INEQUALITIES OF HERMITE-HADAMARD TYPE FOR K-BOUNDED MODULUS CONVEX COMPLEX FUNCTIONS

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Abstract. Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. We say that the function $f : D \subset \mathbb{C} \to \mathbb{C}$ is called K-bounded modulus convex, for the given $K > 0$, if it satisfies the condition

$$\left|(1-\lambda) f(x) + \lambda f(y) - f((1-\lambda) x + \lambda y)\right| \leq \frac{1}{2} K\lambda (1-\lambda) |x-y|^2$$

for any $x, y \in D$ and $\lambda \in [0,1]$.

In this paper we establish some new Hermite-Hadamard type inequalities for the complex integral on γ, a smooth path from $\mathbb{C}$ and K-bounded modulus convex functions. Some examples for integrals on segments and circular paths are also given.

1. Introduction

Let $(X; \|\cdot\|_X)$ and $(Y; \|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$. Let C be a convex set in X. In the recent paper [3] we introduced the following class of functions:

Definition 1. *A mapping $f : C \subset X \to Y$ is called K-bounded norm convex, for some given $K > 0$, if it satisfies the condition*

$$\|(1-\lambda) f(x) + \lambda f(y) - f((1-\lambda) x + \lambda y)\|_Y \leq \frac{1}{2} K\lambda (1-\lambda) \|x-y\|_X^2 \tag{1.1}$$

for any $x, y \in C$ and $\lambda \in [0,1]$. For simplicity, we denote this by $f \in \mathcal{BN}_K(C)$.

We have from (1.1) for $\lambda = \frac{1}{2}$ the Jensen's inequality

$$\left\| \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \right\|_Y \leq \frac{1}{8} K \|x-y\|_X^2$$

for any $x, y \in C$.

We observe that $\mathcal{BN}_K(C)$ is a convex subset in the linear space of all functions defined on C and with values in Y.

In the same paper [3], we obtained the following result which provides a large class of examples of such functions.

Theorem 1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces, C an open convex subset of X and $f : C \to Y$ a twice-differentiable mapping on C. Then for any $x, y \in C$ and $\lambda \in [0,1]$ we have*

$$\|(1-\lambda) f(x) + \lambda f(y) - f((1-\lambda) x + \lambda y)\|_Y \leq \frac{1}{2} K\lambda (1-\lambda) \|y-x\|_X^2, \tag{1.2}$$

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where

$$K_{f''} := \sup_{z \in C} \|f''(z)\|_{\mathcal{L}(X^2;Y)} \tag{1.3}$$

is assumed to be finite, namely $f \in \mathcal{BN}_{K_{f''}}(C)$.

We have the following inequalities of Hermite-Hadamard type [3]:

Theorem 2. *Let* $(X; \|\cdot\|_X)$ *and* $(Y; \|\cdot\|_Y)$ *be two normed linear spaces over the complex number field* $\mathbb{C}$ *with* Y *complete. Assume that the mapping* $f : C \subset X \to Y$ *is continuous on the convex set* C *in the norm topology. If* $f \in \mathcal{BN}_K(C)$ *for some* $K > 0$, *then we have*

$$\left\| \frac{f(x) + f(y)}{2} - \int_0^1 f((1-\lambda)x + \lambda y)\, d\lambda \right\|_Y \le \frac{1}{12} K \|x - y\|_X^2 \tag{1.4}$$

and

$$\left\| \int_0^1 f((1-\lambda)x + \lambda y)\, d\lambda - f\left(\frac{x+y}{2}\right) \right\|_Y \le \frac{1}{24} K \|x - y\|_X^2 \tag{1.5}$$

for any x, $y \in C$.

The constants $\frac{1}{12}$ *and* $\frac{1}{24}$ *are best possible.*

For a monograph devoted to Hermite-Hadamard type inequalities see [5] and the recent survey paper [4].

Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. Following Definition 1, we say that the function $f : D \subset \mathbb{C} \to \mathbb{C}$ is called K-bounded modulus convex, for the given $K > 0$, if it satisfies the condition

$$|(1-\lambda) f(x) + \lambda f(y) - f((1-\lambda)x + \lambda y)| \le \frac{1}{2} K \lambda (1-\lambda) |x - y|^2 \tag{1.6}$$

for any x, $y \in D$ and $\lambda \in [0,1]$. For simplicity, we denote this by $f \in \mathcal{BM}_K(D)$.

All the above results can be translated for complex functions defined on convex subsets $D \subset \mathbb{C}$.

In the following, in order to obtain several inequalities for the complex integral, we need the following facts.

Suppose γ is a smooth path from $\mathbb{C}$ parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with u, $w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\, dz = \int_{\gamma_{u,w}} f(z)\, dz := \int_a^b f(z(t))\, z'(t)\, dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\, dz := \int_{\gamma_{u,v}} f(z)\, dz + \int_{\gamma_{v,w}} f(z)\, dz$$

where $v := z(s)$ for some $s \in (a,b)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in D, and open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.7}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right| \leq \int_\gamma |f(z)|\,|dz| \leq \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.8}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \geq 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\,|dz|\right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|.$$

If $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \leq [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}.$$

Motivated by the above results, in this paper we establish some new Hermite-Hadamard type inequalities for the complex integral on γ, a smooth path from $\mathbb{C}$ and K-bounded modulus convex functions. Some examples for integrals on segments and circular paths are also given.

2. Integral Inequalities

We have:

Theorem 3. *Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. Assume that f is holomorphic on D and $f \in \mathcal{BM}_K(D)$. If $\gamma \subset D$ parametrized by $z(t)$, $t \in [a,b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v \in D$, then*

$$\left|\int_\gamma f(z)\,dz - \left[f(v) + f'(v)\left(\frac{w+u}{2} - v\right)\right](w-u)\right| \leq \frac{1}{2}K\int_\gamma |z-v|^2\,|dz|. \tag{2.1}$$

In particular, we have for $v = \frac{w+u}{2}$ that

$$\left|\int_\gamma f(z)\,dz - f\left(\frac{w+u}{2}\right)(w-u)\right| \leq \frac{1}{2}K\int_\gamma \left|z - \frac{w+u}{2}\right|^2 |dz|. \tag{2.2}$$

Proof. Let x, $y \in D$. Since $f \in \mathcal{BM}_K(D)$, then we have

$$\left|f\left((1-\lambda)x+\lambda y\right)-f(x)+\lambda\left[f(x)-f(y)\right]\right| \leq \frac{1}{2}K\lambda(1-\lambda)|x-y|^2$$

that implies that

$$\left|\frac{f(x+\lambda(y-x))-f(x)}{\lambda}+f(x)-f(y)\right| \leq \frac{1}{2}K(1-\lambda)|x-y|^2$$

for $\lambda \in (0,1)$.

Since f is holomorphic on D, then by letting $\lambda \to 0+$, we get

$$|f'(x)(y-x)+f(x)-f(y)| \leq \frac{1}{2}K|x-y|^2$$

that is equivalent to

$$|f(y)-f(x)-f'(x)(y-x)| \leq \frac{1}{2}K|y-x|^2 \tag{2.3}$$

for all x, $y \in D$.

We have

$$\begin{aligned}
&\int_\gamma [f(z)-f(v)-f'(v)(z-v)]\,dz \\
&= \int_\gamma f(z)\,dz - f(v)\int_\gamma dz - f'(v)\left(\int_\gamma z dz - v\int_\gamma dz\right) \\
&= \int_\gamma f(z)\,dz - f(v)(w-u) - f'(v)\left[\frac{1}{2}\left(w^2-u^2\right)-v(w-u)\right] \\
&= \int_\gamma f(z)\,dz - \left[f(v)+f'(v)\left(\frac{w+u}{2}-v\right)\right](w-u)
\end{aligned} \tag{2.4}$$

for any $v \in D$.

By using (2.3) we get

$$\begin{aligned}
&\left|\int_\gamma f(z)\,dz - \left[f(v)+f'(v)\left(\frac{w+u}{2}-v\right)\right](w-u)\right| \\
&\leq \int_\gamma |f(z)-f(v)-f'(v)(z-v)|\,|dz| \leq \frac{1}{2}K\int_\gamma |z-v|^2\,|dz|
\end{aligned}$$

for any $v \in D$, which proves the inequality (2.1). $\square$

If the path γ is a segment $[u,w] \subset G$ connecting two distinct points u and w in G then we write $\int_\gamma f(z)\,dz$ as $\int_u^w f(z)\,dz$.

Corollary 1. *With the assumptions of Theorem 3 and suppose $[u,w] \subset D$ is a segment connecting two distinct points u and w in D and $v \in [u,w]$. Then for $v=(1-s)u+sw$ with $s \in [0,1]$, we have*

$$\begin{aligned}
&\left|\int_u^w f(z)\,dz - f((1-s)u+sw)(w-u)\right. \\
&\qquad \left. -f'((1-s)u+sw)\left(\frac{1}{2}-s\right)(w-u)^2\right| \\
&\qquad\qquad \leq \frac{1}{6}K|w-u|^3\left[(1-s)^3+s^3\right].
\end{aligned} \tag{2.5}$$

In particular, we have, see also (1.5),

$$\left|\int_u^w f(z)\,dz - f\left(\frac{w+u}{2}\right)(w-u)\right| \le \frac{1}{24}K\,|w-u|^3. \tag{2.6}$$

Proof. It follows by Theorem 3 by observing that

$$\begin{aligned}\int_u^w |z-v|^2\,|dz| &= |w-u|\int_0^1 |(1-t)\,u + tw - (1-s)\,u - sw|^2\,dt \\ &= |w-u|\int_0^1 |(1-t)\,u + tw - (1-s)\,u - sw|^2\,dt \\ &= |w-u|^3\int_0^1 (t-s)^2\,dt = \frac{1}{3}\,|w-u|^3\left[(1-s)^3 + s^3\right]\end{aligned}$$

for $s \in [0,1]$. □

Theorem 4. *Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. Assume that f is holomorphic on D and $f \in \mathcal{BM}_K(D)$. If $\gamma \subset D$ parametrized by $z(t)$, $t \in [a,b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v \in D$, then*

$$\begin{aligned}\left|\frac{1}{2}\left[f(w)(w-v) + f(u)(v-u) + f(v)(w-u)\right] - \int_\gamma f(z)\,dz\right| \\ \le \frac{1}{4}K\int_\gamma |z-v|^2\,|dz|.\end{aligned} \tag{2.7}$$

In particular, we have for $v = \frac{w+u}{2}$ that

$$\begin{aligned}\left|\frac{1}{2}\left[\frac{f(w)+f(u)}{2} + f\left(\frac{w+u}{2}\right)\right](w-u) - \int_\gamma f(z)\,dz\right| \\ \le \frac{1}{4}K\int_\gamma \left|z - \frac{w+u}{2}\right|^2 |dz|.\end{aligned} \tag{2.8}$$

Proof. By using (2.3) we get

$$\int_\gamma |f(v) - f(z) - f'(z)(v-z)|\,|dz| \le \frac{1}{2}K\int_\gamma |v-z|^2\,|dz| \tag{2.9}$$

for $v \in D$.

By the complex integral properties, we have

$$\begin{aligned}\left|\int_\gamma [f(v) - f(z) - f'(z)(v-z)]\,dz\right| \\ \le \int_\gamma |f(v) - f(z) - f'(z)(v-z)|\,|dz|\end{aligned} \tag{2.10}$$

for $v \in D$.

Using integration by parts, we get

$$\begin{aligned}
&\int_\gamma [f(v) - f(z) - f'(z)(v-z)]\,dz \\
&= f(v)\int_\gamma dz - \int_\gamma f(z)\,dz - \int_\gamma f'(z)(v-z)\,dz \\
&= f(v)(w-u) - \int_\gamma f(z)\,dz - \left[f(z)(v-z)|_u^w + \int_\gamma f(z)\,dz \right] \\
&= f(v)(w-u) - \int_\gamma f(z)\,dz - f(w)(v-w) + f(u)(v-u) - \int_\gamma f(z)\,dz \\
&= f(w)(w-v) + f(u)(v-u) + f(v)(w-u) - 2\int_\gamma f(z)\,dz,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{1}{2}[f(w)(w-v) + f(u)(v-u) + f(v)(w-u)] - \int_\gamma f(z)\,dz \\
&\qquad = \frac{1}{2}\int_\gamma [f(v) - f(z) - f'(z)(v-z)]\,dz
\end{aligned} \tag{2.11}$$

for $v \in D$.

By utilising (2.9)-(2.11) we get the desired result (2.7). $\square$

We have:

Corollary 2. *With the assumptions of Theorem 3 and suppose $[u,w] \subset D$ is a segment connecting two distinct points u and w in D and $v \in [u,w]$. Then for $v = (1-s)u + sw$ with $s \in [0,1]$, we have*

$$\begin{aligned}
&\left| \frac{1}{2}[(1-s)f(w) + sf(u) + f((1-s)u + sw)](w-u) - \int_u^w f(z)\,dz \right| \\
&\qquad \leq \frac{1}{12} K |w-u|^3 \left[(1-s)^3 + s^3 \right].
\end{aligned} \tag{2.12}$$

In particular, we have for $v = \frac{w+u}{2}$ that

$$\begin{aligned}
&\left| \frac{1}{2}\left[\frac{f(w)+f(u)}{2} + f\left(\frac{w+u}{2}\right) \right](w-u) - \int_u^w f(z)\,dz \right| \\
&\qquad \leq \frac{1}{48} K |w-u|^3.
\end{aligned} \tag{2.13}$$

We observe that, if f is holomorphic on D and $K = \sup_{z\in D} |f''(z)|$ is finite, then by (2.1) and (2.2) we have

$$\begin{aligned}
&\left| \int_\gamma f(z)\,dz - \left[f(v) + f'(v)\left(\frac{w+u}{2} - v\right) \right](w-u) \right| \\
&\qquad \leq \frac{1}{2} \sup_{z\in D} |f''(z)| \int_\gamma |z-v|^2 |dz|
\end{aligned} \tag{2.14}$$

for all $v \in D$. In particular,

$$\left|\int_\gamma f(z)\,dz - f\left(\frac{w+u}{2}\right)(w-u)\right| \le \frac{1}{2}\sup_{z\in D}|f''(z)|\int_\gamma \left|z-\frac{w+u}{2}\right|^2 |dz|. \tag{2.15}$$

From (2.7) and (2.8) we get

$$\left|\frac{1}{2}\left[f(w)(w-v)+f(u)(v-u)+f(v)(w-u)\right]-\int_\gamma f(z)\,dz\right| \le \frac{1}{4}\sup_{z\in D}|f''(z)|\int_\gamma |z-v|^2|dz|. \tag{2.16}$$

for all $v \in D$. In particular,

$$\left|\frac{1}{2}\left[\frac{f(w)+f(u)}{2}+f\left(\frac{w+u}{2}\right)\right](w-u) - \int_\gamma f(z)\,dz\right| \le \frac{1}{4}\sup_{z\in D}|f''(z)|\int_\gamma \left|z-\frac{w+u}{2}\right|^2 |dz|. \tag{2.17}$$

The inequalities (2.14)-(2.17) provide many examples of interest as follows.

If we consider the function $f(z) = \exp z$, $z \in \mathbb{C}$ and $\gamma \subset \mathbb{C}$ parametrized by $z(t)$, $t \in [a,b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ then by (2.14)-(2.17) we have by the inequalities

$$\left|\exp w - \exp u - \left(1+\frac{w+u}{2}-v\right)(w-u)\exp v\right| \le \frac{1}{2}\sup_{z\in D}|\exp z|\int_\gamma |z-v|^2|dz| \tag{2.18}$$

for all $v \in \mathbb{C}$. In particular,

$$\left|\exp w - \exp u - \exp\left(\frac{w+u}{2}\right)(w-u)\right| \le \frac{1}{2}\sup_{z\in D}|\exp z|\int_\gamma \left|z-\frac{w+u}{2}\right|^2 |dz|. \tag{2.19}$$

We also have

$$\left|\frac{1}{2}\left[(w-v)\exp w+(v-u)\exp u+(w-u)\exp v\right]-\exp w+\exp u\right| \le \frac{1}{4}\sup_{z\in D}|\exp z|\int_\gamma |z-v|^2|dz|. \tag{2.20}$$

for all $v \in \mathbb{C}$. In particular,

$$\left|\frac{1}{2}\left[\frac{\exp w+\exp u}{2}+\exp\left(\frac{w+u}{2}\right)\right](w-u) - \exp w + \exp u\right| \le \frac{1}{4}\sup_{z\in D}|\exp z|\int_\gamma \left|z-\frac{w+u}{2}\right|^2 |dz|. \tag{2.21}$$

Consider the function $F(z) = \operatorname{Log}(z)$ where $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$ and $\operatorname{Arg}(z)$ is such that $0 < \operatorname{Arg}(z) < 2\pi$. Log is called the *"principal branch"* of the complex logarithmic function. F is analytic on all of $\mathbb{C}\setminus\{x+iy : x \geq 0,\ y = 0\}$ and $F'(z) = \frac{1}{z}$ on this set.

If we consider $f : D \to \mathbb{C}$, $f(z) = \frac{1}{z}$ where $D \subset \mathbb{C}\setminus\{x+iy : x \geq 0,\ y = 0\}$, then F is a primitive of f on D and if $\gamma \subset D$ parametrized by $z(t)$, $t \in [a,b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$, then

$$\int_\gamma f(z)\,dz = \operatorname{Log}(w) - \operatorname{Log}(u).$$

For $D \subset \mathbb{C}\setminus\{x+iy : x \geq 0,\ y = 0\}$, define $d := \inf_{z\in D}|z|$ and assume that $d \in (0,\infty)$. By the inequalities (2.14)-(2.17) we then have

$$\left|\operatorname{Log}(w) - \operatorname{Log}(u) - \left[\frac{1}{v} - \frac{1}{v^2}\left(\frac{w+u}{2} - v\right)\right](w-u)\right| \leq \frac{1}{d^3}\int_\gamma |z-v|^2\,|dz| \tag{2.22}$$

for all $v \in D$. In particular,

$$\left|\operatorname{Log}(w) - \operatorname{Log}(u) - \left(\frac{w+u}{2}\right)^{-1}(w-u)\right| \leq \frac{1}{d^3}\int_\gamma \left|z - \frac{w+u}{2}\right|^2 |dz|. \tag{2.23}$$

We also have

$$\left|\frac{1}{2}\left(\frac{w-v}{w} + \frac{v-u}{u} + \frac{w-u}{v}\right) - \operatorname{Log}(w) + \operatorname{Log}(u)\right| \leq \frac{1}{2d^3}\int_\gamma |z-v|^2\,|dz|. \tag{2.24}$$

for all $v \in D$. In particular,

$$\left|\frac{1}{2}\left[\frac{u+w}{2wu} + \left(\frac{w+u}{2}\right)^{-1}\right](w-u) - \operatorname{Log}(w) + \operatorname{Log}(u)\right| \leq \frac{1}{2d^3}\int_\gamma \left|z - \frac{w+u}{2}\right|^2 |dz|. \tag{2.25}$$

3. Examples for Circular Paths

Let $[a,b] \subseteq [0,2\pi]$ and the circular path $\gamma_{[a,b],R}$ centered in 0 and with radius $R > 0$

$$z(t) = R\exp(it) = R(\cos t + i\sin t),\ t \in [a,b].$$

If $[a,b] = [0,\pi]$ then we get a half circle while for $[a,b] = [0,2\pi]$ we get the full circle.

Since

$$\begin{aligned}\left|e^{is}-e^{it}\right|^2 &= \left|e^{is}\right|^2 - 2\operatorname{Re}\left(e^{i(s-t)}\right) + \left|e^{it}\right|^2 \\ &= 2-2\cos(s-t) = 4\sin^2\left(\frac{s-t}{2}\right)\end{aligned}$$

for any $t,\ s\in\mathbb{R}$, then

$$\left|e^{is}-e^{it}\right|^r = 2^r\left|\sin\left(\frac{s-t}{2}\right)\right|^r \tag{3.1}$$

for any $t,\ s\in\mathbb{R}$ and $r>0$. In particular,

$$\left|e^{is}-e^{it}\right| = 2\left|\sin\left(\frac{s-t}{2}\right)\right|$$

for any $t,\ s\in\mathbb{R}$.

For $s=a$ and $s=b$ we have

$$\left|e^{ia}-e^{it}\right| = 2\left|\sin\left(\frac{a-t}{2}\right)\right| \text{ and } \left|e^{ib}-e^{it}\right| = 2\left|\sin\left(\frac{b-t}{2}\right)\right|.$$

If $u=R\exp(ia)$ and $w=R\exp(ib)$ then

$$\begin{aligned}w-u &= R\left[\exp(ib)-\exp(ia)\right] = R\left[\cos b + i\sin b - \cos a - i\sin a\right] \\ &= R\left[\cos b - \cos a + i\left(\sin b - \sin a\right)\right].\end{aligned}$$

Since

$$\cos b - \cos a = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

and

$$\sin b - \sin a = 2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right),$$

hence

$$\begin{aligned}w-u &= R\left[-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right) + 2i\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right)\right] \\ &= 2R\sin\left(\frac{b-a}{2}\right)\left[-\sin\left(\frac{a+b}{2}\right) + i\cos\left(\frac{a+b}{2}\right)\right] \\ &= 2Ri\sin\left(\frac{b-a}{2}\right)\left[\cos\left(\frac{a+b}{2}\right) + i\sin\left(\frac{a+b}{2}\right)\right] \\ &= 2Ri\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right].\end{aligned}$$

We also have

$$z'(t) = Ri\exp(it) \text{ and } \left|z'(t)\right| = R$$

for $t\in[a,b]$.

In what follows we assume that f is defined on a domain containing the circular path $\gamma_{[a,b],R}$ and that f is holomorphic on that domain.

Consider the circular path $\gamma_{[a,b],R}$ and assume that $v=R\exp(is)\in\gamma_{[a,b],R}$ with $s\in[a,b]$. Then by using the inequality (2.1) we get

$$\begin{aligned}
&\left|Ri\int_a^b f(R\exp(it))\exp(it)\,dt\right. \\
&-\left[f(R\exp(is))+f'(R\exp(is))\left(\frac{R\exp(ib)+R\exp(ia)}{2}-R\exp(is)\right)\right] \\
&\qquad\left.\times 2Ri\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right]\right| \\
&\leq\frac{1}{2}\sup_{t\in[a,b]}|f''(R\exp(it))|\,R\int_a^b|R\exp(it)-R\exp(is)|^2\,dt \\
&=\frac{1}{2}\sup_{t\in[a,b]}|f''(R\exp(it))|\,R^3\int_a^b 4\sin^2\left(\frac{s-t}{2}\right)dt \\
&=2\sup_{t\in[a,b]}|f''(R\exp(it))|\,R^3\int_a^b\sin^2\left(\frac{s-t}{2}\right)dt,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\left|\int_a^b f(R\exp(it))\exp(it)\,dt\right. \\
&-2R\left[f(R\exp(is))+f'(R\exp(is))\left(\frac{\exp(ib)+\exp(ia)}{2}-\exp(is)\right)\right] \\
&\qquad\left.\times\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right]\right| \\
&\leq 2\sup_{t\in[a,b]}|f''(R\exp(it))|\,R^2\int_a^b\sin^2\left(\frac{s-t}{2}\right)dt
\end{aligned}\tag{3.2}$$

for $s\in[a,b]$.

Since

$$\sin^2\left(\frac{s-t}{2}\right)=\frac{1-\cos(s-t)}{2},$$

hence

$$\begin{aligned}
&\int_a^b\sin^2\left(\frac{s-t}{2}\right)dt \\
&=\int_a^b\frac{1-\cos(s-t)}{2}dt=\frac{1}{2}[b-a-\sin(b-s)-\sin(s-a)] \\
&=\frac{1}{2}\left[b-a-2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}-s\right)\right] \\
&=\frac{b-a}{2}-\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}-s\right)
\end{aligned}$$

for $s\in[a,b]$.

Therefore by (3.2) we get

$$
\begin{aligned}
(3.3)\quad & \left|\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right. \\
& - 2R\left[f\left(R\exp\left(is\right)\right) + f'\left(R\exp\left(is\right)\right)\left(\frac{\exp\left(ib\right)+\exp\left(ia\right)}{2} - \exp\left(is\right)\right)\right] \\
& \qquad \left.\times \sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right]\right| \\
& \leq 2R^2 \sup_{t\in[a,b]}\left|f''\left(R\exp\left(it\right)\right)\right|\left[\frac{b-a}{2} - \sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}-s\right)\right]
\end{aligned}
$$

for $s\in[a,b]$.

In particular, for $s=\frac{a+b}{2}$, we obtain from (3.3) the best possible inequality

$$
\begin{aligned}
(3.4)\quad & \left|\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right. \\
& - 2R\left[f\left(R\exp\left(\frac{a+b}{2}i\right)\right) + f'\left(R\exp\left(\frac{a+b}{2}i\right)\right)\right. \\
& \quad \left.\times\left(\frac{\exp\left(ib\right)+\exp\left(ia\right)}{2} - \exp\left(\frac{a+b}{2}i\right)\right)\right] \\
& \qquad \left.\times \sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right]\right| \\
& \leq 2R^2 \sup_{t\in[a,b]}\left|f''\left(R\exp\left(it\right)\right)\right|\left[\frac{b-a}{2} - \sin\left(\frac{b-a}{2}\right)\right].
\end{aligned}
$$

By utilising the inequality (2.16) for the circular path $\gamma_{[a,b],R}$ and $v=R\exp\left(is\right)\in\gamma_{[a,b],R}$ with $s\in[a,b]$, we also get

$$
\begin{aligned}
(3.5)\quad & \left|f\left(R\exp\left(ib\right)\right)\sin\left(\frac{b-s}{2}\right)\exp\left[\left(\frac{s+b}{2}\right)i\right]\right. \\
& + f\left(R\exp\left(ia\right)\right)\sin\left(\frac{s-a}{2}\right)\exp\left[\left(\frac{a+s}{2}\right)i\right] \\
& + f\left(R\exp\left(is\right)\right)\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right] \\
& \left. - \int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right| \\
& \leq R^2 \sup_{t\in[a,b]}\left|f''\left(R\exp\left(it\right)\right)\right|\left[\frac{b-a}{2} - \sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}-s\right)\right].
\end{aligned}
$$

In particular, for $s=\frac{a+b}{2}$, we get from (3.5) best possible inequality

$$\begin{aligned}(3.6)\quad &\left| f\left(R\exp\left(bi\right)\right)\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{a+3b}{4}\right)i\right]\right.\\ &+f\left(R\exp\left(ia\right)\right)\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{3a+b}{4}\right)i\right]\\ &+f\left(R\exp\left(\frac{a+b}{2}i\right)\right)\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right]\\ &\left.-\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right|\\ &\leq R^2\sup_{t\in[a,b]}\left|f''\left(R\exp\left(it\right)\right)\right|\left[\frac{b-a}{2}-\sin\left(\frac{b-a}{2}\right)\right].\end{aligned}$$

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