TWO PARAMETERS WEIGHTED INEQUALITIES FOR THE COMPLEX INTEGRAL

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Abstract. In this paper we provide some upper bounds for the quantity

$$\left|\int_{\gamma} f(z)\, g(z)\, dz-\lambda \int_{\gamma_{u, v}} g(z)\, dz-\mu \int_{\gamma_{v, w}} g(z)\, dz\right|$$

with λ, $\mu \in \mathbb{C}$ and under the assumptions that f and g are continuous in D, an open domain and $\gamma \subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x \in(a, b)$. The cases when either f is bounded or Lipschitzian in certain sense are analyzed in some details. Some examples for circular paths are also given.

1. Introduction

Suppose γ is a smooth path parametrized by $z(t)$, $t \in[a, b]$ and f is a complex function which is continuous on γ. Put $z(a)=u$ and $z(b)=w$ with u, $w \in \mathbb{C}$. We define the integral of f on $\gamma_{u, w}=\gamma$ as

$$\int_{\gamma} f(z)\, dz=\int_{\gamma_{u, w}} f(z)\, dz:=\int_{a}^{b} f(z(t))\, z^{\prime}(t)\, dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in[a, b]$, which is differentiable on the intervals $[a, c]$ and $[c, b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u, w}} f(z)\, dz:=\int_{\gamma_{u, v}} f(z)\, dz+\int_{\gamma_{v, w}} f(z)\, dz$$

where $v:=z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u, w}} f(z)\,|dz|:=\int_{a}^{b} f(z(t))\left|z^{\prime}(t)\right| dt$$

and the length of the curve γ is then

$$\ell(\gamma)=\int_{\gamma_{u, w}}|dz|=\int_{a}^{b}\left|z^{\prime}(t)\right| dt.$$

Let f and g be holomorphic in D, and open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$. Then we have the *integration*

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by parts formula

$$\int_{\gamma_{u,w}} f(z)\, g'(z)\, dz = f(w)\, g(w) - f(u)\, g(u) - \int_{\gamma_{u,w}} f'(z)\, g(z)\, dz. \tag{1.1}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left| \int_{\gamma} f(z)\, dz \right| \leq \int_{\gamma} |f(z)|\, |dz| \leq \|f\|_{\gamma,\infty}\, \ell(\gamma) \tag{1.2}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \geq 1$ by

$$\|f\|_{\gamma,p} := \left(\int_{\gamma} |f(z)|^p\, |dz| \right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_{\gamma} |f(z)|\, |dz|.$$

If $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \leq [\ell(\gamma)]^{1/q}\, \|f\|_{\gamma,p}.$$

In the recent paper [8], we established the following Ostrowski type inequality for the complex integral:

Theorem 1. *Let f be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(x)$ with $x \in (a, b)$, then $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$,*

$$\begin{aligned} &\left| f(v)(w-u) - \int_{\gamma} f(z)\, dz \right| \\ &\leq \|f'\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-u|\, |dz| + \|f'\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-w|\, |dz| \\ &\leq \left[\int_{\gamma_{u,v}} |z-u|\, |dz| + \int_{\gamma_{v,w}} |z-w|\, |dz| \right] \|f'\|_{\gamma_{u,w};\infty} \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} &\left| f(v)(w-u) - \int_{\gamma} f(z)\, dz \right| \\ &\leq \max_{z\in\gamma_{u,v}} |z-u|\, \|f'\|_{\gamma_{u,v};1} + \max_{z\in\gamma_{v,w}} |z-w|\, \|f'\|_{\gamma_{v,w};1} \\ &\leq \max\left\{ \max_{z\in\gamma_{u,v}} |z-u|,\ \max_{z\in\gamma_{v,w}} |z-w| \right\} \|f'\|_{\gamma_{u,w};1}. \end{aligned} \tag{1.4}$$

If $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1,$ *then*

$$
\begin{aligned}
(1.5)\quad & \left| f(v)(w-u)-\int_{\gamma} f(z)\,dz \right| \\
&\leq \left(\int_{\gamma_{u,v}} |z-u|^{q} |dz| \right)^{1/q} \|f'\|_{\gamma_{u,v};p} + \left(\int_{\gamma_{v,w}} |z-w|^{q} |dz| \right)^{1/q} \|f'\|_{\gamma_{v,w};p} \\
&\leq \left(\int_{\gamma_{u,v}} |z-u|^{q} |dz| + \int_{\gamma_{v,w}} |z-w|^{q} |dz| \right)^{1/q} \|f'\|_{\gamma_{u,w};p} .
\end{aligned}
$$

The corresponding trapezoid inequality for complex integral was obtained in [9]:

Theorem 2. *Let* f *be holomorphic in* G, *an open domain and suppose* $\gamma\subset G$ *is a smooth path from* $z(a)=u$ *to* $z(b)=w$. *If* $v=z(x)$ *with* $x\in(a,b)$, *then* $\gamma_{u,w}=\gamma_{u,v}\cup\gamma_{v,w}$,

$$
\begin{aligned}
(1.6)\quad & \left| (v-u) f(u)+(w-v) f(w)-\int_{\gamma} f(z)\,dz \right| \\
&\leq \|f'\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-v|\,|dz| + \|f'\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-v|\,|dz| \\
&\leq \|f'\|_{\gamma_{u,w};\infty} \int_{\gamma_{u,w}} |z-v|\,|dz| ,
\end{aligned}
$$

and

$$
\begin{aligned}
(1.7)\quad & \left| (v-u) f(u)+(w-v) f(w)-\int_{\gamma} f(z)\,dz \right| \\
&\leq \|f'\|_{\gamma_{u,v};1} \max_{z\in\gamma_{u,v}} |z-v| + \|f'\|_{\gamma_{v,w};1} \max_{z\in\gamma_{v,w}} |z-v| \\
&\leq \|f'\|_{\gamma_{u,w};1} \max_{z\in\gamma_{u,w}} |z-v| .
\end{aligned}
$$

If $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1,$ *then*

$$
\begin{aligned}
(1.8)\quad & \left| (v-u) f(u)+(w-v) f(w)-\int_{\gamma} f(z)\,dz \right| \\
&\leq \|f'\|_{\gamma_{u,v};p} \left(\int_{\gamma_{u,v}} |z-v|^{q} |dz| \right)^{1/q} + \|f'\|_{\gamma_{v,w};p} \left(\int_{\gamma_{v,w}} |z-v|^{q} |dz| \right)^{1/q} \\
&\leq \|f'\|_{\gamma_{u,w};p} \left(\int_{\gamma_{u,w}} |z-v|^{q} |dz| \right)^{1/q} .
\end{aligned}
$$

For several previous results concerning three points inequalities, see [1], [2] and [10]-[16]. For some trapezoid, Ostrowski, Grüss and quasi-Grüss type inequalities for complex functions defined on the unit circle centered in zero, see [3]-[7].

Motivated by the above results, in this paper we provide some upper bounds for the quantity

$$
\left| \int_{\gamma} f(z) g(z)\,dz-\lambda \int_{\gamma_{u,v}} g(z)\,dz-\mu \int_{\gamma_{v,w}} g(z)\,dz \right|
$$

with λ, $\mu \in \mathbb{C}$ and under the assumptions that f and g are continuous in D, an open domain and $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v = z(x)$ with $x \in (a, b)$. The cases when either f is bounded or Lipschitzian in certain sense are analyzed in some details. Some examples for circular paths are also given.

2. Some Preliminary Facts

We have:

Lemma 1. *Let f and g be continuous in D, an open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v = z(x)$ with $x \in (a, b)$. Then for any complex numbers λ, μ we have*

$$\begin{aligned}\int_{\gamma} f(z) g(z) dz &= \lambda \int_{\gamma_{u,v}} g(z) dz + \mu \int_{\gamma_{v,w}} g(z) dz \\ &\quad + \int_{\gamma_{u,v}} [f(z) - \lambda] g(z) dz + \int_{\gamma_{v,w}} [f(z) - \mu] g(z) dz \\ &= \mu \int_{\gamma} g(z) dz + (\lambda - \mu) \int_{\gamma_{u,v}} g(z) dz \\ &\quad + \int_{\gamma_{u,v}} [f(z) - \lambda] g(z) dz + \int_{\gamma_{v,w}} [f(z) - \mu] g(z) dz.\end{aligned} \tag{2.1}$$

In particular, for $\mu = \lambda$, we have

$$\begin{aligned}\int_{\gamma} f(z) g(z) dz &= \lambda \int_{\gamma} g(z) dz \\ &\quad + \int_{\gamma_{u,v}} [f(z) - \lambda] g(z) dz + \int_{\gamma_{v,w}} [f(z) - \lambda] g(z) dz \\ &= \lambda \int_{\gamma} g(z) dz + \int_{\gamma} [f(z) - \lambda] g(z) dz.\end{aligned} \tag{2.2}$$

Proof. Using the properties of the complex integral, we have

$$\begin{aligned}&\int_{\gamma_{u,v}} [f(z) - \lambda] g(z) dz + \int_{\gamma_{v,w}} [f(z) - \mu] g(z) dz \\ &= \int_{\gamma_{u,v}} f(z) g(z) dz - \lambda \int_{\gamma_{u,v}} g(z) dz + \int_{\gamma_{v,w}} f(z) g(z) dz - \mu \int_{\gamma_{v,w}} g(z) dz \\ &= \int_{\gamma} f(z) g(z) dz - \lambda \int_{\gamma_{u,v}} g(z) dz - \mu \int_{\gamma_{v,w}} g(z) dz,\end{aligned}$$

which is equivalent to the first equality in (2.1).

The rest is obvious. □

Corollary 1. *Let f be continuous in D, an open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v = z(x)$ with $x \in (a, b)$. Then for any complex numbers λ, μ we have*

$$\int_{\gamma} f(z) dz = \lambda (v - u) + \mu (w - v) + \int_{\gamma_{u,v}} [f(z) - \lambda] dz + \int_{\gamma_{v,w}} [f(z) - \mu] dz. \tag{2.3}$$

In particular, for $\mu = \lambda$, we have

$$\int_\gamma f(z) g(z) dz = \lambda (w-u) + \int_{\gamma_{u,v}} [f(z) - \lambda] dz + \int_{\gamma_{v,w}} [f(z) - \lambda] dz = \lambda (w-u) + \int_\gamma [f(z) - \lambda] g(z) dz. \tag{2.4}$$

If we use the equality (2.2) for $\lambda = f(v)$, $\lambda = \frac{1}{w-u}\int_\gamma f(z) dz$ and $\lambda = \frac{f(u)+f(w)}{2}$, then we have

$$\int_\gamma f(z) g(z) dz = f(v) \int_\gamma g(z) dz - \int_{\gamma_{u,v}} [f(v) - f(z)] g(z) dz + \int_{\gamma_{v,w}} [f(z) - f(v)] g(z) dz = \int_\gamma [f(z) - f(v)] g(z) dz, \tag{2.5}$$

$$\int_\gamma f(z) g(z) dz = \frac{1}{w-u} \int_\gamma f(z) dz \int_\gamma g(z) dz + \int_\gamma \left[f(z) - \frac{1}{w-u} \int_\gamma f(w) dw \right] g(z) dz, \tag{2.6}$$

and

$$\int_\gamma f(z) g(z) dz = \frac{f(u)+f(w)}{2} \int_\gamma g(z) dz + \int_\gamma \left[f(z) - \frac{f(u)+f(w)}{2} \right] g(z) dz, \tag{2.7}$$

respectively.

In particular, for $g(z) = 1$, $z \in \gamma$, we have for any $v \in \gamma$ that

$$\int_\gamma f(z) dz = (w-u) f(v) - \int_{\gamma_{u,v}} [f(v) - f(z)] dz + \int_{\gamma_{v,w}} [f(z) - f(v)] dz = (w-u) f(v) + \int_\gamma [f(z) - f(v)] dz, \tag{2.8}$$

and

$$\int_\gamma f(z) dz = (w-u) \frac{f(u)+f(w)}{2} + \int_\gamma \left[f(z) - \frac{f(u)+f(w)}{2} \right] dz, \tag{2.9}$$

respectively.

If we take $\lambda = f(u)$ and $\mu = f(w)$ in (2.1) we get for $v \in \gamma$ that

$$\int_\gamma f(z) g(z) dz = f(u) \int_{\gamma_{u,v}} g(z) dz + f(w) \int_{\gamma_{v,w}} g(z) dz + \int_{\gamma_{u,v}} [f(z) - f(u)] g(z) dz + \int_{\gamma_{v,w}} [f(z) - f(w)] g(z) dz. \tag{2.10}$$

Also, we take $\lambda = \frac{f(u)+f(v)}{2}$ and $\mu = \frac{f(v)+f(w)}{2}$ in (2.1) we get for $v \in \gamma$ that

$$\begin{aligned}(2.11)\quad &\int_\gamma f(z)\,g(z)\,dz \\ &= \frac{f(u)+f(v)}{2}\int_{\gamma_{u,v}} g(z)\,dz + \frac{f(v)+f(w)}{2}\int_{\gamma_{v,w}} g(z)\,dz \\ &\quad + \int_{\gamma_{u,v}}\left[f(z) - \frac{f(u)+f(v)}{2}\right] g(z)\,dz \\ &\quad + \int_{\gamma_{v,w}}\left[f(z) - \frac{f(v)+f(w)}{2}\right] g(z)\,dz\end{aligned}$$

while for $\lambda = \frac{1}{v-u}\int_{\gamma_{u,v}} f(y)\,dy$ and $\mu = \frac{1}{w-v}\int_{\gamma_{v,w}} f(y)\,dy$ we get

$$\begin{aligned}(2.12)\quad &\int_\gamma f(z)\,g(z)\,dz \\ &= \frac{1}{v-u}\int_{\gamma_{u,v}} f(y)\,dy\int_{\gamma_{u,v}} g(z)\,dz + \frac{1}{w-v}\int_{\gamma_{v,w}} f(y)\,dy\int_{\gamma_{v,w}} g(z)\,dz \\ &\quad + \int_{\gamma_{u,v}}\left[f(z) - \frac{1}{v-u}\int_{\gamma_{u,v}} f(y)\,dy\right] g(z)\,dz \\ &\quad + \int_{\gamma_{v,w}}\left[f(z) - \frac{1}{w-v}\int_{\gamma_{v,w}} f(y)\,dy\right] g(z)\,dz\end{aligned}$$

for $v \in \gamma$ with $v \neq u, w$.

In particular, for $g(z) = 1$, $z \in \gamma$, we have for $v \in \gamma$ that

$$\begin{aligned}(2.13)\quad \int_\gamma f(z)\,dz &= (v-u)\,f(u) + (w-v)\,f(w) \\ &\quad + \int_{\gamma_{u,v}}[f(z) - f(u)]\,dz + \int_{\gamma_{v,w}}[f(z) - f(w)]\,dz,\end{aligned}$$

and

$$\begin{aligned}(2.14)\quad \int_\gamma f(z)\,dz &= (v-u)\,\frac{f(u)+f(v)}{2} + (w-v)\,\frac{f(v)+f(w)}{2} \\ &\quad + \int_{\gamma_{u,v}}\left[f(z) - \frac{f(u)+f(v)}{2}\right] dz + \int_{\gamma_{v,w}}\left[f(z) - \frac{f(v)+f(w)}{2}\right] dz.\end{aligned}$$

3. Some Inequalities for Bounded Functions

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$. Now, for $\phi, \Phi \in \mathbb{C}$ and γ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_\gamma(\phi, \Phi) := \left\{ f : \gamma \to \mathbb{C} \,|\, \operatorname{Re}\left[(\Phi - f(z))\left(\overline{f(z)} - \overline{\phi}\right)\right] \geq 0 \text{ for each } z \in \gamma \right\}$$

and

$$\bar{\Delta}_\gamma(\phi, \Phi) := \left\{ f : \gamma \to \mathbb{C} \,|\, \left| f(z) - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2}|\Phi - \phi| \text{ for each } z \in \gamma \right\}.$$

The following representation result may be stated.

Proposition 1. *For any $\phi,\ \Phi\in\mathbb{C},\ \phi\neq\Phi$, we have that $\bar{U}_{\gamma}(\phi,\Phi)$ and $\bar{\Delta}_{\gamma}(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_{\gamma}(\phi,\Phi)=\bar{\Delta}_{\gamma}(\phi,\Phi)\,. \tag{3.1}$$

Proof. We observe that for any $w\in\mathbb{C}$ we have the equivalence

$$\left|w-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}\left|\Phi-\phi\right|$$

if and only if

$$\operatorname{Re}\left[\left(\Phi-w\right)\left(\overline{w}-\overline{\phi}\right)\right]\geq0.$$

This follows by the equality

$$\frac{1}{4}\left|\Phi-\phi\right|^{2}-\left|w-\frac{\phi+\Phi}{2}\right|^{2}=\operatorname{Re}\left[\left(\Phi-w\right)\left(\overline{w}-\overline{\phi}\right)\right]$$

that holds for any $w\in\mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 2. *For any $\phi,\ \Phi\in\mathbb{C},\ \phi\neq\Phi$, we have that*

$$\begin{aligned}\bar{U}_{\gamma}(\phi,\Phi)=\{f:\gamma\to\mathbb{C}\mid\ &(\operatorname{Re}\Phi-\operatorname{Re}f(z))(\operatorname{Re}f(z)-\operatorname{Re}\phi)\\&+(\operatorname{Im}\Phi-\operatorname{Im}f(z))(\operatorname{Im}f(z)-\operatorname{Im}\phi)\geq0\ \textit{for each}\ z\in\gamma\}\,.\end{aligned} \tag{3.2}$$

Now, if we assume that $\operatorname{Re}(\Phi)\geq\operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi)\geq\operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_{\gamma}(\phi,\Phi):=\{f:\gamma\to\mathbb{C}\mid\ &\operatorname{Re}(\Phi)\geq\operatorname{Re}f(z)\geq\operatorname{Re}(\phi)\\&\text{and}\ \operatorname{Im}(\Phi)\geq\operatorname{Im}f(z)\geq\operatorname{Im}(\phi)\ \text{for each}\ \ z\in\gamma\}\,.\end{aligned} \tag{3.3}$$

One can easily observe that $\bar{S}_{\gamma}(\phi,\Phi)$ is closed, convex and

$$\emptyset\neq\bar{S}_{\gamma}(\phi,\Phi)\subseteq\bar{U}_{\gamma}(\phi,\Phi)\,. \tag{3.4}$$

Theorem 3. *Let f be continuous in D, an open domain and suppose $\gamma\subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x\in(a,b)$. Assume that $\phi_{k},\ \Phi_{k}\in\mathbb{C},\ \phi_{k}\neq\Phi_{k}$, with $k\in\{1,2\}$ and $f\in\bar{\Delta}_{\gamma_{u,v}}(\phi_{1},\Phi_{1})\cap\bar{\Delta}_{\gamma_{v,w}}(\phi_{2},\Phi_{2})$, then*

$$\begin{aligned}&\left|\int_{\gamma}f(z)\,g(z)\,dz-\frac{\phi_{1}+\Phi_{1}}{2}\int_{\gamma_{u,v}}g(z)\,dz-\frac{\phi_{2}+\Phi_{2}}{2}\int_{\gamma_{v,w}}g(z)\,dz\right|\\&\quad\leq\frac{1}{2}\left|\Phi_{1}-\phi_{1}\right|\int_{\gamma_{u,v}}\left|g(z)\right|\left|dz\right|+\frac{1}{2}\left|\Phi_{2}-\phi_{2}\right|\int_{\gamma_{v,w}}\left|g(z)\right|\left|dz\right|\\&\qquad\leq\frac{1}{2}\max\left\{\left|\Phi_{1}-\phi_{1}\right|,\left|\Phi_{2}-\phi_{2}\right|\right\}\int_{\gamma}\left|g(z)\right|\left|dz\right|.\end{aligned} \tag{3.5}$$

Proof. Using the identity (2.1) we get

$$\begin{aligned}
&\left|\int_\gamma f(z)\,g(z)\,dz-\lambda\int_{\gamma_{u,v}} g(z)\,dz-\mu\int_{\gamma_{v,w}} g(z)\,dz\right| \quad (3.6)\\
&\leq\left|\int_{\gamma_{u,v}}[f(z)-\lambda]\,g(z)\,dz\right|+\left|\int_{\gamma_{v,w}}[f(z)-\mu]\,g(z)\,dz\right|\\
&\leq\int_{\gamma_{u,v}}\left|[f(z)-\lambda]\,g(z)\right||dz|+\int_{\gamma_{v,w}}\left|[f(z)-\mu]\,g(z)\right||dz|\\
&=\int_{\gamma_{u,v}}|f(z)-\lambda|\,|g(z)|\,|dz|+\int_{\gamma_{v,w}}|f(z)-\mu|\,|g(z)|\,|dz|
\end{aligned}$$

where $v\in\gamma$, for any $\lambda,\ \mu\in\mathbb{C}$.

Since $f\in\bar{\Delta}_{\gamma_{u,v}}(\phi_1,\Phi1)\cap\bar{\Delta}_{\gamma_{v,w}}(\phi_2,\Phi_2)$, hence by taking $\lambda=\frac{\phi_1+\Phi_1}{2}$ and $\mu=\frac{\phi_2+\Phi_2}{2}$ in (3.6) we get

$$\begin{aligned}
&\left|\int_\gamma f(z)\,g(z)\,dz-\frac{\phi_1+\Phi_1}{2}\int_{\gamma_{u,v}} g(z)\,dz-\frac{\phi_2+\Phi_2}{2}\int_{\gamma_{v,w}} g(z)\,dz\right|\\
&\leq\int_{\gamma_{u,v}}\left|f(z)-\frac{\phi_1+\Phi_1}{2}\right||g(z)|\,|dz|+\int_{\gamma_{v,w}}\left|f(z)-\frac{\phi_2+\Phi_2}{2}\right||g(z)|\,|dz|\\
&\leq\frac{1}{2}|\Phi_1-\phi_1|\int_{\gamma_{u,v}}|g(z)|\,|dz|+\frac{1}{2}|\Phi_2-\phi_2|\int_{\gamma_{v,w}}|g(z)|\,|dz|,
\end{aligned}$$

which proves the first inequality in (3.5). □

Corollary 3. *Let f be continuous in D, an open domain and suppose $\gamma\subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x\in(a,b)$. Assume that $\phi,\ \Phi\in\mathbb{C}$, $\phi\neq\Phi$ and $f\in\bar{\Delta}_{\gamma_{u,v}}(\phi,\Phi)$, then*

$$\left|\int_\gamma f(z)\,g(z)\,dz-\frac{\phi+\Phi}{2}\int_\gamma g(z)\,dz\right|\leq\frac{1}{2}|\Phi-\phi|\int_{\gamma_{u,v}}|g(z)|\,|dz|. \tag{3.7}$$

Remark 1. *If we take $g(z)=1$ in (3.5), we get*

$$\begin{aligned}
&\left|\int_\gamma f(z)\,dz-\frac{\phi_1+\Phi_1}{2}(v-u)-\frac{\phi_2+\Phi_2}{2}(w-v)\right| \quad (3.8)\\
&\leq\frac{1}{2}|\Phi_1-\phi_1|\,\ell(\gamma_{u,v})+\frac{1}{2}|\Phi_2-\phi_2|\,\ell(\gamma_{v,w})\leq\frac{1}{2}\max\{|\Phi_1-\phi_1|,|\Phi_2-\phi_2|\}\,\ell(\gamma),
\end{aligned}$$

while from (3.7) we get

$$\left|\int_\gamma f(z)\,dz-\frac{\phi+\Phi}{2}(w-u)\right|\leq\frac{1}{2}|\Phi-\phi|\,\ell(\gamma). \tag{3.9}$$

4. Inequalities for Lipschitzian Functions

Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t\in[a,b]$ from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x\in(a,b)$. Let $p_1,\ p_2>0$ and L_{p_1}, $L_{p_1}>0$. We say that $f\in\mathfrak{Lip}_{p_1,p_2}(L_{p_1},L_{p_1};\gamma,v)$ if

$$|f(z)-f(v)|\leq L_{p_1}|z-v|^{p_1}\ \text{ for all } z\in\gamma_{u,v}$$

and

$$|f(z)-f(v)| \leq L_{p_2} |z-v|^{p_2} \text{ for all } z \in \gamma_{v,w}.$$

If $p_1 = p_2 = p > 0$ and $L_{p_1} = L_{p_1} = L > 0$, then we have $f \in \mathfrak{Lip}_p(L_p; \gamma, v)$ if

$$|f(z)-f(v)| \leq L |z-v|^{p} \text{ for all } z \in \gamma.$$

We have the following weighted Ostrowski type inequality:

Theorem 4. *Let f and g be continuous in D, an open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v = z(x)$ with $x \in (a,b)$. Let p_1, $p_2 > 0$ and L_{p_1}, $L_{p_1} > 0$ and assume that $f \in \mathfrak{Lip}_{p_1,p_2}(L_{p_1}, L_{p_1}; \gamma, v)$, then*

$$\begin{aligned}
(4.1)\quad & \left| \int_\gamma f(z) g(z) dz - f(v) \int_\gamma g(z) dz \right| \\
& \leq L_{p_1} \int_{\gamma_{u,v}} |z-v|^{p_1} |g(z)| |dz| + L_{p_2} \int_{\gamma_{v,w}} |z-v|^{p_2} |g(z)| |dz| \\
& \leq L_{p_1} \times \begin{cases} \max_{z\in\gamma_{u,v}} |z-v|^{p_1} \int_{\gamma_{u,v}} |g(z)| |dz| \\ \\ \left(\int_{\gamma_{u,v}} |z-v|^{mp_1} |dz|\right)^{1/m} \left(\int_{\gamma_{u,v}} |g(z)|^n |dz|\right)^{1/n} \\ m,\ n>1 \text{ with } \frac{1}{m}+\frac{1}{n}=1, \\ \\ \max_{z\in\gamma_{u,v}} |g(z)| \int_{\gamma_{u,v}} |z-v|^{p_1} |dz| \end{cases} \\
& \quad + L_{p_2} \begin{cases} \max_{z\in\gamma_{v,w}} |z-v|^{p_2} \int_{\gamma_{v,w}} |g(z)| |dz| \\ \\ \left(\int_{\gamma_{v,w}} |z-v|^{mp_2} |dz|\right)^{1/m} \left(\int_{\gamma_{v,w}} |g(z)|^n |dz|\right)^{1/n} \\ m,\ n>1 \text{ with } \frac{1}{m}+\frac{1}{n}=1, \\ \\ \max_{z\in\gamma_{v,w}} |g(z)| \int_{\gamma_{v,w}} |z-v|^{p_2} |dz|. \end{cases}
\end{aligned}$$

Proof. From the identity (2.5) we have

$$\begin{aligned}
(4.2)\quad & \left| \int_\gamma f(z) g(z) dz - f(v) \int_\gamma g(z) dz \right| \\
& \leq \left| \int_{\gamma_{u,v}} [f(v)-f(z)] g(z) dz \right| + \left| \int_{\gamma_{v,w}} [f(z)-f(v)] g(z) dz \right| \\
& \leq \int_{\gamma_{u,v}} |f(v)-f(z)| |g(z)| |dz| + \int_{\gamma_{v,w}} |f(z)-f(v)| |g(z)| |dz|.
\end{aligned}$$

Since $f \in \mathfrak{Lip}_{p_1,p_2}(L_{p_1}, L_{p_1}; \gamma, v)$, hence

$$\begin{aligned}
(4.3)\quad & \int_{\gamma_{u,v}} |f(v)-f(z)| |g(z)| |dz| + \int_{\gamma_{v,w}} |f(z)-f(v)| |g(z)| |dz| \\
& \leq L_{p_1} \int_{\gamma_{u,v}} |z-v|^{p_1} |g(z)| |dz| + L_{p_2} \int_{\gamma_{v,w}} |z-v|^{p_2} |g(z)| |dz|.
\end{aligned}$$

By utilising the inequalities (4.2) and (4.3) we get the first part of (4.1).

Using Hölder's integral inequality we have

$$\int_{\gamma_{u,v}} |z-v|^{p_1} |g(z)| |dz| \leq \begin{cases} \max_{z\in\gamma_{u,v}} |z-v|^{p_1} \int_{\gamma_{u,v}} |g(z)| |dz| \\ \\ \left(\int_{\gamma_{u,v}} |z-v|^{mp_1} |dz|\right)^{1/m} \left(\int_{\gamma_{u,v}} |g(z)|^n |dz|\right)^{1/n} \\ m,\ n>1 \text{ with } \frac{1}{m}+\frac{1}{n}=1, \\ \\ \max_{z\in\gamma_{u,v}} |g(z)| \int_{\gamma_{u,v}} |z-v|^{p_1} |dz| \end{cases}$$

and

$$\int_{\gamma_{v,w}} |z-v|^{p_2} |g(z)| |dz| \leq \begin{cases} \max_{z\in\gamma_{v,w}} |z-v|^{p_2} \int_{\gamma_{v,w}} |g(z)| |dz| \\ \\ \left(\int_{\gamma_{v,w}} |z-v|^{mp_2} |dz|\right)^{1/m} \left(\int_{\gamma_{v,w}} |g(z)|^n |dz|\right)^{1/n} \\ m,\ n>1 \text{ with } \frac{1}{m}+\frac{1}{n}=1, \\ \\ \max_{z\in\gamma_{v,w}} |g(z)| \int_{\gamma_{v,w}} |z-v|^{p_2} |dz|, \end{cases}$$

which proves the last part of (4.1). □

Corollary 4. *Let f and g be continuous in D, an open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x\in(a,b)$. Let $p>0$ and $L_p>0$ and assume that $f\in \mathfrak{Lip}_p(L_p;\gamma,v)$, then*

$$\left|\int_\gamma f(z) g(z) dz - f(v)\int_\gamma g(z) dz\right| \leq L_p \int_\gamma |z-v|^p |g(z)| |dz| \tag{4.4}$$

$$\leq L_p \times \begin{cases} \max_{z\in\gamma} |z-v|^p \int_\gamma |g(z)| |dz| \\ \\ \left(\int_\gamma |z-v|^{mp} |dz|\right)^{1/m} \left(\int_\gamma |g(z)|^n |dz|\right)^{1/n} \\ m,\ n>1 \textit{ with } \frac{1}{m}+\frac{1}{n}=1, \\ \\ \max_{z\in\gamma} |g(z)| \int_\gamma |z-v|^p |dz|. \end{cases}$$

Remark 2. *If we take $g(z)=1$ in (4.1), we get the following Ostrowski type inequality*

$$\left|\int_\gamma f(z) dz - f(v)(w-u)\right| \tag{4.5}$$

$$\leq L_{p_1}\int_{\gamma_{u,v}} |z-v|^{p_1} |dz| + L_{p_2}\int_{\gamma_{v,w}} |z-v|^{p_2} |dz|$$

$$\leq L_{p_1}\times\begin{cases}\max_{z\in\gamma_{u,v}}\left|z-v\right|^{p_1}\ell\left(\gamma_{u,v}\right)\\ \\ \left(\int_{\gamma_{u,v}}\left|z-v\right|^{mp_1}\left|dz\right|\right)^{1/m}\left[\ell\left(\gamma_{u,v}\right)\right]^{1/n}\\ m,\ n>1\ \mathit{with}\ \frac{1}{m}+\frac{1}{n}=1,\\ \\ \int_{\gamma_{u,v}}\left|z-v\right|^{p_1}\left|dz\right|\end{cases}$$

$$+L_{p_2}\begin{cases}\max_{z\in\gamma_{v,w}}\left|z-v\right|^{p_2}\ell\left(\gamma_{v,w}\right)\\ \\ \left(\int_{\gamma_{v,w}}\left|z-v\right|^{mp_2}\left|dz\right|\right)^{1/m}\left[\ell\left(\gamma_{v,w}\right)\right]^{1/n}\\ m,\ n>1\ \mathit{with}\ \frac{1}{m}+\frac{1}{n}=1,\\ \\ \int_{\gamma_{v,w}}\left|z-v\right|^{p_2}\left|dz\right|,\end{cases}$$

while from (4.4) we get

$$\left|\int_{\gamma}f\left(z\right)dz-f\left(v\right)\left(w-u\right)\right|\leq L_p\int_{\gamma}\left|z-v\right|^p\left|dz\right| \tag{4.6}$$

$$\leq L_p\times\begin{cases}\max_{z\in\gamma}\left|z-v\right|^{p}\ell\left(\gamma\right)\\ \\ \left(\int_{\gamma}\left|z-v\right|^{mp}\left|dz\right|\right)^{1/m}\left[\ell\left(\gamma\right)\right]^{1/n}\\ m,\ n>1\ \mathit{with}\ \frac{1}{m}+\frac{1}{n}=1,\\ \\ \int_{\gamma}\left|z-v\right|^{p}\left|dz\right|.\end{cases}$$

Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z\left(t\right),\ t\in\left[a,b\right]$ from $z\left(a\right)=u$ to $z\left(b\right)=w$ and $v=z\left(x\right)$ with $x\in\left(a,b\right)$. Let $p_1,\ p_2>0$ and L_{p_1}, $L_{p_1}>0$. We say that $f\in\mathfrak{Lipe}_{p_1,p_2}\left(L_{p_1},L_{p_1};\gamma,u,v,w\right)$ if

$$\left|f\left(z\right)-f\left(u\right)\right|\leq L_{p_1}\left|z-u\right|^{p_1}\ \text{ for all } z\in\gamma_{u,v}$$

and

$$\left|f\left(z\right)-f\left(w\right)\right|\leq L_{p_2}\left|z-w\right|^{p_2}\ \text{ for all } z\in\gamma_{v,w}.$$

If $p_1=p_2=p>0$ and $L_{p_1}=L_{p_1}=L>0$, then we have $f\in\mathfrak{Lipe}_{p}\left(L_p;\gamma,u,v,w\right)$ if

$$\left|f\left(z\right)-f\left(u\right)\right|\leq L_{p}\left|z-u\right|^{p}\ \text{ for all } z\in\gamma_{u,v}$$

and

$$\left|f\left(z\right)-f\left(w\right)\right|\leq L_{p}\left|z-w\right|^{p}\ \text{ for all } z\in\gamma_{v,w}.$$

We have the following weighted trapezoid type inequality:

Theorem 5. *Let f and g be continuous in D, an open domain and suppose $\gamma\subset D$ is a piecewise smooth path from $z\left(a\right)=u$ to $z\left(b\right)=w$ and $v=z\left(x\right)$ with $x\in\left(a,b\right)$. Let $p_1,\ p_2>0$ and L_{p_1}, $L_{p_1}>0$ and assume that $f\in\mathfrak{Lipe}_{p_1,p_2}\left(L_{p_1},L_{p_1};\gamma,u,v,w\right)$,* *then*

$$\left|\int_{\gamma} f(z)\,g(z)\,dz - f(u)\int_{\gamma_{u,v}} g(z)\,dz - f(w)\int_{\gamma_{v,w}} g(z)\,dz\right| \tag{4.7}$$
$$\leq L_{p_1}\int_{\gamma_{u,v}} |z-u|^{p_1}|g(z)||dz| + L_{p_2}\int_{\gamma_{v,w}} |z-w|^{p_2}|g(z)||dz|$$
$$\leq L_{p_1}\times\begin{cases} \max_{z\in\gamma_{u,v}}|z-u|^{p_1}\int_{\gamma_{u,v}}|g(z)||dz| \\ \\ \left(\int_{\gamma_{u,v}}|z-u|^{mp_1}|dz|\right)^{1/m}\left(\int_{\gamma_{u,v}}|g(z)|^n|dz|\right)^{1/n} \\ m,\ n>1 \textit{ with } \frac{1}{m}+\frac{1}{n}=1, \\ \\ \max_{z\in\gamma_{u,v}}|g(z)|\int_{\gamma_{u,v}}|z-u|^{p_1}|dz| \end{cases}$$
$$+L_{p_2}\times\begin{cases} \max_{z\in\gamma_{v,w}}|z-w|^{p_2}\int_{\gamma_{v,w}}|g(z)||dz| \\ \\ \left(\int_{\gamma_{v,w}}|z-w|^{mp_2}|dz|\right)^{1/m}\left(\int_{\gamma_{v,w}}|g(z)|^n|dz|\right)^{1/n} \\ m,\ n>1 \textit{ with } \frac{1}{m}+\frac{1}{n}=1, \\ \\ \max_{z\in\gamma_{v,w}}|g(z)|\int_{\gamma_{v,w}}|z-w|^{p_2}|dz|. \end{cases}$$

Proof. Using the equality (2.10) we have

$$\left|\int_{\gamma} f(z)\,g(z)\,dz - f(u)\int_{\gamma_{u,v}} g(z)\,dz - f(w)\int_{\gamma_{v,w}} g(z)\,dz\right|$$
$$\leq \left|\int_{\gamma_{u,v}}[f(z)-f(u)]\,g(z)\,dz\right| + \left|\int_{\gamma_{v,w}}[f(z)-f(w)]\,g(z)\,dz\right|$$
$$\leq \int_{\gamma_{u,v}}|f(z)-f(u)||g(z)||dz| + \int_{\gamma_{v,w}}|f(z)-f(w)||g(z)||dz|$$
$$\leq L_{p_1}\int_{\gamma_{u,v}}|z-u|^{p_1}|g(z)||dz| + L_{p_2}\int_{\gamma_{v,w}}|z-w|^{p_2}|g(z)||dz|,$$

which proves the first inequality in (4.7).

Using Hölder's integral inequality we also have

$$\int_{\gamma_{u,v}}|z-u|^{p_1}|g(z)||dz| \leq \begin{cases} \max_{z\in\gamma_{u,v}}|z-u|^{p_1}\int_{\gamma_{u,v}}|g(z)||dz| \\ \\ \left(\int_{\gamma_{u,v}}|z-u|^{mp_1}|dz|\right)^{1/m}\left(\int_{\gamma_{u,v}}|g(z)|^n|dz|\right)^{1/n} \\ m,\ n>1 \text{ with } \frac{1}{m}+\frac{1}{n}=1, \\ \\ \max_{z\in\gamma_{u,v}}|g(z)|\int_{\gamma_{u,v}}|z-u|^{p_1}|dz| \end{cases}$$

and

$$\int_{\gamma_{v,w}} |z-w|^{p_2} |g(z)| |dz| \le \begin{cases} \max_{z\in\gamma_{v,w}} |z-w|^{p_2} \int_{\gamma_{v,w}} |g(z)| |dz| \\ \\ \left(\int_{\gamma_{v,w}} |z-w|^{mp_2} |dz|\right)^{1/m} \left(\int_{\gamma_{v,w}} |g(z)|^n |dz|\right)^{1/n} \\ m,\ n>1 \text{ with } \frac{1}{m}+\frac{1}{n}=1, \\ \\ \max_{z\in\gamma_{v,w}} |g(z)| \int_{\gamma_{v,w}} |z-w|^{p_2} |dz|, \end{cases}$$

which proves the last part of (4.1). □

Corollary 5. *Let f and g be continuous in D, an open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x\in(a,b)$. Let $p>0$ and $L_p>0$ and assume that $f\in \mathfrak{Lipe}_p(L_p;\gamma,u,v,w)$, then*

$$\left|\int_\gamma f(z)g(z)\,dz - f(u)\int_{\gamma_{u,v}} g(z)\,dz - f(w)\int_{\gamma_{v,w}} g(z)\,dz\right| \tag{4.8}$$
$$\le L_p\left[\int_{\gamma_{u,v}} |z-u|^p |g(z)| |dz| + \int_{\gamma_{v,w}} |z-w|^p |g(z)| |dz|\right]$$
$$\le L_p\times \begin{cases} \max_{z\in\gamma_{u,v}} |z-u|^p \int_{\gamma_{u,v}} |g(z)| |dz| \\ \\ \left(\int_{\gamma_{u,v}} |z-u|^{mp} |dz|\right)^{1/m} \left(\int_{\gamma_{u,v}} |g(z)|^n |dz|\right)^{1/n} \\ m,\ n>1 \textit{ with } \frac{1}{m}+\frac{1}{n}=1, \\ \\ \max_{z\in\gamma_{u,v}} |g(z)| \int_{\gamma_{u,v}} |z-u|^p |dz| \end{cases}$$
$$+ L_p\times \begin{cases} \max_{z\in\gamma_{v,w}} |z-w|^p \int_{\gamma_{v,w}} |g(z)| |dz| \\ \\ \left(\int_{\gamma_{v,w}} |z-w|^{mp} |dz|\right)^{1/m} \left(\int_{\gamma_{v,w}} |g(z)|^n |dz|\right)^{1/n} \\ m,\ n>1 \textit{ with } \frac{1}{m}+\frac{1}{n}=1, \\ \\ \max_{z\in\gamma_{v,w}} |g(z)| \int_{\gamma_{v,w}} |z-w|^p |dz|. \end{cases}$$

Remark 3. *If we take $g(z)=1$ then by (4.7) we get*

$$\left|\int_\gamma f(z)\,dz - f(u)(v-u) - f(w)(w-v)\right| \tag{4.9}$$
$$\le L_{p_1}\int_{\gamma_{u,v}} |z-u|^{p_1} |dz| + L_{p_2}\int_{\gamma_{v,w}} |z-w|^{p_2} |dz|$$

$$\leq L_{p_1}\times\begin{cases}\max_{z\in\gamma_{u,v}}|z-u|^{p_1}\,\ell\left(\gamma_{u,v}\right)\\ \\ \left(\int_{\gamma_{u,v}}|z-u|^{mp_1}|dz|\right)^{1/m}\left[\ell\left(\gamma_{u,v}\right)\right]^{1/n}\\ m,\ n>1\ \textit{with}\ \frac{1}{m}+\frac{1}{n}=1,\\ \\ \max_{z\in\gamma_{u,v}}|g(z)|\int_{\gamma_{u,v}}|z-u|^{p_1}|dz|\end{cases}$$

$$+L_{p_2}\times\begin{cases}\max_{z\in\gamma_{v,w}}|z-w|^{p_2}\,\ell\left(\gamma_{v,w}\right)\\ \\ \left(\int_{\gamma_{v,w}}|z-w|^{mp_2}|dz|\right)^{1/m}\left[\ell\left(\gamma_{v,w}\right)\right]^{1/n}\\ m,\ n>1\ \textit{with}\ \frac{1}{m}+\frac{1}{n}=1,\\ \\ \max_{z\in\gamma_{v,w}}|g(z)|\int_{\gamma_{v,w}}|z-w|^{p_2}|dz|.\end{cases}$$

while from (4.8) we get

$$\left|\int_{\gamma}f(z)\,dz-f(u)(v-u)-f(w)(w-v)\right| \tag{4.10}$$

$$\leq L_p\left[\int_{\gamma_{u,v}}|z-u|^p|dz|+\int_{\gamma_{v,w}}|z-w|^p|dz|\right]$$

$$\leq L_p\times\begin{cases}\max_{z\in\gamma_{u,v}}|z-u|^{p}\,\ell\left(\gamma_{u,v}\right)\\ \\ \left(\int_{\gamma_{u,v}}|z-u|^{mp}|dz|\right)^{1/m}\left[\ell\left(\gamma_{u,v}\right)\right]^{1/n}\\ m,\ n>1\ \textit{with}\ \frac{1}{m}+\frac{1}{n}=1,\\ \\ \int_{\gamma_{u,v}}|z-u|^{p}|dz|\end{cases}$$

$$+L_p\times\begin{cases}\max_{z\in\gamma_{v,w}}|z-w|^{p}\,\ell\left(\gamma_{v,w}\right)\\ \\ \left(\int_{\gamma_{v,w}}|z-w|^{mp}|dz|\right)^{1/m}\left[\ell\left(\gamma_{v,w}\right)\right]^{1/n}\\ m,\ n>1\ \textit{with}\ \frac{1}{m}+\frac{1}{n}=1,\\ \\ \int_{\gamma_{v,w}}|z-w|^{p}|dz|.\end{cases}$$

5. Examples for Circular Paths

Let $[a,b]\subseteq[0,2\pi]$ and the circular path $\gamma_{[a,b],R}$ centered in 0 and with radius $R>0$

$$z(t)=R\exp(it)=R(\cos t+i\sin t),\ t\in[a,b].$$

If $[a,b]=[0,\pi]$ then we get a half circle while for $[a,b]=[0,2\pi]$ we get the full circle.

Since

$$\begin{aligned}\left|e^{is}-e^{it}\right|^2&=\left|e^{is}\right|^2-2\operatorname{Re}\left(e^{i(s-t)}\right)+\left|e^{it}\right|^2\\ &=2-2\cos(s-t)=4\sin^2\left(\frac{s-t}{2}\right)\end{aligned}$$

for any $t,\, s\in\mathbb{R}$, then

$$\left|e^{is}-e^{it}\right|^{r}=2^{r}\left|\sin\left(\frac{s-t}{2}\right)\right|^{r}$$

for any $t,\, s\in\mathbb{R}$ and $r>0$. In particular,

$$\left|e^{is}-e^{it}\right|=2\left|\sin\left(\frac{s-t}{2}\right)\right|$$

for any $t,\, s\in\mathbb{R}$.

For $s=a$ and $s=b$ we have

$$\left|e^{ia}-e^{it}\right|=2\left|\sin\left(\frac{a-t}{2}\right)\right| \text{ and } \left|e^{ib}-e^{it}\right|=2\left|\sin\left(\frac{b-t}{2}\right)\right|.$$

If $u=R\exp(ia)$ and $w=R\exp(ib)$ then

$$\begin{aligned}w-u&=R\left[\exp(ib)-\exp(ia)\right]=R\left[\cos b+i\sin b-\cos a-i\sin a\right]\\&=R\left[\cos b-\cos a+i\left(\sin b-\sin a\right)\right].\end{aligned}$$

Since

$$\cos b-\cos a=-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

and

$$\sin b-\sin a=2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right),$$

hence

$$\begin{aligned}w-u&=R\left[-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)+2i\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right)\right]\\&=2R\sin\left(\frac{b-a}{2}\right)\left[-\sin\left(\frac{a+b}{2}\right)+i\cos\left(\frac{a+b}{2}\right)\right]\\&=2Ri\sin\left(\frac{b-a}{2}\right)\left[\cos\left(\frac{a+b}{2}\right)+i\sin\left(\frac{a+b}{2}\right)\right]\\&=2Ri\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right].\end{aligned}$$

Moreover,

$$|z-u|=R\left|\exp(it)-\exp(ia)\right|=2R\left|\sin\left(\frac{t-a}{2}\right)\right|$$

and

$$|z-w|=R\left|\exp(it)-\exp(ib)\right|=2R\left|\sin\left(\frac{b-t}{2}\right)\right|$$

for $t\in[a,b]$.

If $[a,b]\subseteq[0,2\pi]$ then $0\le\frac{t-a}{2},\ \frac{b-t}{2}\le\pi$ for $t\in[a,b]$, therefore

$$|z-u|=2R\sin\left(\frac{t-a}{2}\right) \text{ and } |z-w|=2R\sin\left(\frac{b-t}{2}\right).$$

We also have

$$z'(t)=Ri\exp(it) \text{ and } |z'(t)|=R$$

for $t\in[a,b]$.

Let f and g be continuous in D, an open domain and suppose $\gamma_{[a,b],R} \subset D$ is a circular path from $z(a) = u$ to $z(b) = w$ and $v = z(x)$ with $x \in (a,b)$. Let $p > 0$ and $L_p > 0$ and assume that $f \in \mathfrak{Lip}_p \left(L_p; \gamma_{[a,b],R}, v \right)$, then by (4.4) we get

$$\begin{aligned}
(5.1)\quad & \left| \int_a^b f(R\exp(it))\, g(R\exp(it)) \exp(it)\, dt \right. \\
& \qquad \left. - f(R\exp(ix)) \int_a^b g(R\exp(it)) \exp(it)\, dt \right| \\
& \leq 2^p R^p L_p \int_a^b \left| \sin\left(\frac{x-t}{2}\right) \right|^p |g(R\exp(it))|\, dt \\
& \leq 2^p R^p L_p \times \begin{cases} \max_{t\in[a,b]} \left|\sin\left(\frac{x-t}{2}\right)\right|^p \int_a^b |g(R\exp(it))|\, dt \\ \\ \left(\int_a^b \left|\sin\left(\frac{x-t}{2}\right)\right|^{pm} dt\right)^{1/m} \left(\int_a^b |g(R\exp(it))|^n\, dt\right)^{1/n} \\ m,\ n > 1 \text{ with } \frac{1}{m} + \frac{1}{n} = 1, \\ \\ \max_{t\in[a,b]} |g(R\exp(it))| \int_a^b \left|\sin\left(\frac{x-t}{2}\right)\right|^p dt \end{cases}
\end{aligned}$$

for all $x \in [a,b]$.

The case $p = 1$, namely $f \in \mathfrak{Lip}\left(L; \gamma_{[a,b],R} \right)$, which means that

$$|f(z) - f(w)| \leq L|z - w| \text{ for all } z,\ w \in \gamma_{[a,b],R}$$

is of particular interest.

In this case, we have by (5.1) that

$$\begin{aligned}
(5.2)\quad & \left| \int_a^b f(R\exp(it))\, g(R\exp(it)) \exp(it)\, dt \right. \\
& \qquad \left. - f(R\exp(ix)) \int_a^b g(R\exp(it)) \exp(it)\, dt \right| \\
& \leq 2RL \int_a^b \left| \sin\left(\frac{x-t}{2}\right) \right| |g(R\exp(it))|\, dt \\
& \leq 2RL \times \begin{cases} \max_{t\in[a,b]} \left|\sin\left(\frac{x-t}{2}\right)\right| \int_a^b |g(R\exp(it))|\, dt \\ \\ \left(\int_a^b \left|\sin\left(\frac{x-t}{2}\right)\right|^{m} dt\right)^{1/m} \left(\int_a^b |g(R\exp(it))|^n\, dt\right)^{1/n} \\ m,\ n > 1 \text{ with } \frac{1}{m} + \frac{1}{n} = 1, \\ \\ \max_{t\in[a,b]} |g(R\exp(it))| \int_a^b \left|\sin\left(\frac{x-t}{2}\right)\right| dt \end{cases}
\end{aligned}$$

Observe that

$$\begin{aligned}
\int_a^x \left| \sin\left(\frac{t-x}{2}\right) \right| dt &= \int_a^x \sin\left(\frac{x-t}{2}\right) dt = 2 - 2\cos\left(\frac{x-a}{2}\right) \\
&= 4\sin^2\left(\frac{x-a}{4}\right)
\end{aligned}$$

and

$$\int_x^b \left|\sin\left(\frac{t-x}{2}\right)\right| dt = \int_x^b \sin\left(\frac{t-x}{2}\right) dt = 2 - 2\cos\left(\frac{b-t}{2}\right)$$
$$= 4\sin^2\left(\frac{b-x}{4}\right),$$

then

$$\begin{aligned}\int_a^b \left|\sin\left(\frac{x-t}{2}\right)\right| dt &= \int_a^{\ x} \left|\sin\left(\frac{t-x}{2}\right)\right| dt + \int_x^b \left|\sin\left(\frac{t-x}{2}\right)\right| \\ &= 4\left[\sin^2\left(\frac{x-a}{4}\right) + \sin^2\left(\frac{b-x}{4}\right)\right]\end{aligned}$$

for $x \in [a,b]$.

Using (5.2) we get the following simple inequality of interest:

$$\begin{aligned}&\left|\int_a^b f(R\exp(it))\, g(R\exp(it))\exp(it)\, dt\right. \\ &\qquad \left. -f(R\exp(ix))\int_a^b g(R\exp(it))\exp(it)\, dt\right| \\ &\qquad \le 2RL\int_a^b \left|\sin\left(\frac{x-t}{2}\right)\right| |g(R\exp(it))|\, dt \\ &\qquad \le 8RL\max_{t\in[a,b]} |g(R\exp(it))| \left[\sin^2\left(\frac{x-a}{4}\right) + \sin^2\left(\frac{b-x}{4}\right)\right]\end{aligned} \tag{5.3}$$

for $x \in [a,b]$.

Let f and g be continuous in D, an open domain and suppose $\gamma_{[a,b],R} \subset D$ is a circular path from $z(a) = u$ to $z(b) = w$ and $v = z(x)$ with $x \in (a,b)$. Let $p > 0$ and $L_p > 0$ and assume that $f \in \mathfrak{Lipe}_p\left(L_p; \gamma_{[a,b],R}, u, v, w\right)$, then

$$\begin{aligned}&\left|\int_a^b f(R\exp(it))\, g(R\exp(it))\exp(it)\, dt\right. \\ &\qquad - f(R\exp(ia))\int_a^x g(R\exp(it))\exp(it)\, dt \\ &\qquad \left. -f(R\exp(ib))\int_x^b g(R\exp(it))\exp(it)\, dt\right| \\ &\le 2^p R^p L_p\left[\int_a^x \sin^p\left(\frac{t-a}{2}\right)|g(R\exp(it))|\, dt + \int_x^b \sin^p\left(\frac{b-t}{2}\right)|g(R\exp(it))|\, dt\right]\end{aligned} \tag{5.4}$$

$$\leq 2^p R^p L_p \times \begin{cases} \max_{t\in[a,x]}\left[\sin^p\left(\frac{t-a}{2}\right)\right]\int_a^x |g(R\exp(it))|\,dt \\ \\ \left(\int_a^x \sin^{pm}\left(\frac{t-a}{2}\right)dt\right)^{1/m}\left(\int_a^x |g(R\exp(it))|^n\,dt\right)^{1/n} \\ m,\ n>1 \text{ with } \frac{1}{m}+\frac{1}{n}=1, \\ \\ \max_{t\in[a,x]}|g(R\exp(it))|\int_a^x \sin^p\left(\frac{t-a}{2}\right)dt \end{cases}$$
$$+\, 2^p R^p L_p \times \begin{cases} \max_{t\in[x,b]}\left[\sin^p\left(\frac{b-t}{2}\right)\right]\int_x^b |g(R\exp(it))|\,dt \\ \\ \left(\int_x^b \sin^{pm}\left(\frac{b-t}{2}\right)dt\right)^{1/m}\left(\int_x^b |g(R\exp(it))|^n\,dt\right)^{1/n} \\ m,\ n>1 \text{ with } \frac{1}{m}+\frac{1}{n}=1, \\ \\ \max_{t\in[x,b]}|g(R\exp(it))|\int_x^b \sin^p\left(\frac{b-t}{2}\right)dt \end{cases}$$

for $x\in[a,b]$.

The case $p=1$, namely $f\in\mathfrak{Lip}\left(L;\gamma_{[a,b],R}\right)$, then by (5.4) we get

$$\begin{aligned} &\left|\int_a^b f(R\exp(it))\,g(R\exp(it))\exp(it)\,dt\right. \\ &\qquad - f(R\exp(ia))\int_a^x g(R\exp(it))\exp(it)\,dt \\ &\qquad \left. - f(R\exp(ib))\int_x^b g(R\exp(it))\exp(it)\,dt\right| \\ &\leq 2RL\left[\int_a^x \sin\left(\frac{t-a}{2}\right)|g(R\exp(it))|\,dt+\int_x^b \sin\left(\frac{b-t}{2}\right)|g(R\exp(it))|\,dt\right] \end{aligned} \tag{5.5}$$

$$\leq 2RL \times \begin{cases} \max_{t\in[a,x]}\left[\sin\left(\frac{t-a}{2}\right)\right]\int_a^x |g(R\exp(it))|\,dt \\ \\ \left(\int_a^x \sin^{m}\left(\frac{t-a}{2}\right)dt\right)^{1/m}\left(\int_a^x |g(R\exp(it))|^n\,dt\right)^{1/n} \\ m,\ n>1 \text{ with } \frac{1}{m}+\frac{1}{n}=1, \\ \\ \max_{t\in[a,x]}|g(R\exp(it))|\int_a^x \sin\left(\frac{t-a}{2}\right)dt \end{cases}$$
$$+\, 2RL \times \begin{cases} \max_{t\in[x,b]}\left[\sin\left(\frac{b-t}{2}\right)\right]\int_x^b |g(R\exp(it))|\,dt \\ \\ \left(\int_x^b \sin^{m}\left(\frac{b-t}{2}\right)dt\right)^{1/m}\left(\int_x^b |g(R\exp(it))|^n\,dt\right)^{1/n} \\ m,\ n>1 \text{ with } \frac{1}{m}+\frac{1}{n}=1, \\ \\ \max_{t\in[x,b]}|g(R\exp(it))|\int_x^b \sin\left(\frac{b-t}{2}\right)dt \end{cases}$$

for $x\in[a,b]$.

Observe that

$$\int_a^x \sin\left(\frac{s-a}{2}\right)ds = 2-2\cos\left(\frac{x-a}{2}\right)=4\sin^2\left(\frac{x-a}{4}\right)$$

and

$$\int_x^b \sin\left(\frac{b-s}{2}\right) ds = 2 - 2\cos\left(\frac{b-t}{2}\right) = 4\sin^2\left(\frac{b-x}{4}\right)$$

for $x \in [a,b]$.

By using (5.5) we get

$$\begin{aligned}
(5.6) \quad & \left| \int_a^b f(R\exp(it))\, g(R\exp(it)) \exp(it)\, dt \right. \\
& - f(R\exp(ia)) \int_a^x g(R\exp(it)) \exp(it)\, dt \\
& \left. - f(R\exp(ib)) \int_x^b g(R\exp(it)) \exp(it)\, dt \right| \\
\leq 2RL & \left[\int_a^x \sin\left(\frac{t-a}{2}\right) |g(R\exp(it))|\, dt + \int_x^b \sin\left(\frac{b-t}{2}\right) |g(R\exp(it))|\, dt \right] \\
& \leq 8RL \max_{t\in[a,x]} |g(R\exp(it))| \sin^2\left(\frac{x-a}{4}\right) \\
& + 8RL \max_{t\in[x,b]} |g(R\exp(it))| \sin^2\left(\frac{b-x}{4}\right) \\
& \leq 8RL \max_{t\in[a,b]} |g(R\exp(it))| \left[\sin^2\left(\frac{x-a}{4}\right) + \sin^2\left(\frac{b-x}{4}\right) \right]
\end{aligned}$$

for $x \in [a,b]$.

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