NEW INTEGRAL INEQUALITIES OF THE TYPE OF SIMPSON'S AND HERMITE-HADAMARD'S FOR TWICE DIFFERENTIABLE QUASI-GEOMETRICALLY CONVEX MAPPINGS

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ABSTRACT. In this paper, we define a new identity for twice differentiable mappings and obtained some new estimates on the generalization of Hadamard's and Simpson's type inequalities for quasi-geometrically convex mappings using of this identity.

1. INTRODUCTION AND PRELIMINARY RESULTS

A convex function is a continuous function whose value at the midpoint of every interval in its domain does not exceed the arithmetic mean of its value at the ends of the interval. An important mathematical problem is to investigate how function behaves under the action of means. The best known case is that of midpoint convex (or Jensen convex) functions, which deals with the arithmetic mean [7, pp.2].

More generally, a function $f(x)$ is convex on an interval $[a, b]$, if for any two points $x, y \in [a, b]$ and $\alpha, \beta \in (0, 1]$, we have

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$$

A real valued function defined on nonempty subinterval I of $\mathbb{R}$ is called convex if we replace $\alpha + \beta = 1$ for all points x and $y \in I$ in above inequality. It is called strictly convex if the above inequality holds strictly whenever x and y are distinct points. If $-f$ is convex (respectively, strictly convex) then we say that f is concave (respectively, strictly concave). A function is called affine if it is both convex and concave.

The appearance of the new mathematical inequality often puts on firm foundation for the heuristic algorithms and procedures used in applied sciences. Among others one of the main inequality, which gives us an explicit error bounds in the trapezoidal and midpoint

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rules of a smooth function, called Hermit-Hadamard's inequality defined as

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f:[a,b]\to\mathbb{R}$ is a convex function. Both inequalities hold in the reversed direction for f to be concave. We note that Hermit-Hadamard's inequality (1.1) may be regarded as a refinement of the concept of convexity and it follows easily from Jensens inequality. Inequality (1.1) has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found in literature and the references cited therein. The second well known inequality in literature as Simpson's Inequality defined as

$$\left|\frac{1}{3}\left[2f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\parallel f^{(iv)}\parallel_\infty (b-a)^4 \tag{1.2}$$

where $f:[a,b]\to\mathbb{R}$ is a four times continuous differentiable mapping on (a,b) and $\parallel f^{(iv)}\parallel_\infty=\sup_{x\in(a,b)}\left|f^{(iv)}(x)\right|<\infty$. It is well known that if the mapping f is neither four times differentiable nor is the fourth derivative $f^{(iv)}$ bounded on (a,b), then we cannot apply the classical Simpson quadrature formula.

The notion of quasi-convex functions generalizes the notion of convex functions. More exactly, a function $f:[a,b]\to\mathbb{R}$ is said quasi-convex on $[a,b]$ if

$$f(\alpha x + \beta y) \leq \sup\{f(x),f(y)\}$$

where $x,y\in[a,b]$, $\alpha,\beta\in(0,1]$ and $\alpha+\beta=1$. The notion of geometrically convex first introduce by Niculescu, C. P. in [5] and [6] and produced as

Definition 1. *A function* $f:I\subseteq\mathbb{R}^+\to\mathbb{R}^+$ *is said to be GG-convex (called geometrically convex function) if*

$$f\left(x^\alpha y^\beta\right) \leq f^\alpha(x)f^\beta(y)$$

where $x,y\in[a,b]$, $\alpha,\beta\in(0,1]$ *and* $\alpha+\beta=1$.

Niculescu in same article defined the term $geometric-arithmatically$ convex with notation GA-convex as

Definition 2. *A function* $f:I\subseteq\mathbb{R}^+\to\mathbb{R}$ *is said to be GG-convex (called geometrically convex function) if*

$$f\left(x^\alpha y^\beta\right) \leq \alpha f(x) + \beta f(y)$$

where $x,y\in[a,b]$, $\alpha,\beta\in(0,1]$ *and* $\alpha+\beta=1$.

In [1], İşcan, İ. gave definition of quasi-geometrically convexity as follows:

Definition 3. *A function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ is said to be quasi-convex if*

$$f\left(x^\alpha y^\beta\right) \leq \sup\{f(x), f(y)\}$$

where $x, y \in [a, b]$, $\alpha, \beta \in (0, 1]$ and $\alpha + \beta = 1$.

Clearly, any GA-convex and geometrically convex functions are quasi-geometrically convex functions. Furthermore, there exist quasi-geometrically convex functions which are neither GA-convex nor GG-convex [1].

Recently, İşcan, İ. et.al in [4] established some results based on single differentiability for quasi-geometrically convex functions using the identity

Lemma 1. *A function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I^o such that $f \in L^1([a, b])$, where $a, b \in I$ with $a < b$. Then for all $\lambda, \mu \in \mathbb{R}$, we have:*

$$I_f(\lambda, \mu, a, b) = ln\left(\frac{b}{a}\right) \left\{ \int_0^{\frac{1}{2}} (t - \mu)a^{1-t}b^t f'\left(a^{1-t}b^t\right) dt \right.$$

$$\left. + \int_{\frac{1}{2}}^1 (t - \lambda)a^{1-t}b^t f'\left(a^{1-t}b^t\right) dt \right\} \qquad (1.3)$$

where

$$I_f(\lambda, \mu, a, b) = (\lambda - \mu) f\left(\sqrt{ab}\right) + \mu f(a) + (1 - \lambda) f(b) - \frac{1}{ln\left(\frac{b}{a}\right)} \int_a^b \frac{f(u)}{u} du$$

where $a, b \in I$ with $a < b$ and $\lambda, \mu \in \mathbb{R}$.

This article is in the continuation of [4]. The main purpose of this article is to establish some new general integral inequalities of Hermite-Hadamard and Simpson type for twice differentiable quasi-geometrically convex functions by using a new integral identity.

2. MAIN RESULTS

In order to prove our main results we need the following identity.

Lemma 2. *A function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I^o such that $f \in L^1([a, b])$, where $a, b \in I$ with $a < b$. Then for all $\lambda, \mu \in \mathbb{R}$, we have*

$$|M_f(\lambda, \mu, a, b)| = \left(ln\left(\frac{b}{a}\right)\right)^2 \left\{ \int_0^{\frac{1}{2}} t(t - \mu)a^{2(1-t)}b^{2t} f''\left(a^{1-t}b^t\right) dt \right.$$

$$\left. + \int_{\frac{1}{2}}^1 (1 - t)(t - \lambda)a^{2(1-t)}b^{2t} f''\left(a^{1-t}b^t\right) dt \right\} (2.4)$$

where

$$M_f\left(\lambda, \mu, a, b\right) = \left(\lambda - \mu + 1\right) f\left(\sqrt{ab}\right) + \mu f(a) + \left(1 - \lambda\right) f(b)$$

$$+ \frac{\sqrt{ab}\left(\lambda + \mu - 1\right)}{2} ln\left(\frac{b}{a}\right) f'\left(\sqrt{ab}\right) - \frac{2}{ln\left(b/a\right)} \int_a^b \frac{f(u)}{u} du$$

where $a, b \in I$ with $a < b$ and $\lambda, \mu \in \mathbb{R}$.

Proof. Using integration rules and changing the parameter, we can easily prove the above result. $\square$

Theorem 1. *A function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function on I^o such that $f'' \in L^1\left([a, b]\right)$, where $a, b \in I$ with $a < b$. If $\left|f''\right|^q$ is $quasi-geometrically$ convex on $[a, b]$ for some fixed $q \geq 1$ and $0 \leq \mu \leq 1/2 \leq \lambda \leq 1$, then the following inequality holds*

$$\left|M_f\left(\lambda, \mu, a, b\right)\right| \leq \left(\ln\left(\frac{b}{a}\right)\right)\left(\sup\left\{\left|f''(a)\right|^q, \left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}\left\{c_1^{1-\frac{1}{q}}(\mu)c_3^{\frac{1}{q}}\left(\mu, q, a, b\right)\right.$$

$$\left. + c_2^{1-\frac{1}{q}}(\lambda)c_4^{\frac{1}{q}}\left(\lambda, q, a, b\right)\right\} \qquad (2.5)$$

where

$$c_1(\mu) = \frac{\mu^3}{3} - \frac{\mu}{8} + \frac{1}{24}$$

$$c_2(\lambda) = -\frac{\lambda^3}{3} + \lambda^2 - \frac{7\lambda}{8} + \frac{1}{4}$$

$$c_3(\mu, q, a, b) = \frac{1}{2q\left(2\ln\left(\frac{b}{a}\right)\right)}\left[8\mu^2 a^{2q(1-\mu)}\mathrm{L}\left(a^{2q\mu}, b^{2q\mu}\right) - \frac{8\mu a^{2q(1-\mu)}}{q\left(2\ln\left(\frac{b}{a}\right)\right)}\times\right.$$

$$\mathrm{L}\left(a^{2q\mu}, b^{2q\mu}\right) + \frac{a^q}{q\left(\ln\left(\frac{b}{a}\right)\right)}\mathrm{L}\left(a^q, b^q\right) + \frac{10\mu a^{2q}}{q\left(2\ln\left(\frac{b}{a}\right)\right)}$$

$$\left. + \frac{2(\mu - 1)}{q\left(2\ln\left(\frac{b}{a}\right)\right)}(ab)^q + \frac{1 - 2\mu}{2}(ab)^q\right]$$

$$c_4(\lambda, q, a, b) = \frac{1}{2q\left(2\ln\left(\frac{b}{a}\right)\right)}\left[\frac{2(1-\lambda)b^{2q}}{q\left(2\ln\left(\frac{b}{a}\right)\right)} - \frac{(2\lambda - 1)(ab)^q}{2} + 4\lambda(1-\lambda)a^{2q(1-\lambda)}\times\right.$$

$$\mathrm{L}\left(a^{2q\lambda}, b^{2q\lambda}\right) - 2\lambda a^q\mathrm{L}\left(a^q, b^q\right) + \frac{8a^{2q(1-\lambda)}b^{2q\lambda}}{q^2\left(2\ln\left(\frac{b}{a}\right)\right)^2} + \frac{2\lambda(ab)^q}{q\left(2\ln\left(\frac{b}{a}\right)\right)}$$

$$\left. - \frac{4b^{2q}}{q^2\left(2\ln\left(\frac{b}{a}\right)\right)^2} - \frac{4(ab)^q}{q^2\left(2\ln\left(\frac{b}{a}\right)\right)^2}\right]$$

Proof. Since $\left|f''\right|^q$ is quasi-geometrically convex on $[a, b]$ for all $t \in [0.1]$

$$\left|f''\left(a^{1-t}b^t\right)\right|^q \leq \sup\left\{\left|f''(a)\right|^q, \left|f''(b)\right|^q\right\}$$

From lemma 2 and using power mean inequality, we have

$$|M_f(\lambda,\mu,a,b)| = \left(ln\left(\frac{b}{a}\right)\right)^2 \left\{\left(\int_0^{\frac{1}{2}}|t(t-\mu)|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}|t(t-\mu)|\left(a^{2(1-t)}b^{2t}\right)^q\times\right.\right.$$

$$\left.\sup\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1|(1-t)(t-\lambda)|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1|(1-t)(t-\lambda)|\right.$$

$$\left.\left.\left(a^{2(1-t)}b^{2t}\right)^q\sup\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}\right\} \qquad (2.6)$$

Let here

$$c_1^{1-\frac{1}{q}}(\mu) = \int_0^{\frac{1}{2}}|t(t-\mu)|dt = \int_0^{\mu}(t(\mu-t))dt + \int_\mu^{\frac{1}{2}}(t(t-\mu))dt = \frac{\mu^3}{3}-\frac{\mu}{8}+\frac{1}{24}$$

$$c_2^{1-\frac{1}{q}}(\lambda) = \int_{\frac{1}{2}}^1|(1-t)(t-\lambda)|dt = \int_{\frac{1}{2}}^\lambda(1-t)(\lambda-t)dt + \int_\lambda^1(1-t)(t-\lambda)dt = -\frac{\lambda^3}{3}+\lambda^2-\frac{7\lambda}{8}+\frac{1}{4}$$

$$c_3^{\frac{1}{q}}(\mu,q,a,b) = \int_0^{\frac{1}{2}}|t(\mu-t)|\left(a^{2(1-t)}b^{2t}\right)^q dt = \int_0^{\mu}t(\mu-t)\left(a^{2(1-t)}b^{2t}\right)^q dt + \int_\mu^{\frac{1}{2}}t(t-\mu)\left(a^{2(1-t)}b^{2t}\right)^q dt$$

Using substitution $u = a^{2(1-t)}b^{2t}$ in $c_3^{\frac{1}{q}}(\mu,q,a,b)$, we have

$$\int_0^{\mu}t(\mu-t)\left(a^{2(1-t)}b^{2t}\right)dt = \frac{\mu}{\left(2\ln\left(\frac{b}{a}\right)\right)^2}\int_{a^2}^{a^{2(1-\mu)}b^{2\mu}}u^{q-1}\ln\left(\frac{u}{a^2}\right)du$$

$$-\frac{\mu}{\left(2\ln\left(\frac{b}{a}\right)\right)^3}\int_{a^2}^{a^{2(1-\mu)}b^{2\mu}}u^{q-1}\ln\left(\frac{u}{a^2}\right)du$$

$$\int_0^{\mu}t(\mu-t)\left(a^{2(1-t)}b^{2t}\right)dt = \frac{\mu a^{2q(1-\mu)}b^{2q\mu}}{q^2\left(2\ln\left(\frac{b}{a}\right)\right)^2} - \frac{2a^{2q(1-\mu)}b^{2q\mu}}{q^3\left(2\ln\left(\frac{b}{a}\right)\right)^3} + \frac{\mu a^{2q}}{q^2\left(2\ln\left(\frac{b}{a}\right)\right)^2} + \frac{\mu a^{2q}}{q^3\left(2\ln\left(\frac{b}{a}\right)\right)^3}$$

$$\int_\mu^{\frac{1}{2}}t(t-\mu)\left(a^{2(1-t)}b^{2t}\right)dt = \frac{1}{\left(2\ln\left(\frac{b}{a}\right)\right)^3}\int_{a^{2(1-\mu)}b^{2\mu}}^{ab}u^{q-1}\left(\ln\left(\frac{u}{a^2}\right)\right)^2 du$$

$$-\frac{\mu}{\left(2\ln\left(\frac{b}{a}\right)\right)^2}\int_{a^{2(1-\mu)}b^{2\mu}}^{ab}u^{q-1}\ln\left(\frac{u}{a^2}\right)du$$

$$\int_\mu^{\frac{1}{2}}t(t-\mu)\left(a^{2(1-t)}b^{2t}\right)dt = \frac{(1-2\mu)(ab)^q}{8q\ln\left(\frac{b}{a}\right)} + \frac{(\mu-1)(ab)^q}{q^2\left(2\ln\left(\frac{b}{a}\right)\right)^2} + \frac{3\mu a^{2q(1-\mu)}b^{2qu}}{q^2\left(2\ln\left(\frac{b}{a}\right)\right)^2}$$

$$+\frac{2(ab)^q}{q^3\left(2\ln\left(\frac{b}{a}\right)\right)^3} - \frac{2a^{2q(1-\mu)}b^{2q\mu}}{q^3\left(2\ln\left(\frac{b}{a}\right)\right)^3}$$

finally, we get

$$
\begin{aligned}
c_3(\mu, q, a, b) \;=\; & \frac{1}{2q\left(2\ln\left(\frac{b}{a}\right)\right)}\left[8\mu^2 a^{2q(1-\mu)}\mathrm{L}\left(a^{2q\mu}, b^{2q\mu}\right) - \frac{8\mu a^{2q(1-\mu)}}{q\left(2\ln\left(\frac{b}{a}\right)\right)}\times\right. \\
& \mathrm{L}\left(a^{2q\mu}, b^{2q\mu}\right) + \frac{a^q}{q\left(\ln\left(\frac{b}{a}\right)\right)}\mathrm{L}\left(a^q, b^q\right) + \frac{10\mu a^{2q}}{q\left(2\ln\left(\frac{b}{a}\right)\right)} \\
& \left. + \frac{2(\mu-1)}{q\left(2\ln\left(\frac{b}{a}\right)\right)}(ab)^q + \frac{1-2\mu}{2}(ab)^q\right]
\end{aligned}
$$

And

$$
\begin{aligned}
c_4^{\frac{1}{q}}(\lambda, q, a, b) \;=\; & \int_{\frac{1}{2}}^{1}\left|(1-t)(t-\lambda)\right|\left(a^{2(1-t)}b^{2t}\right)^q dt \\
\;=\; & \int_{\frac{1}{2}}^{\lambda} t(\mu-t)\left(a^{2(1-t)}b^{2t}\right)^q dt + \int_{\lambda}^{1} t(t-\mu)\left(a^{2(1-t)}b^{2t}\right)^q dt
\end{aligned}
$$

Using same substitution $u = a^{2(1-t)}b^{2t}$ in $c_4^{\frac{1}{q}}(\lambda, q, a, b)$, we have

$$
\begin{aligned}
c_4(\lambda, q, a, b) \;=\; & \frac{1}{2q\left(2\ln\left(\frac{b}{a}\right)\right)}\left[\frac{2(1-\lambda)b^{2q}}{q\left(2\ln\left(\frac{b}{a}\right)\right)} - \frac{(2\lambda-1)(ab)^q}{2} + 4\lambda(1-\lambda)a^{2q(1-\lambda)}\times\right. \\
& \mathrm{L}\left(a^{2q\lambda}, b^{2q\lambda}\right) - 2\lambda a^q \mathrm{L}\left(a^q, b^q\right) + \frac{8a^{2q(1-\lambda)}b^{2q\lambda}}{q^2\left(2\ln\left(\frac{b}{a}\right)\right)^2} + \frac{2\lambda(ab)^q}{q\left(2\ln\left(\frac{b}{a}\right)\right)} \\
& \left. - \frac{4b^{2q}}{q^2\left(2\ln\left(\frac{b}{a}\right)\right)^2} - \frac{4(ab)^q}{q^2\left(2\ln\left(\frac{b}{a}\right)\right)^2}\right]
\end{aligned}
$$

This completes the proof. □

Corollary 1. *A function* $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ *be a twice differentiable function on* I^o *such that* $f'' \in L^1([a, b])$, *where* $a, b \in I$ *with* $a < b$. *If* $|f''|^q$ *is* $quasi-geometrically$ *convex on* $[a, b]$ *for some fixed* $q \geq 1$ *and for* $l, m \in \mathbb{R}$ *with* $l < m$, *then the following inequality holds*

$$
\begin{aligned}
\left|m_f\left(\frac{l}{m}, a, b\right)\right| \leq & \frac{8l^3 - 3m^2 l + m^3}{24m^3}\left(\ln\left(\frac{b}{a}\right)\right)\left(\sup\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}}\times \\
& \left\{c_3^{\frac{1}{q}}\left(\frac{l}{m}, q, a, b\right) + c_4^{\frac{1}{q}}\left(\frac{l}{m}, q, a, b\right)\right\} \quad (2.7)
\end{aligned}
$$

where

$$
m_f\left(\frac{l}{m}, a, b\right) = 2\left(\frac{m-l}{m}\right)f\left(\sqrt{ab}\right) + \frac{l}{m}\left(f(a) + f(b)\right) - \frac{2}{ln\left(\frac{b}{a}\right)}\int_a^b \frac{f(u)}{u}du
$$

and

$$
\begin{aligned}
c_1\left(\frac{l}{m}\right) = \; & c_2\left(\frac{l}{m}\right) = \frac{8l^3 - 3m^2l + m^3}{24m^3} \\
c_3\left(\frac{l}{m}, q, a, b\right) = \; & \frac{1}{2q\left(2\ln\left(\frac{b}{a}\right)\right)}\left[\frac{8l}{m}a^{\frac{2q(m-l)}{m}}\left(\frac{lq\left(2\ln\left(\frac{b}{a}\right)\right)-m}{mq\left(2\ln\left(\frac{b}{a}\right)\right)}\right)\mathrm{L}\left(a^{2q\frac{l}{m}}, b^{2q\frac{l}{m}}\right)\right. \\
& + \frac{a^q}{q\left(\ln\left(\frac{b}{a}\right)\right)}\mathrm{L}\left(a^q, b^q\right) + \frac{10la^{2q}}{mq\left(2\ln\left(\frac{b}{a}\right)\right)} - \frac{2(m-l)}{mq\left(2\ln\left(\frac{b}{a}\right)\right)}(ab)^q \\
& \left. + \frac{(m-2l)}{2m}(ab)^q\right] \\
c_4\left(\frac{l}{m}, q, a, b\right) = \; & \frac{1}{2q\left(2\ln\left(\frac{b}{a}\right)\right)}\left[\frac{8l}{m}b^{\frac{2q(m-l)}{m}}\left(\frac{lq\left(2\ln\left(\frac{b}{a}\right)\right)-m}{mq\left(2\ln\left(\frac{b}{a}\right)\right)}\right)\mathrm{L}\left(a^{2q\frac{l}{m}}, b^{2q\frac{l}{m}}\right)\right. \\
& + \frac{b^q}{q\left(\ln\left(\frac{b}{a}\right)\right)}\mathrm{L}\left(a^q, b^q\right) + \frac{10lb^{2q}}{mq\left(2\ln\left(\frac{b}{a}\right)\right)} - \frac{2(m-l)}{mq\left(2\ln\left(\frac{b}{a}\right)\right)}(ab)^q \\
& \left. + \frac{(m-2l)}{2m}(ab)^q\right]
\end{aligned}
$$

Proof. Proof is exchangeable with Theorem 1 with the substitution $\mu = \frac{l}{m}$ and $\lambda = 1 - \frac{l}{m}$. ☐

Theorem 2. *A function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function on I^o such that $f'' \in L^1([a,b])$, where $a, b \in I$ with $a < b$. If $|f''|^q$ is $quasi-geometrically$ convex on $[a,b]$ for some fixed conjugate numbers $p, q \geq 0$ where $q > 1$ and $0 \leq \mu \leq 1/2 \leq \lambda \leq 1$, then the following inequality holds*

$$
|M_f(\lambda, \mu, a, b)| \leq \left(\ln\left(\frac{b}{a}\right)\right)^2 \left(\sup\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}}
$$
$$
\left\{c_5^{\frac{1}{p}}(p, \mu)c_7^{\frac{1}{q}}(q, a, b) + c_6^{\frac{1}{p}}(p, \lambda)c_8^{\frac{1}{q}}(q, a, b)\right\} \quad (2.8)
$$

where

$$
c_5(p, \mu) = \frac{1}{1-\mu}\left[\frac{(1-2\mu)^{p+1}}{4^{p+1}(p+1)}\right], c_6(p, \lambda) = \frac{(2\lambda-1)^{p+1}}{\lambda 4^{p+1}(p+1)}
$$

$$
c_7(q, a, b) = \frac{a^q}{2}\mathrm{L}\left(a^q, b^q\right), c_8(q, a, b) = \frac{b^q}{2}\mathrm{L}\left(a^q, b^q\right)
$$

Proof. From lemma 2 by applying Hölder inequality and using the $quasi-geometrically$ convexity on $[a,b]$ of $|f''|^q$, we have

$$|M_f(\lambda,\mu,a,b)| = \left(ln\left(\frac{b}{a}\right)\right)^2 \left\{ \left(\int_0^{\frac{1}{2}} |t(\mu-t)|^p dt\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left(a^{2(1-t)}b^{2t}\right)^q \times \right.$$

$$\left. \sup\left\{|f''(a)|^q, |f''(b)|^q\right\} dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 |(1-t)(t-\lambda)|^p dt\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left(a^{2(1-t)}b^{2t}\right)^q\right.$$

$$\left. \sup\left\{|f''(a)|^q, |f''(b)|^q\right\} dt\right)^{\frac{1}{q}} \right\} \qquad (2.9)$$

$$|M_f(\lambda,\mu,a,b)| = \left(ln\left(\frac{b}{a}\right)\right)^2 \left(\sup\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}} \left\{ \left(\int_0^{\frac{1}{2}} |t(\mu-t)|^p dt\right)^{\frac{1}{p}} \right.$$

$$\left(\int_0^{\frac{1}{2}} \left(a^{2(1-t)}b^{2t}\right)^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 |(1-t)(t-\lambda)|^p dt\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left(a^{2(1-t)}b^{2t}\right)^q dt\right)^{\frac{1}{q}} \left. \right\} (2.10)$$

where

$$c_5(p,\mu) = \int_0^{\frac{1}{2}} |t(\mu-t)|^p dt = \int_0^\mu t^p(\mu-t)^p dt + \int_\mu^{\frac{1}{2}} t^p(t-\mu)^p dt = \frac{1}{1-\mu}\left[\frac{(1-2\mu)^{p+1}}{4^{p+1}(p+1)}\right]$$

$$c_6(p,\mu) = \int_{\frac{1}{2}}^1 |(1-t)(\lambda-t)|^p dt = \int_{\frac{1}{2}}^\lambda (1-t)^p(t-\lambda)^p dt + \int_\lambda^1 (1-t)^p(t-\lambda)^p dt = \frac{(2\lambda-1)^{p+1}}{\lambda 4^{p+1}(p+1)}$$

Here we use $u = a^{2(1-t)}b^{2t}$ to calculate $c_7^{\frac{1}{q}}(q,a,b)$ and $c_8^{\frac{1}{q}}(q,a,b)$

$$c_7(q,a,b) = \int_0^{\frac{1}{2}} \left(a^{2(1-t)}b^{2t}\right)^q dt = \frac{a^q}{2}\text{L}(a^q,b^q)$$

$$c_8(q,a,b) = \int_{\frac{1}{2}}^1 \left(a^{2(1-t)}b^{2t}\right)^q dt = \frac{b^q}{2}\text{L}(a^q,b^q)$$

Hence (2.8) easily found from (2.10). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Corollary 2. *A function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function on I^o such that $f'' \in L^1([a,b])$, where $a,b \in I$ with $a < b$. If $|f''|^q$ is $quasi-geometrically$ convex on $[a,b]$ for some fixed conjugate numbers $p,q \geq 0$ with $q > 1$ and for $l,m \in \mathbb{R}$ with $l < m$, then the following inequality holds*

$$\left|m_f\left(\frac{l}{m},a,b\right)\right| \leq \left(\ln\left(\frac{b}{a}\right)\right)^2 \frac{1}{m^p(p+1)}\left[\frac{(m-2l)^{p+1}}{4^{p+1}(m-l)}\right] \left(\sup\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}} \times$$

$$\left\{c_7^{\frac{1}{q}}(q,a,b) + c_8^{\frac{1}{q}}(q,a,b)\right\} \quad (2.11)$$

where

$$c_5\left(p,\frac{l}{m}\right) = c_6\left(p,\frac{l}{m}\right) = \frac{1}{m^p(p+1)}\left[\frac{(m-2l)^{p+1}}{4^{p+1}(m-l)}\right],$$

$$c_7(q, a, b) = \frac{a^q}{2} \mathrm{L}\left(a^q, b^q\right), c_8(q, a, b) = \frac{b^q}{2} \mathrm{L}\left(a^q, b^q\right)$$

and $m_f\left(\frac{l}{m}, a, b\right)$ *fixed in Corollary 1.*

Proof. Proof is exchangeable with Theorem 2 with the substitution $\mu = \frac{l}{m}$ and $\lambda = 1 - \frac{l}{m}$.
□

Theorem 3. *A function* $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ *be a twice differentiable function on* I^o *such that* $f'' \in L^1([a, b])$ *, where* $a, b \in I$ *with* $a < b$. *If* $|f''|^q$ *is* $quasi - geometrically$ *convex on* $[a, b]$ *for some fixed conjugate numbers* $p, q \geq 0$ *where* $q > 1$ *and* $0 \leq \mu \leq 1/2 \leq \lambda \leq 1$, *then the following inequality holds*

$$|M_f\left(\lambda, \mu, a, b\right)| \leq \left(\ln\left(\frac{b}{a}\right)\right)^2 \left(\sup\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}}$$

$$\left\{c_7^{\frac{1}{p}}(p, a, b) c_5^{\frac{1}{q}}(q, \mu) + c_8^{\frac{1}{p}}(p, a, b) c_6^{\frac{1}{q}}(q, \lambda)\right\} \quad (2.12)$$

where

$$c_5(q, \mu) = \frac{1}{1 - \mu}\left[\frac{(1 - 2\mu)^{q+1}}{4^{q+1}(q + 1)}\right], c_6(q, \lambda) = \frac{(2\lambda - 1)^{q+1}}{\lambda 4^{q+1}(q + 1)}$$

$$c_7(p, a, b) = \frac{a^p}{2} \mathrm{L}\left(a^p, b^p\right), c_8(p, a, b) = \frac{b^p}{2} \mathrm{L}\left(a^p, b^p\right)$$

and $\frac{1}{p} + \frac{1}{q} = 1$

Proof. From lemma 2 by applying Hölder inequality and using the $quasi - geometrically$ convexity on $[a, b]$ of $|f''|^q$, we have

$$|M_f(\lambda, \mu, a, b)| = \left(ln\left(\frac{b}{a}\right)\right)^2 \left\{\left(\int_0^{\frac{1}{2}}\left(a^{2(1-t)}b^{2t}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|t(\mu - t)|^q \times\right.\right.$$

$$\left.\sup\left\{|f''(a)|^q, |f''(b)|^q\right\} dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1\left(a^{2(1-t)}b^{2t}\right)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1|(1-t)(t-\lambda)|^q\right.$$

$$\left.\left.\sup\left\{|f''(a)|^q, |f''(b)|^q\right\} dt\right)^{\frac{1}{q}}\right\} \quad (2.13)$$

$$|M_f(\lambda, \mu, a, b)| = \left(ln\left(\frac{b}{a}\right)\right)^2 \left(\sup\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}}\left\{\left(\int_0^{\frac{1}{2}}|t(\mu - t)|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.\left(\int_0^{\frac{1}{2}}\left(a^{2(1-t)}b^{2t}\right)^p dt\right)^{\frac{1}{p}} + \left(\int_{\frac{1}{2}}^1|(1-t)(t-\lambda)|^q dt\right)^{\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left(a^{2(1-t)}b^{2t}\right)^p dt\right)^{\frac{1}{p}}\right\}(2.14)$$

where

$$c_5(q, \mu) = \int_0^{\frac{1}{2}}|t(\mu - t)|^q dt = \frac{1}{1 - \mu}\left[\frac{(1 - 2\mu)^{q+1}}{4^{q+1}(q+1)}\right]$$

$$c_6(p, \lambda) = \int_{\frac{1}{2}}^1 |(1-t)(\lambda-t)|^q dt = \frac{(2\lambda-1)^{q+1}}{\lambda 4^{q+1}(q+1)}$$

Here we use $u = a^{2(1-t)}b^{2t}$ to calculate $c_7^{\frac{1}{q}}(q, a, b)$ and $c_8^{\frac{1}{q}}(q, a, b)$

$$c_7(p, a, b) = \int_0^{\frac{1}{2}} \left(a^{2(1-t)}b^{2t}\right)^p dt = \frac{a^p}{2} \mathrm{L}(a^p, b^p)$$

$$c_8(p, a, b) = \int_{\frac{1}{2}}^1 \left(a^{2(1-t)}b^{2t}\right)^p dt = \frac{b^p}{2} \mathrm{L}(a^p, b^p)$$

Hence (2.12) easily found from (2.14). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 3. *A function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function on I^o such that $f'' \in L^1([a, b])$, where $a, b \in I$ with $a < b$. If $|f''|^q$ is $quasi-geometrically$ convex on $[a, b]$ for some fixed conjugate numbers $p, q \geq 0$ and for $l, m \in \mathbb{R}$ with $l < m$, then the following inequality holds*

$$\left| m_f\left(\frac{l}{m}, a, b\right) \right| \leq \left(\ln\left(\frac{b}{a}\right)\right)^2 \frac{1}{m^q(q+1)} \left[\frac{(m-2l)^{q+1}}{4^{q+1}(m-l)}\right] \left(\sup\left\{|f''(a)|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}} \times$$

$$\left\{c_7^{\frac{1}{p}}(p, a, b) + c_8^{\frac{1}{p}}(p, a, b)\right\} \quad (2.15)$$

where

$$c_5\left(q, \frac{l}{m}\right) = c_6\left(q, \frac{l}{m}\right) = \frac{1}{m^q(q+1)} \left[\frac{(m-2l)^{q+1}}{4^{q+1}(m-l)}\right],$$

$$c_7(p, a, b) = \frac{a^p}{2} \mathrm{L}(a^p, b^p), c_8(p, a, b) = \frac{b^p}{2} \mathrm{L}(a^p, b^p)$$

and $m_f\left(\frac{l}{m}, a, b\right)$ fixed in Corollary 1.

Proof. Proof is exchangeable with Theorem 3 with the substitution $\mu = \frac{l}{m}$ and $\lambda = 1 - \frac{l}{m}$. $\square$

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