}|f'(z)|\left|\int_{\gamma_{z,w}}g(y)\,dy\right||dz|\le\begin{cases}\max_{z\in\gamma_{v,w}}|f'(z)|\int_{\gamma_{v,w}}\left|\int_{\gamma_{z,w}}g(y)\,dy\right||dz|\\[2ex]\left(\int_{\gamma_{v,w}}|f'(z)|^p\right)^{1/p}\left(\int_{\gamma_{v,w}}\left|\int_{\gamma_{z,w}}g(y)\,dy\right|^q|dz|\right)^{1/q}\\ p,q>1\text{ and }\frac{1}{p}+\frac{1}{q}=1;\\[2ex]\max_{z\in\gamma_{v,w}}\left|\int_{\gamma_{z,w}}g(y)\,dy\right|\int_{\gamma_{v,w}}|f'(z)||dz|,\end{cases}$$

and

$$\int_{\gamma_{u,w}}|f'(z)|\left|\int_{\gamma_{z,v}}g(y)\,dy\right||dz|\le\begin{cases}\max_{z\in\gamma_{u,w}}|f'(z)|\int_{\gamma_{u,w}}\left|\int_{\gamma_{z,v}}g(y)\,dy\right||dz|\\[2ex]\left(\int_{\gamma_{u,w}}|f'(z)|^p\right)^{1/p}\left(\int_{\gamma_{u,w}}\left|\int_{\gamma_{z,v}}g(y)\,dy\right|^q|dz|\right)^{1/q}\\ p,q>1\text{ and }\frac{1}{p}+\frac{1}{q}=1;\\[2ex]\max_{z\in\gamma_{u,w}}\left|\int_{\gamma_{z,v}}g(y)\,dy\right|\int_{\gamma_{u,w}}|f'(z)||dz|,\end{cases}$$

for any $v\in\gamma$. □

Remark 1. *If we take $s=0$ above, then we get the generalized trapezoid type inequalities*

$$\left|f(w)\int_{\gamma_{v,w}}g(z)\,dz+f(u)\int_{\gamma_{u,v}}g(z)\,dz-\int_{\gamma}f(z)\,g(z)\,dz\right|\le B(0,v)\tag{3.5}$$

where

$$B(0,v) := \int_{\gamma_{u,v}} |f'(z)| \left| \int_{\gamma_{z,v}} g(y)\,dy \right| |dz| + \int_{\gamma_{v,w}} |f'(z)| \left| \int_{\gamma_{v,z}} g(y)\,dy \right| |dz|.$$

Moreover, we have

$$B(0,v) \le \int_{\gamma_{u,w}} |f'(z)| \left| \int_{\gamma_{z,v}} g(y)\,dy \right| |dz| + \begin{cases} \max_{z\in\gamma_{u,w}} |f'(z)| \int_{\gamma_{u,w}} \left| \int_{\gamma_{z,v}} g(y)\,dy \right| |dz| \\ \left(\int_{\gamma_{u,w}} |f'(z)|^p \right)^{1/p} \left(\int_{\gamma_{u,w}} \left| \int_{\gamma_{z,v}} g(y)\,dy \right|^q |dz| \right)^{1/q} \\ p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1; \\ \max_{z\in\gamma_{u,w}} \left| \int_{\gamma_{z,v}} g(y)\,dy \right| \int_{\gamma_{u,w}} |f'(z)|\,|dz|, \end{cases} \tag{3.6}$$

for any $v \in \gamma$.

For $s = 1$, *we get*

$$\left| f(v) \int_{\gamma_{u,w}} g(z)\,dz - \int_{\gamma} f(z)\,g(z)\,dz \right| \le B(1,v) \tag{3.7}$$

where

$$B(1,v) := \int_{\gamma_{u,v}} |f'(z)| \left| \int_{\gamma_{u,z}} g(y)\,dy \right| |dz| + \int_{\gamma_{v,w}} |f'(z)| \left| \int_{\gamma_{z,w}} g(y)\,dy \right| |dz|.$$

Moreover, we have the Ostrowski type inequality

$$B(1,v) \le \int_{\gamma_{u,v}} |f'(z)| \left| \int_{\gamma_{u,z}} g(y)\,dy \right| |dz| + \int_{\gamma_{v,w}} |f'(z)| \left| \int_{\gamma_{z,w}} g(y)\,dy \right| |dz| \tag{3.8}$$

$$\leq \begin{cases} \max_{z\in\gamma_{u,v}} |f'(z)| \int_{\gamma_{u,v}} \left| \int_{\gamma_{u,z}} g(y)\,dy \right| |dz| \\ \\ \left(\int_{\gamma_{u,v}} |f'(z)|^p \right)^{1/p} \left(\int_{\gamma_{u,v}} \left| \int_{\gamma_{u,z}} g(y)\,dy \right|^q |dz| \right)^{1/q} \\ p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \max_{z\in\gamma_{u,v}} \left| \int_{\gamma_{u,z}} g(y)\,dy \right| \int_{\gamma_{u,v}} |f'(z)|\,|dz|, \end{cases}$$

$$+ \begin{cases} \max_{z\in\gamma_{v,w}} |f'(z)| \int_{\gamma_{v,w}} \left| \int_{\gamma_{z,w}} g(y)\,dy \right| |dz| \\ \\ \left(\int_{\gamma_{v,w}} |f'(z)|^p \right)^{1/p} \left(\int_{\gamma_{v,w}} \left| \int_{\gamma_{z,w}} g(y)\,dy \right|^q |dz| \right)^{1/q} \\ p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \max_{z\in\gamma_{v,w}} \left| \int_{\gamma_{z,w}} g(y)\,dy \right| \int_{\gamma_{v,w}} |f'(z)|\,|dz|, \end{cases}$$

for any $v\in\gamma$.

Finally, by taking $s=\frac{1}{2}$ above, we get

$$\left| \frac{1}{2} \left\{ f(w) \int_{\gamma_{v,w}} g(z)\,dz + f(u) \int_{\gamma_{u,v}} g(z)\,dz + f(v) \int_{\gamma_{u,w}} g(z)\,dz \right\} - \int_{\gamma} f(z)\,g(z)\,dz \right| \leq B\left(\frac{1}{2}s, v\right) \tag{3.9}$$

where

$$B(s,v) := \frac{1}{2} \int_{\gamma_{u,v}} |f'(z)| \left| \int_{\gamma_{u,z}} g(y)\,dy - \int_{\gamma_{z,v}} g(y)\,dy \right| |dz| + \frac{1}{2} \int_{\gamma_{v,w}} |f'(z)| \left| \int_{\gamma_{v,z}} g(y)\,dy - \int_{\gamma_{z,w}} g(y)\,dy \right| |dz|.$$

Moreover, we have

$$B\left(\frac{1}{2}, v\right) \leq \frac{1}{2} \left\{ \int_{\gamma_{u,v}} |f'(z)| \left| \int_{\gamma_{u,z}} g(y)\,dy \right| |dz| + \int_{\gamma_{v,w}} |f'(z)| \left| \int_{\gamma_{z,w}} g(y)\,dy \right| |dz| \right\} + \frac{1}{2} \int_{\gamma_{u,w}} |f'(z)| \left| \int_{\gamma_{z,v}} g(y)\,dy \right| |dz| \tag{3.10}$$

$$\leq \frac{1}{2}\begin{cases} \max_{z\in\gamma_{u,v}} |f'(z)| \int_{\gamma_{u,v}} \left|\int_{\gamma_{u,z}} g(y)\,dy\right| |dz| \\ \\ \left(\int_{\gamma_{u,v}} |f'(z)|^p\right)^{1/p} \left(\int_{\gamma_{u,v}} \left|\int_{\gamma_{u,z}} g(y)\,dy\right|^q |dz|\right)^{1/q} \\ p,q>1 \ \ and \ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \max_{z\in\gamma_{u,v}} \left|\int_{\gamma_{u,z}} g(y)\,dy\right| \int_{\gamma_{u,v}} |f'(z)|\,|dz|, \end{cases}$$

$$+\frac{1}{2}\begin{cases} \max_{z\in\gamma_{v,w}} |f'(z)| \int_{\gamma_{v,w}} \left|\int_{\gamma_{z,w}} g(y)\,dy\right| |dz| \\ \\ \left(\int_{\gamma_{v,w}} |f'(z)|^p\right)^{1/p} \left(\int_{\gamma_{v,w}} \left|\int_{\gamma_{z,w}} g(y)\,dy\right|^q |dz|\right)^{1/q} \\ p,q>1 \ \ and \ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \max_{z\in\gamma_{v,w}} \left|\int_{\gamma_{z,w}} g(y)\,dy\right| \int_{\gamma_{v,w}} |f'(z)|\,|dz|, \end{cases}$$

$$+\frac{1}{2}\begin{cases} \max_{z\in\gamma_{u,w}} |f'(z)| \int_{\gamma_{u,w}} \left|\int_{\gamma_{z,v}} g(y)\,dy\right| |dz| \\ \\ \left(\int_{\gamma_{u,w}} |f'(z)|^p\right)^{1/p} \left(\int_{\gamma_{u,w}} \left|\int_{\gamma_{z,v}} g(y)\,dy\right|^q |dz|\right)^{1/q} \\ p,q>1 \ \ and \ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \max_{z\in\gamma_{u,w}} \left|\int_{\gamma_{z,v}} g(y)\,dy\right| \int_{\gamma_{u,w}} |f'(z)|\,|dz|, \end{cases}$$

for $v\in\gamma$.

4. Related Results

Let f and g be holomorphic in D, an open domain and suppose $\gamma\subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)\neq u,\ w$ with $x\in(a,b)$. If G is a primitive for the function g on γ, we can consider the functional

$$\begin{aligned}(4.1)\quad G(f,g,\gamma,v) &:= f(w)\left[G(w)-\frac{1}{w-v}\int_{\gamma_{v,w}} G(y)\,dy\right] \\ &\quad + f(u)\left[\frac{1}{v-u}\int_{\gamma_{u,v}} G(y)\,dy - G(u)\right] \\ &\quad + \left(\frac{1}{w-v}\int_{\gamma_{v,w}} G(y)\,dy - \frac{1}{v-u}\int_{\gamma_{u,v}} G(y)\,dy\right) f(v) - \int_\gamma f(z)\,g(z)\,dz.\end{aligned}$$

We observe that, by (2.7)

$$\begin{aligned}(4.2)\quad G(f,g,\gamma,v) &= \int_{\gamma_{u,v}} f'(z)\left[G(z)-\frac{1}{v-u}\int_{\gamma_{u,v}} G(y)\,dy\right] dz \\ &\quad + \int_{\gamma_{v,w}} f'(z)\left[G(z)-\frac{1}{w-v}\int_{\gamma_{v,w}} G(y)\,dy\right] dz.\end{aligned}$$

Since, obviously

$$\int_{\gamma_{u,v}}\left[G(z)-\frac{1}{v-u}\int_{\gamma_{u,v}}G(y)\,dy\right]dz=0$$

and

$$\int_{\gamma_{v,w}}\left[G(z)-\frac{1}{w-v}\int_{\gamma_{v,w}}G(y)\,dy\right]dz=0,$$

then from (4.3) we get in fact the more general identity in terms of two complex parameters $\alpha,\ \beta\in\mathbb{C}$

$$\begin{aligned}G(f,g,\gamma,v)&=\int_{\gamma_{u,v}}(f'(z)-\alpha)\left[G(z)-\frac{1}{v-u}\int_{\gamma_{u,v}}G(y)\,dy\right]dz\\&\quad+\int_{\gamma_{v,w}}(f'(z)-\beta)\left[G(z)-\frac{1}{w-v}\int_{\gamma_{v,w}}G(y)\,dy\right]dz.\end{aligned}\tag{4.3}$$

Theorem 2. *Let f and g be holomorphic in D, an open domain and suppose $\gamma\subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)\neq u,\ w$ with $x\in(a,b)$. If G is a primitive for the function g on γ, then*

$$\begin{aligned}|G(f,g,\gamma,v)|&\le\int_{\gamma_{u,v}}|f'(z)-\alpha|\left|G(z)-\frac{1}{v-u}\int_{\gamma_{u,v}}G(y)\,dy\right||dz|\\&\quad+\int_{\gamma_{v,w}}|f'(z)-\beta|\left|G(z)-\frac{1}{w-v}\int_{\gamma_{v,w}}G(y)\,dy\right||dz|=:B(\alpha,\beta,v).\end{aligned}\tag{4.4}$$

Moreover, we have

$$\begin{aligned}&B(\alpha,\beta,v)\\&\le\begin{cases}\max_{z\in\gamma_{u,v}}|f'(z)-\alpha|\int_{\gamma_{u,v}}\left|G(z)-\frac{1}{v-u}\int_{\gamma_{u,v}}G(y)\,dy\right||dz|,\\[1ex]\left(\int_{\gamma_{u,v}}|f'(z)-\alpha|^p|dz|\right)^{1/p}\left(\int_{\gamma_{u,v}}\left|G(z)-\frac{1}{v-u}\int_{\gamma_{u,v}}G(y)\,dy\right|^q|dz|\right)^{1/q}\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[1ex]\max_{z\in\gamma_{u,v}}\left|G(z)-\frac{1}{v-u}\int_{\gamma_{u,v}}G(y)\,dy\right|\int_{\gamma_{u,v}}|f'(z)-\alpha||dz|\end{cases}\\&+\begin{cases}\max_{\gamma_{v,w}}|f'(z)-\beta|\int_{\gamma_{v,w}}\left|G(z)-\frac{1}{w-v}\int_{\gamma_{v,w}}G(y)\,dy\right||dz|\\[1ex]\left(\int_{\gamma_{v,w}}|f'(z)-\beta|^p|dz|\right)^{1/p}\left(\int_{\gamma_{v,w}}\left|G(z)-\frac{1}{w-v}\int_{\gamma_{v,w}}G(y)\,dy\right|^q|dz|\right)^{1/q}\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[1ex]\max_{\gamma_{v,w}}\left|G(z)-\frac{1}{w-v}\int_{\gamma_{v,w}}G(y)\,dy\right|\int_{\gamma_{v,w}}|f'(z)-\beta||dz|.\end{cases}\end{aligned}\tag{4.5}$$

The proof follows by the identity (4.3) on taking the modulus, using the triangle and Holder's integral inequalities. We omit the details.

Remark 2. *If we take $\beta = \alpha$ in the above Theorem 2, we get*

$$|G(f,g,\gamma,v)| \leq \int_{\gamma_{u,v}} |f'(z)-\alpha| \left| G(z) - \frac{1}{v-u}\int_{\gamma_{u,v}} G(y)\,dy \right| |dz| + \int_{\gamma_{v,w}} |f'(z)-\alpha| \left| G(z) - \frac{1}{w-v}\int_{\gamma_{v,w}} G(y)\,dy \right| |dz| =: B(\alpha,v). \tag{4.6}$$

Moreover,

$$B(\alpha,v) \tag{4.7}$$

$$\leq \begin{cases} \max_{z\in\gamma_{u,v}} |f'(z)-\alpha| \int_{\gamma_{u,v}} \left| G(z) - \frac{1}{v-u}\int_{\gamma_{u,v}} G(y)\,dy \right| |dz|, \\ \left(\int_{\gamma_{u,v}} |f'(z)-\alpha|^p |dz|\right)^{1/p} \left(\int_{\gamma_{u,v}} \left| G(z) - \frac{1}{v-u}\int_{\gamma_{u,v}} G(y)\,dy \right|^q |dz|\right)^{1/q} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \max_{z\in\gamma_{u,v}} \left| G(z) - \frac{1}{v-u}\int_{\gamma_{u,v}} G(y)\,dy \right| \int_{\gamma_{u,v}} |f'(z)-\alpha| |dz| \end{cases}$$

$$+ \begin{cases} \max_{\gamma_{v,w}} |f'(z)-\alpha| \int_{\gamma_{v,w}} \left| G(z) - \frac{1}{w-v}\int_{\gamma_{v,w}} G(y)\,dy \right| |dz| \\ \left(\int_{\gamma_{v,w}} |f'(z)-\alpha|^p |dz|\right)^{1/p} \left(\int_{\gamma_{v,w}} \left| G(z) - \frac{1}{w-v}\int_{\gamma_{v,w}} G(y)\,dy \right|^q |dz|\right)^{1/q} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \max_{\gamma_{v,w}} \left| G(z) - \frac{1}{w-v}\int_{\gamma_{v,w}} G(y)\,dy \right| \int_{\gamma_{v,w}} |f'(z)-\alpha| |dz| \end{cases}$$

$$\leq \begin{cases} \max_{z\in\gamma_{u,w}} |f'(z)-\alpha| \left(\int_{\gamma_{u,v}} \left| G(z) - \frac{1}{v-u}\int_{\gamma_{u,v}} G(y)\,dy \right| |dz| \right. \\ \left. + \int_{\gamma_{v,w}} \left| G(z) - \frac{1}{w-v}\int_{\gamma_{v,w}} G(y)\,dy \right| |dz|\right), \\ \left(\int_{\gamma_{u,w}} |f'(z)-\alpha|^p |dz|\right)^{1/p} \left(\int_{\gamma_{u,v}} \left| G(z) - \frac{1}{v-u}\int_{\gamma_{u,v}} G(y)\,dy \right|^q |dz| \right. \\ \left. + \int_{\gamma_{v,w}} \left| G(z) - \frac{1}{w-v}\int_{\gamma_{v,w}} G(y)\,dy \right|^q |dz|\right)^{1/q} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \int_{\gamma_{u,w}} |f'(z)-\alpha| |dz| \left(\max_{z\in\gamma_{u,v}} \left| G(z) - \frac{1}{v-u}\int_{\gamma_{u,v}} G(y)\,dy \right| \right. \\ \left. + \max_{\gamma_{v,w}} \left| G(z) - \frac{1}{w-v}\int_{\gamma_{v,w}} G(y)\,dy \right|\right). \end{cases}$$

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$. Now, for ϕ, $\Phi \in \mathbb{C}$ and γ an interval of real numbers, define the sets of complex-valued functions [8]

$$\bar{U}_\gamma(\phi,\Phi) := \left\{ h : \gamma \to \mathbb{C} \middle| \operatorname{Re}\left[(\Phi - h(z)) \left(\overline{h(z)} - \overline{\phi} \right) \right] \geq 0 \text{ for each } z \in \gamma \right\}$$

and

$$\bar{\Delta}_\gamma(\phi,\Phi) := \left\{ h : \gamma \to \mathbb{C} \middle| \left| h(z) - \frac{\phi+\Phi}{2} \right| \leq \frac{1}{2}|\Phi - \phi| \text{ for each } z \in \gamma \right\}.$$

The following representation result may be stated.

Proposition 1. *For any ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that $\bar{U}_{\gamma}(\phi,\Phi)$ and $\bar{\Delta}_{\gamma}(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_{\gamma}(\phi,\Phi)=\bar{\Delta}_{\gamma}(\phi,\Phi). \tag{4.8}$$

Proof. We observe that for any $w\in\mathbb{C}$ we have the equivalence

$$\left|w-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}\left|\Phi-\phi\right|$$

if and only if

$$\operatorname{Re}\left[(\Phi-w)\left(\overline{w}-\overline{\phi}\right)\right]\geq 0.$$

This follows by the equality

$$\frac{1}{4}\left|\Phi-\phi\right|^2-\left|w-\frac{\phi+\Phi}{2}\right|^2=\operatorname{Re}\left[(\Phi-w)\left(\overline{w}-\overline{\phi}\right)\right]$$

that holds for any $w\in\mathbb{C}$.

The equality (4.8) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 2. *For any ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that*

$$\begin{aligned}\bar{U}_{\gamma}(\phi,\Phi)=\{h:\gamma\to\mathbb{C}\mid\ &(\operatorname{Re}\Phi-\operatorname{Re}h(z))(\operatorname{Re}h(z)-\operatorname{Re}\phi)\\&+(\operatorname{Im}\Phi-\operatorname{Im}h(z))(\operatorname{Im}h(z)-\operatorname{Im}\phi)\geq 0 \textit{ for each } z\in\gamma\}.\end{aligned} \tag{4.9}$$

Now, if we assume that $\operatorname{Re}(\Phi)\geq\operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi)\geq\operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_{\gamma}(\phi,\Phi):=\{h:\gamma\to\mathbb{C}\mid\ &\operatorname{Re}(\Phi)\geq\operatorname{Re}h(z)\geq\operatorname{Re}(\phi)\\&\text{and }\operatorname{Im}(\Phi)\geq\operatorname{Im}h(z)\geq\operatorname{Im}(\phi)\ \text{ for each }\ z\in\gamma\}.\end{aligned} \tag{4.10}$$

One can easily observe that $\bar{S}_{\gamma}(\phi,\Phi)$ is closed, convex and

$$\emptyset\neq\bar{S}_{\gamma}(\phi,\Phi)\subseteq\bar{U}_{\gamma}(\phi,\Phi). \tag{4.11}$$

Corollary 3. *Let f and g be holomorphic in D, an open domain and suppose $\gamma\subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)\neq u$, w with $x\in(a,b)$. If G is a primitive for the function g on γ and there exists the constants ϕ_i, $\Phi_i\in\mathbb{C}$, $\phi_i\neq\Phi_i$, $i\in\{1,2\}$ with $f'\in\bar{\Delta}_{\gamma_{u,v}}(\phi_1,\Phi_1)\cap\bar{\Delta}_{\gamma_{v,w}}(\phi_2,\Phi_2)$, then*

$$\begin{aligned}|G(f,g,\gamma,v)|\leq\ &\frac{1}{2}\left|\Phi_1-\phi_1\right|\int_{\gamma_{u,v}}\left|G(z)-\frac{1}{v-u}\int_{\gamma_{u,v}}G(y)\,dy\right||dz|\\&+\frac{1}{2}\left|\Phi_2-\phi_2\right|\int_{\gamma_{v,w}}\left|G(z)-\frac{1}{w-v}\int_{\gamma_{v,w}}G(y)\,dy\right||dz|.\end{aligned} \tag{4.12}$$

In particular, for $\Phi_1=\Phi_2=\Phi$ and $\phi_1=\phi_2=\phi$ we get

$$\begin{aligned}|G(f,g,\gamma,v)|\leq\ &\frac{1}{2}\left|\Phi-\phi\right|\left[\int_{\gamma_{u,v}}\left|G(z)-\frac{1}{v-u}\int_{\gamma_{u,v}}G(y)\,dy\right||dz|\right.\\&\left.+\int_{\gamma_{v,w}}\left|G(z)-\frac{1}{w-v}\int_{\gamma_{v,w}}G(y)\,dy\right||dz|\right].\end{aligned} \tag{4.13}$$

5. Some Unweighted Inequalities

Let f be holomorphic in D, an open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v = z(x)$ with $x \in (a,b)$. The case $g(z) = 1$, $z \in \mathbb{C}$ in the inequality (3.7) gives simple unweighted inequalities as follows:

$$\left|(1-s)\{f(w)(w-v) + f(u)(v-u)\} + sf(v)(w-u) - \int_\gamma f(z)\,dz\right| \leq B(s,v) \tag{5.1}$$

where

$$B(s,v) := \int_{\gamma_{u,v}} |f'(z)|\,|z - su - (1-s)v|\,|dz| + \int_{\gamma_{v,w}} |f'(z)|\,|z - (1-s)v - sw|\,|dz|. \tag{5.2}$$

Moreover, we have

$$\begin{aligned} B(s,v) &\leq s\left\{\int_{\gamma_{u,v}} |f'(z)|\,|z-u|\,|dz| + \int_{\gamma_{v,w}} |f'(z)|\,|w-z|\,|dz|\right\} \\ &\quad + (1-s)\int_{\gamma_{u,w}} |f'(z)|\,|v-z|\,|dz| \\ &\leq s\begin{cases} \max_{z\in\gamma_{u,v}} |f'(z)| \int_{\gamma_{u,v}} |z-u|\,|dz| \\ \\ \left(\int_{\gamma_{u,v}} |f'(z)|^p\right)^{1/p} \left(\int_{\gamma_{u,v}} |z-u|^q\,|dz|\right)^{1/q} \\ p,\ q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \max_{z\in\gamma_{u,v}} |z-u| \int_{\gamma_{u,v}} |f'(z)|\,|dz|, \end{cases} \\ &\quad + s\begin{cases} \max_{z\in\gamma_{v,w}} |f'(z)| \int_{\gamma_{v,w}} |w-z|\,|dz| \\ \\ \left(\int_{\gamma_{v,w}} |f'(z)|^p\right)^{1/p} \left(\int_{\gamma_{v,w}} |w-z|^q\,|dz|\right)^{1/q} \\ p,\ q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \max_{z\in\gamma_{v,w}} |w-z| \int_{\gamma_{v,w}} |f'(z)|\,|dz|, \end{cases} \\ &\quad + (1-s)\begin{cases} \max_{z\in\gamma_{u,w}} |f'(z)| \int_{\gamma_{u,w}} |v-z|\,|dz| \\ \\ \left(\int_{\gamma_{u,w}} |f'(z)|^p\right)^{1/p} \left(\int_{\gamma_{u,w}} |v-z|^q\,|dz|\right)^{1/q} \\ p,\ q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \max_{z\in\gamma_{u,w}} |v-z| \int_{\gamma_{u,w}} |f'(z)|\,|dz|, \end{cases} \end{aligned} \tag{5.3}$$

for any $v \in \gamma$ and $s \in [0,1]$.

For $s=0$ we get the trapezoid type inequalities

$$\left|f(w)(w-v)+f(u)(v-u)-\int_{\gamma}f(z)\,dz\right|\leq B(0,v) \tag{5.4}$$

where

$$B(0,v):=\int_{\gamma_{u,w}}|f'(z)|\,|z-v|\,|dz|. \tag{5.5}$$

Moreover, we have

$$B(0,v)\leq\begin{cases}\max_{z\in\gamma_{u,w}}|f'(z)|\int_{\gamma_{u,w}}|v-z|\,|dz|\\[2ex] \left(\int_{\gamma_{u,w}}|f'(z)|^{p}\right)^{1/p}\left(\int_{\gamma_{u,w}}|v-z|^{q}\,|dz|\right)^{1/q}\\ p,\ q>1\text{ and }\frac{1}{p}+\frac{1}{q}=1;\\[2ex] \max_{z\in\gamma_{u,w}}|v-z|\int_{\gamma_{u,w}}|f'(z)|\,|dz|,\end{cases} \tag{5.6}$$

for any $v\in\gamma$.

For $s=1$ we get the Ostrowski type inequalities

$$\left|f(v)(w-u)-\int_{\gamma}f(z)\,dz\right|\leq B(1,v) \tag{5.7}$$

where

$$B(1,v):=\int_{\gamma_{u,v}}|f'(z)|\,|z-u|\,|dz|+\int_{\gamma_{v,w}}|f'(z)|\,|z-w|\,|dz|. \tag{5.8}$$

Moreover, we have

$$\begin{aligned}B(1,v)\leq&\begin{cases}\max_{z\in\gamma_{u,v}}|f'(z)|\int_{\gamma_{u,v}}|z-u|\,|dz|\\[2ex] \left(\int_{\gamma_{u,v}}|f'(z)|^{p}\right)^{1/p}\left(\int_{\gamma_{u,v}}|z-u|^{q}\,|dz|\right)^{1/q}\\ p,\ q>1\text{ and }\frac{1}{p}+\frac{1}{q}=1;\\[2ex] \max_{z\in\gamma_{u,v}}|z-u|\int_{\gamma_{u,v}}|f'(z)|\,|dz|,\end{cases}\\ &+\begin{cases}\max_{z\in\gamma_{v,w}}|f'(z)|\int_{\gamma_{v,w}}|w-z|\,|dz|\\[2ex] \left(\int_{\gamma_{v,w}}|f'(z)|^{p}\right)^{1/p}\left(\int_{\gamma_{v,w}}|w-z|^{q}\,|dz|\right)^{1/q}\\ p,\ q>1\text{ and }\frac{1}{p}+\frac{1}{q}=1;\\[2ex] \max_{z\in\gamma_{v,w}}|w-z|\int_{\gamma_{v,w}}|f'(z)|\,|dz|,\end{cases}\end{aligned} \tag{5.9}$$

for any $v\in\gamma$.

For $s=\frac{1}{2}$ we get the mixed inequality

$$\begin{aligned}\bigg|\frac{1}{2}\{f(w)(w-v)+f(u)(v-u)+f(v)(w-u)\}&\\ -\int_{\gamma}f(z)\,dz\bigg|&\leq B\left(\frac{1}{2},v\right)\end{aligned} \tag{5.10}$$

where

$$B\left(\frac{1}{2},v\right):=\int_{\gamma_{u,w}}\left|f'(z)\right|\left|z-\frac{u+v}{2}\right|\left|dz\right|. \tag{5.11}$$

Moreover, we have

$$\begin{aligned}
&B\left(\frac{1}{2},v\right) \qquad (5.12)\\
&\leq\frac{1}{2}\left\{\int_{\gamma_{u,v}}\left|f'(z)\right|\left|z-u\right|\left|dz\right|+\int_{\gamma_{v,w}}\left|f'(z)\right|\left|w-z\right|\left|dz\right|+\int_{\gamma_{u,w}}\left|f'(z)\right|\left|v-z\right|\left|dz\right|\right\}\\
&\leq\frac{1}{2}\begin{cases}\max_{z\in\gamma_{u,v}}\left|f'(z)\right|\int_{\gamma_{u,v}}\left|z-u\right|\left|dz\right|\\ \\ \left(\int_{\gamma_{u,v}}\left|f'(z)\right|^{p}\right)^{1/p}\left(\int_{\gamma_{u,v}}\left|z-u\right|^{q}\left|dz\right|\right)^{1/q}\\ p,\ q>1\text{ and }\frac{1}{p}+\frac{1}{q}=1;\\ \\ \max_{z\in\gamma_{u,v}}\left|z-u\right|\int_{\gamma_{u,v}}\left|f'(z)\right|\left|dz\right|,\end{cases}\\
&+\frac{1}{2}\begin{cases}\max_{z\in\gamma_{v,w}}\left|f'(z)\right|\int_{\gamma_{v,w}}\left|w-z\right|\left|dz\right|\\ \\ \left(\int_{\gamma_{v,w}}\left|f'(z)\right|^{p}\right)^{1/p}\left(\int_{\gamma_{v,w}}\left|w-z\right|^{q}\left|dz\right|\right)^{1/q}\\ p,\ q>1\text{ and }\frac{1}{p}+\frac{1}{q}=1;\\ \\ \max_{z\in\gamma_{v,w}}\left|w-z\right|\int_{\gamma_{v,w}}\left|f'(z)\right|\left|dz\right|,\end{cases}\\
&+\frac{1}{2}\begin{cases}\max_{z\in\gamma_{u,w}}\left|f'(z)\right|\int_{\gamma_{u,w}}\left|v-z\right|\left|dz\right|\\ \\ \left(\int_{\gamma_{u,w}}\left|f'(z)\right|^{p}\right)^{1/p}\left(\int_{\gamma_{u,w}}\left|v-z\right|^{q}\left|dz\right|\right)^{1/q}\\ p,\ q>1\text{ and }\frac{1}{p}+\frac{1}{q}=1;\\ \\ \max_{z\in\gamma_{u,w}}\left|v-z\right|\int_{\gamma_{u,w}}\left|f'(z)\right|\left|dz\right|.\end{cases}
\end{aligned}$$

for any $v\in\gamma$.

Let f be holomorphic in D, an open domain and suppose $\gamma\subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)\neq u,\ w$ with $x\in(a,b)$. If $g(z)=1$ on γ, we can consider the functional, derived from (4.1) for $s=\frac{1}{2}$, defined by

$$\begin{aligned}G(f,\gamma,v):=\frac{1}{2}\left[f(w)(w-v)+f(u)(v-u)+(w-u)f(v)\right]&\\ -\int_{\gamma}f(z)\,g(z)\,dz.&\end{aligned} \tag{5.13}$$

Now, by utilising the inequality (4.4), we get

$$|G(f,g,\gamma,v)| \leq \int_{\gamma_{u,v}} |f'(z)-\alpha| \left|z-\frac{u+v}{2}\right| |dz| + \int_{\gamma_{v,w}} |f'(z)-\beta| \left|z-\frac{v+w}{2}\right| |dz| =: B(\alpha,\beta,v), \tag{5.14}$$

for any $v \in \gamma$ and α, $\beta \in \mathbb{C}$.

Moreover, we have

$$B(\alpha,\beta,v) \leq \begin{cases} \max_{z\in\gamma_{u,v}} |f'(z)-\alpha| \int_{\gamma_{u,v}} \left|z-\frac{v+u}{2}\right| |dz|, \\ \left(\int_{\gamma_{u,v}} |f'(z)-\alpha|^p |dz|\right)^{1/p} \left(\int_{\gamma_{u,v}} \left|z-\frac{v+u}{2}\right|^q |dz|\right)^{1/q} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \max_{z\in\gamma_{u,v}} \left|z-\frac{v+u}{2}\right| \int_{\gamma_{u,v}} |f'(z)-\alpha| |dz| \end{cases} + \begin{cases} \max_{\gamma_{v,w}} |f'(z)-\beta| \int_{\gamma_{v,w}} \left|z-\frac{v+w}{2}\right| |dz| \\ \left(\int_{\gamma_{v,w}} |f'(z)-\beta|^p |dz|\right)^{1/p} \left(\int_{\gamma_{v,w}} \left|z-\frac{v+w}{2}\right|^q |dz|\right)^{1/q} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \max_{\gamma_{v,w}} \left|z-\frac{v+w}{2}\right| \int_{\gamma_{v,w}} |f'(z)-\beta| |dz|, \end{cases} \tag{5.15}$$

for $v \in \gamma$ and α, $\beta \in \mathbb{C}$.

If $f' \in \bar{\Delta}_{\gamma_{u,w}}(\phi,\Phi)$, then by (4.13) we get

$$|G(f,\gamma,v)| \leq \frac{1}{2}|\Phi-\phi| \left[\int_{\gamma_{u,v}} \left|z-\frac{u+v}{2}\right| |dz| + \int_{\gamma_{v,w}} \left|z-\frac{v+w}{2}\right| |dz|\right] \tag{5.16}$$

for $v \in \gamma$.

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