SEVERAL GRÜSS' TYPE INEQUALITIES FOR THE COMPLEX INTEGRAL

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Abstract. Assume that f and g are continuous on γ, $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$ and the *complex Čebyšev functional* is defined by

$$\mathcal{D}_\gamma(f,g) := \frac{1}{w-u}\int_\gamma f(z)g(z)\,dz - \frac{1}{w-u}\int_\gamma f(z)\,dz \frac{1}{w-u}\int_\gamma g(z)\,dz.$$

In this paper we establish some bounds for the magnitude of the functional $\mathcal{D}_\gamma(f,g)$ and a related version of this under various assumptions for the functions f and g and provide some examples for circular paths.

1. Introduction

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in D, and open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration*

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by parts formula

$$\int_{\gamma_{u,w}} f(z)\, g'(z)\, dz = f(w)\, g(w) - f(u)\, g(u) - \int_{\gamma_{u,w}} f'(z)\, g(z)\, dz. \tag{1.1}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left| \int_\gamma f(z)\, dz \right| \le \int_\gamma |f(z)|\, |dz| \le \|f\|_{\gamma,\infty}\, \ell(\gamma) \tag{1.2}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\, |dz| \right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\, |dz|.$$

If $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q}\, \|f\|_{\gamma,p}.$$

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$. Now, for $\phi,\ \Phi \in \mathbb{C}$, define the sets of complex-valued functions

$$\bar{U}_\gamma(\phi,\Phi) := \left\{ f : \gamma \to \mathbb{C} \middle| \operatorname{Re}\left[(\Phi - f(z)) \left(\overline{f(z)} - \overline{\phi} \right) \right] \ge 0 \text{ for each } z \in \gamma \right\}$$

and

$$\bar{\Delta}_\gamma(\phi,\Phi) := \left\{ f : \gamma \to \mathbb{C} \middle| \left| f(z) - \frac{\phi + \Phi}{2} \right| \le \frac{1}{2} |\Phi - \phi| \text{ for each } z \in \gamma \right\}.$$

The following representation result may be stated.

Proposition 1. *For any $\phi,\ \Phi \in \mathbb{C}$, $\phi \ne \Phi$, we have that $\bar{U}_\gamma(\phi,\Phi)$ and $\bar{\Delta}_\gamma(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_\gamma(\phi,\Phi) = \bar{\Delta}_\gamma(\phi,\Phi).$$

On making use of the complex numbers field properties we can also state that:

Corollary 1. *For any $\phi,\ \Phi \in \mathbb{C}$, $\phi \ne \Phi$, we have that*

$$\begin{aligned} \bar{U}_\gamma(\phi,\Phi) = \{ f : \gamma \to \mathbb{C} \mid\ & (\operatorname{Re}\Phi - \operatorname{Re} f(z))(\operatorname{Re} f(z) - \operatorname{Re}\phi) \\ & + (\operatorname{Im}\Phi - \operatorname{Im} f(z))(\operatorname{Im} f(z) - \operatorname{Im}\phi) \ge 0 \textit{ for each } z \in \gamma \}. \end{aligned}$$

Now, if we assume that $\operatorname{Re}(\Phi) \ge \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \ge \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_\gamma(\phi,\Phi) := \{ f : \gamma \to \mathbb{C} \mid\ & \operatorname{Re}(\Phi) \ge \operatorname{Re} f(z) \ge \operatorname{Re}(\phi) \\ & \text{and } \operatorname{Im}(\Phi) \ge \operatorname{Im} f(z) \ge \operatorname{Im}(\phi) \text{ for each } z \in \gamma \}. \end{aligned}$$

One can easily observe that $\bar{S}_\gamma(\phi,\Phi)$ is closed, convex and

$$\emptyset \ne \bar{S}_\gamma(\phi,\Phi) \subseteq \bar{U}_\gamma(\phi,\Phi).$$

Assume that f and g are continuous on γ, $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$ and the *complex Čebyšev functional* is defined by

$$\mathcal{D}_\gamma(f,g) := \frac{1}{w-u}\int_\gamma f(z)\,g(z)\,dz - \frac{1}{w-u}\int_\gamma f(z)\,dz \frac{1}{w-u}\int_\gamma g(z)\,dz.$$

In the recent paper we obtained the following Gruss' type inequality for the complex integral:

Theorem 1. *If f and g are continuous on γ and there exists ϕ, Φ, ψ, $\Psi \in \mathbb{C}$, $\phi \neq \Phi$, $\psi \neq \Psi$ such that $f \in \bar{\Delta}_\gamma(\phi,\Phi)$ and $g \in \bar{\Delta}_\gamma(\psi,\Psi)$ then*

$$|\mathcal{D}_\gamma(f,g)| \leq \frac{1}{4}|\Phi-\phi||\Psi-\psi|\frac{\ell^2(\gamma)}{|w-u|^2}. \tag{1.3}$$

For several previous results concerning three points inequalities, see [1], [2] and [11]-[17]. For some trapezoid, Ostrowski, Grüss and quasi-Grüss type inequalities for complex functions defined on the unit circle centered in zero, see [3]-[7].

Motivated by the above results, in this paper we provide some other Gruss' type inequalities for various assumptions of the functions involved. Examples for circular paths are provided as well.

2. Some Preliminary Facts

We have:

Lemma 1. *Let f and g be continuous in D, an open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v = z(x)$ with $x \in (a,b)$. Then for any complex numbers λ, μ we have*

$$\begin{aligned}\int_\gamma f(z)\,g(z)\,dz &= \lambda\int_{\gamma_{u,v}} g(z)\,dz + \mu\int_{\gamma_{v,w}} g(z)\,dz \\ &\quad + \int_{\gamma_{u,v}}[f(z)-\lambda]\,g(z)\,dz + \int_{\gamma_{v,w}}[f(z)-\mu]\,g(z)\,dz \\ &= \mu\int_\gamma g(z)\,dz + (\lambda-\mu)\int_{\gamma_{u,v}} g(z)\,dz \\ &\quad + \int_{\gamma_{u,v}}[f(z)-\lambda]\,g(z)\,dz + \int_{\gamma_{v,w}}[f(z)-\mu]\,g(z)\,dz.\end{aligned} \tag{2.1}$$

In particular, for $\mu = \lambda$, we have

$$\begin{aligned}\int_\gamma f(z)\,g(z)\,dz &= \lambda\int_\gamma g(z)\,dz \\ &\quad + \int_{\gamma_{u,v}}[f(z)-\lambda]\,g(z)\,dz + \int_{\gamma_{v,w}}[f(z)-\lambda]\,g(z)\,dz \\ &= \lambda\int_\gamma g(z)\,dz + \int_\gamma[f(z)-\lambda]\,g(z)\,dz.\end{aligned} \tag{2.2}$$

Proof. Using the properties of the complex integral, we have

$$\begin{aligned}
&\int_{\gamma_{u,v}} [f(z)-\lambda]\, g(z)\, dz + \int_{\gamma_{v,w}} [f(z)-\mu]\, g(z)\, dz \\
&= \int_{\gamma_{u,v}} f(z)\, g(z)\, dz - \lambda \int_{\gamma_{u,v}} g(z)\, dz + \int_{\gamma_{v,w}} f(z)\, g(z)\, dz - \mu \int_{\gamma_{v,w}} g(z)\, dz \\
&\qquad = \int_{\gamma} f(z)\, g(z)\, dz - \lambda \int_{\gamma_{u,v}} g(z)\, dz - \mu \int_{\gamma_{v,w}} g(z)\, dz,
\end{aligned}$$

which is equivalent to the first equality in (2.1).

The rest is obvious. □

Corollary 2. *Let f be continuous in D, an open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x\in(a,b)$. Then for any complex numbers λ, μ we have*

$$\int_{\gamma} f(z)\, dz = \lambda(v-u)+\mu(w-v)+\int_{\gamma_{u,v}} [f(z)-\lambda]\, dz+\int_{\gamma_{v,w}} [f(z)-\mu]\, dz. \tag{2.3}$$

In particular, for $\mu=\lambda$, we have

$$\begin{aligned}
\int_{\gamma} f(z)\, g(z)\, dz &= \lambda(w-u) + \int_{\gamma_{u,v}} [f(z)-\lambda]\, dz + \int_{\gamma_{v,w}} [f(z)-\lambda]\, dz \\
&= \lambda(w-u) + \int_{\gamma} [f(z)-\lambda]\, g(z)\, dz.
\end{aligned} \tag{2.4}$$

If we use the equality (2.2) for $\lambda = \frac{1}{w-u}\int_{\gamma} f(z)\, dz$, then we have

$$\begin{aligned}
\int_{\gamma} f(z)\, g(z)\, dz &= \frac{1}{w-u}\int_{\gamma} f(z)\, dz \int_{\gamma} g(z)\, dz \\
&\quad + \int_{\gamma}\left[f(z)-\frac{1}{w-u}\int_{\gamma} f(w)\, dw\right] g(z)\, dz.
\end{aligned} \tag{2.5}$$

Moreover, since

$$\int_{\gamma}\left[f(z)-\frac{1}{w-u}\int_{\gamma} f(w)\, dw\right] dz = 0,$$

then from (2.5) we get the more general equality

$$\begin{aligned}
\int_{\gamma} f(z)\, g(z)\, dz &= \frac{1}{w-u}\int_{\gamma} f(z)\, dz \int_{\gamma} g(z)\, dz \\
&\quad + \int_{\gamma}\left[f(z)-\frac{1}{w-u}\int_{\gamma} f(w)\, dw\right] [g(z)-\delta]\, dz.
\end{aligned} \tag{2.6}$$

for any $\delta \in \mathbb{C}$.

Also, we take in (2.1) $\lambda = \frac{1}{v-u}\int_{\gamma_{u,v}} f(y)\,dy$ and $\mu = \frac{1}{w-v}\int_{\gamma_{v,w}} f(y)\,dy$, then we get

$$\begin{aligned}
(2.7)\quad & \int_\gamma f(z)\,g(z)\,dz \\
&= \frac{1}{v-u}\int_{\gamma_{u,v}} f(y)\,dy \int_{\gamma_{u,v}} g(z)\,dz + \frac{1}{w-v}\int_{\gamma_{v,w}} f(y)\,dy \int_{\gamma_{v,w}} g(z)\,dz \\
&\quad + \int_{\gamma_{u,v}} \left[f(z) - \frac{1}{v-u}\int_{\gamma_{u,v}} f(y)\,dy \right] g(z)\,dz \\
&\quad + \int_{\gamma_{v,w}} \left[f(z) - \frac{1}{w-v}\int_{\gamma_{v,w}} f(y)\,dy \right] g(z)\,dz
\end{aligned}$$

for $v \in \gamma$ with $v \neq u$, w.

The identity (2.7) provides the more general equality

$$\begin{aligned}
(2.8)\quad & \int_\gamma f(z)\,g(z)\,dz \\
&= \frac{1}{v-u}\int_{\gamma_{u,v}} f(y)\,dy \int_{\gamma_{u,v}} g(z)\,dz + \frac{1}{w-v}\int_{\gamma_{v,w}} f(y)\,dy \int_{\gamma_{v,w}} g(z)\,dz \\
&\quad + \int_{\gamma_{u,v}} \left[f(z) - \frac{1}{v-u}\int_{\gamma_{u,v}} f(y)\,dy \right] [g(z) - \alpha]\,dz \\
&\quad + \int_{\gamma_{v,w}} \left[f(z) - \frac{1}{w-v}\int_{\gamma_{v,w}} f(y)\,dy \right] [g(z) - \beta]\,dz
\end{aligned}$$

for $v \in \gamma$ with $v \neq u$, w and for any α, $\beta \in \mathbb{C}$.

3. Some Inequalities for Bounded Functions

We start with the following result:

Theorem 2. *Let f and g be continuous in D, an open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v = z(x)$ with $x \in (a,b)$. Let ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$ and assume that $g \in \bar{\Delta}_\gamma(\phi,\Phi)$, then*

$$(3.1)\qquad |\mathcal{D}_\gamma(f,g)| \le \frac{1}{2}\frac{|\Phi-\phi|}{|w-u|}\int_\gamma \left| f(z) - \frac{1}{w-u}\int_\gamma f(w)\,dw \right| |dz|.$$

Proof. Using the identity (2.6) we get

$$\begin{aligned}
(3.2)\quad & \left| \frac{1}{w-u}\int_\gamma f(z)\,g(z)\,dz - \frac{1}{w-u}\int_\gamma f(z)\,dz \frac{1}{w-u}\int_\gamma g(z)\,dz \right| \\
&\qquad \le \frac{1}{|w-u|}\int_\gamma \left| f(z) - \frac{1}{w-u}\int_\gamma f(w)\,dw \right| |g(z) - \delta|\,|dz|
\end{aligned}$$

for any $\delta \in \mathbb{C}$.

Since $g \in \bar{\Delta}_\gamma(\phi,\Phi)$, hence

$$\left| g(z) - \frac{\phi+\Phi}{2} \right| \le \frac{1}{2}|\Phi - \phi|$$

for each $z \in \gamma$.

Therefore

$$\frac{1}{|w-u|}\int_{\gamma}\left|f(z)-\frac{1}{w-u}\int_{\gamma}f(w)\,dw\right|\left|g(z)-\frac{\phi+\Phi}{2}\right||dz|$$
$$\leq\frac{1}{2}\frac{|\Phi-\phi|}{|w-u|}\int_{\gamma}\left|f(z)-\frac{1}{w-u}\int_{\gamma}f(w)\,dw\right||dz|$$

and by (3.2) we get the desired result (3.1). $\square$

We say that $f\in\mathfrak{Lip}_p(L_p;\gamma,v)$ for given $p>0$ and $L_p>0$ if

$$|f(z)-f(v)|\leq L_p|z-v|^p \text{ for all } z\in\gamma.$$

In the recent paper [8] we obtained the following Ostrowski type inequality

$$\left|\int_{\gamma}f(w)\,dw-f(z)(w-u)\right|\leq L_p\int_{\gamma}|w-z|^p|dw| \tag{3.3}$$
$$\leq L_p\times\begin{cases}\max_{w\in\gamma}|w-z|^p\,\ell(\gamma)\\ \\ \left(\int_{\gamma}|w-z|^{mp}|dw|\right)^{1/m}[\ell(\gamma)]^{1/n}\\ m,\ n>1 \text{ with } \frac{1}{m}+\frac{1}{n}=1,\end{cases}$$

Using this inequality and the inequality (3.1), we get:

Corollary 3. *Let f and g be continuous in D, an open domain and suppose $\gamma\subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x\in(a,b)$. Let $\phi,\ \Phi\in\mathbb{C},\ \phi\neq\Phi$ and assume that $g\in\bar{\Delta}_{\gamma}(\phi,\Phi)$ while $f\in\mathfrak{Lip}_p(L_p;\gamma,v)$, then*

$$|\mathcal{D}_{\gamma}(f,g)|\leq\frac{1}{2}L_p\frac{|\Phi-\phi|}{|w-u|}\int_{\gamma}\left(\int_{\gamma}|w-z|^p|dw|\right)|dz| \tag{3.4}$$
$$\leq\frac{1}{2}L_p\frac{|\Phi-\phi|}{|w-u|}\times\begin{cases}\ell(\gamma)\int_{\gamma}\left(\max_{w\in\gamma}|w-z|^p\right)|dz|\\ \\ [\ell(\gamma)]^{1/n}\int_{\gamma}\left(\left(\int_{\gamma}|w-z|^{mp}|dw|\right)^{1/m}\right)|dz|\\ m,\ n>1 \text{ with } \frac{1}{m}+\frac{1}{n}=1.\end{cases}$$

The case $p=1$, namely $f\in\mathfrak{Lip}(L;\gamma)$, which means that

$$|f(z)-f(w)|\leq L|z-w| \text{ for all } z,\ w\in\gamma_{[a,b],R}$$

is of particular interest.

Therefore, by (3.4) we get

$$|\mathcal{D}_{\gamma}(f,g)|\leq\frac{1}{2}L\frac{|\Phi-\phi|}{|w-u|}\int_{\gamma}\left(\int_{\gamma}|w-z||dw|\right)|dz| \tag{3.5}$$
$$\leq\frac{1}{2}L\frac{|\Phi-\phi|}{|w-u|}\times\begin{cases}\ell(\gamma)\int_{\gamma}\left(\max_{w\in\gamma}|w-z|\right)|dz|\\ \\ [\ell(\gamma)]^{1/n}\int_{\gamma}\left(\left(\int_{\gamma}|w-z|^{m}|dw|\right)^{1/m}\right)|dz|\\ m,\ n>1 \text{ with } \frac{1}{m}+\frac{1}{n}=1.\end{cases}$$

Let f and g be continuous in D, an open domain and suppose $\gamma\subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x\in(a,b)$ and assume that $v\neq u,\ w$.

We define the following functional

$$\widetilde{\mathcal{D}}_{\gamma}(f,g,v):=\int_{\gamma}f(z)\,g(z)\,dz$$
$$-\frac{1}{v-u}\int_{\gamma_{u,v}}f(y)\,dy\int_{\gamma_{u,v}}g(z)\,dz-\frac{1}{w-v}\int_{\gamma_{v,w}}f(y)\,dy\int_{\gamma_{v,w}}g(z)\,dz.$$

Further, we have the following result as well:

Theorem 3. *Let f and g be continuous in D, an open domain and suppose $\gamma\subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x\in(a,b)$. Let ϕ_i, $\Phi_i\in\mathbb{C}$, $\phi_i\neq\Phi_i$ with $i\in\{1,2\}$ and assume that $g\in\bar{\Delta}_{\gamma_{u,v}}(\phi_1,\Phi_1)\cap\bar{\Delta}_{\gamma_{v,w}}(\phi_2,\Phi_2)$, then for $v\neq u$, w we have*

$$\left|\widetilde{\mathcal{D}}_{\gamma}(f,g,v)\right|\leq\frac{1}{2}\left|\Phi_1-\phi_1\right|\int_{\gamma_{u,v}}\left|f(z)-\frac{1}{v-u}\int_{\gamma_{u,v}}f(y)\,dy\right|\left|dz\right|$$
$$+\frac{1}{2}\left|\Phi_2-\phi_2\right|\int_{\gamma_{v,w}}\left|f(z)-\frac{1}{w-v}\int_{\gamma_{v,w}}f(y)\,dy\right|\left|dz\right|. \tag{3.6}$$

Proof. Using the identity (2.8) we get

$$\left|\int_{\gamma}f(z)\,g(z)\,dz\right.$$
$$\left.-\frac{1}{v-u}\int_{\gamma_{u,v}}f(y)\,dy\int_{\gamma_{u,v}}g(z)\,dz-\frac{1}{w-v}\int_{\gamma_{v,w}}f(y)\,dy\int_{\gamma_{v,w}}g(z)\,dz\right| \tag{3.7}$$
$$\leq\left|\int_{\gamma_{u,v}}\left[f(z)-\frac{1}{v-u}\int_{\gamma_{u,v}}f(y)\,dy\right]\left[g(z)-\alpha\right]dz\right|$$
$$+\left|\int_{\gamma_{v,w}}\left[f(z)-\frac{1}{w-v}\int_{\gamma_{v,w}}f(y)\,dy\right]\left[g(z)-\beta\right]dz\right|$$
$$\leq\int_{\gamma_{u,v}}\left|\left[f(z)-\frac{1}{v-u}\int_{\gamma_{u,v}}f(y)\,dy\right]\left[g(z)-\alpha\right]\right|\left|dz\right|$$
$$+\int_{\gamma_{v,w}}\left|\left[f(z)-\frac{1}{w-v}\int_{\gamma_{v,w}}f(y)\,dy\right]\left[g(z)-\beta\right]\right|\left|dz\right|$$
$$=\int_{\gamma_{u,v}}\left|f(z)-\frac{1}{v-u}\int_{\gamma_{u,v}}f(y)\,dy\right|\left|g(z)-\alpha\right|\left|dz\right|$$
$$+\int_{\gamma_{v,w}}\left|f(z)-\frac{1}{w-v}\int_{\gamma_{v,w}}f(y)\,dy\right|\left|g(z)-\beta\right|\left|dz\right|$$

for $v\in\gamma$ with $v\neq u$, w and for α, $\beta\in\mathbb{C}$.

If $g\in\bar{\Delta}_{\gamma_{u,v}}(\phi_1,\Phi_1)\cap\bar{\Delta}_{\gamma_{v,w}}(\phi_2,\Phi_2)$, then

$$\left|g(z)-\frac{\phi_1+\Phi_1}{2}\right|\leq\frac{1}{2}\left|\Phi_1-\phi_1\right|\ \text{ for } z\in\gamma_{u,v}$$

and

$$\left|g(z)-\frac{\phi_2+\Phi_2}{2}\right|\le\frac{1}{2}\left|\Phi_2-\phi_2\right| \text{ for } z\in\gamma_{v,w}$$

then

$$\int_{\gamma_{u,v}}\left|f(z)-\frac{1}{v-u}\int_{\gamma_{u,v}}f(y)\,dy\right|\left|g(z)-\frac{\phi_1+\Phi_1}{2}\right||dz| \\ \le\frac{1}{2}\left|\Phi_1-\phi_1\right|\int_{\gamma_{u,v}}\left|f(z)-\frac{1}{v-u}\int_{\gamma_{u,v}}f(y)\,dy\right||dz|$$

and

$$\int_{\gamma_{v,w}}\left|f(z)-\frac{1}{w-v}\int_{\gamma_{v,w}}f(y)\,dy\right|\left|g(z)-\frac{\phi_2+\Phi_2}{2}\right||dz| \\ \le\frac{1}{2}\left|\Phi_2-\phi_2\right|\int_{\gamma_{v,w}}\left|f(z)-\frac{1}{w-v}\int_{\gamma_{v,w}}f(y)\,dy\right||dz|$$

and by (3.7) we get the desired result (3.6). □

Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t\in[a,b]$ from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x\in(a,b)$. Let p_1, $p_2>0$ and L_{p_1}, $L_{p_1}>0$. We say that $f\in\mathfrak{Lip}_{p_1,p_2}(L_{p_1},L_{p_1};\gamma,v)$ if

$$|f(z)-f(v)|\le L_{p_1}|z-v|^{p_1} \text{ for all } z\in\gamma_{u,v}$$

and

$$|f(z)-f(v)|\le L_{p_2}|z-v|^{p_2} \text{ for all } z\in\gamma_{v,w}.$$

Corollary 4. *Let f and g be continuous in D, an open domain and suppose $\gamma\subset D$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x\in(a,b)$. Let ϕ_i, $\Phi_i\in\mathbb{C}$, $\phi_i\ne\Phi_i$ with $i\in\{1,2\}$ and assume that $g\in\bar{\Delta}_{\gamma_{u,v}}(\phi_1,\Phi_1)\cap\bar{\Delta}_{\gamma_{v,w}}(\phi_2,\Phi_2)$ and $f\in\mathfrak{Lip}_{p_1,p_2}(L_{p_1},L_{p_1};\gamma,v)$ where p_1, $p_2>0$ and L_{p_1}, $L_{p_1}>0$, then for $v\ne u$, w we have*

$$\left|\widetilde{\mathcal{D}}_\gamma(f,g,v)\right|\le\frac{1}{2}L_{p_1}\frac{|\Phi_1-\phi_1|}{|v-u|}\int_{\gamma_{u,v}}\left(\int_{\gamma_{u,v}}|z-w|^{p_1}|dw|\right)|dz| \\ +\frac{1}{2}L_{p_2}\frac{|\Phi_2-\phi_2|}{|w-v|}\int_{\gamma_{v,w}}\left(\int_{\gamma_{v,w}}|z-w|^{p_2}|dw|\right)|dz|. \tag{3.8}$$

Proof. By using (3.3) we have

$$\left|\int_{\gamma_{u,v}}f(w)\,dw-f(z)(v-u)\right|\le L_{p_1}\int_{\gamma_{u,v}}|w-z|^{p_1}|dw|,\ z\in\gamma_{u,v}$$

and

$$\left|\int_{\gamma_{v,w}}f(w)\,dw-f(z)(w-v)\right|\le L_{p_1}\int_{\gamma_{v,w}}|w-z|^{p_1}|dw|,\ z\in\gamma_{v,w}.$$

These imply

$$\left|\frac{1}{v-u}\int_{\gamma_{u,v}} f(w)\,dw - f(z)\right| \le \frac{L_{p_1}}{|v-u|}\int_{\gamma_{u,v}} |w-z|^{p_1}\,|dw|\,,\ z\in\gamma_{u,v}$$

and

$$\left|\frac{1}{w-v}\int_{\gamma_{v,w}} f(w)\,dw - f(z)\right| \le \frac{L_{p_1}}{|w-v|}\int_{\gamma_{v,w}} |w-z|^{p_1}\,|dw|\,,\ z\in\gamma_{v,w}.$$

Therefore

$$\int_{\gamma_{u,v}}\left|f(z)-\frac{1}{v-u}\int_{\gamma_{u,v}} f(y)\,dy\right||dz| \le \frac{L_{p_1}}{|v-u|}\int_{\gamma_{u,v}}\left(\int_{\gamma_{u,v}} |w-z|^{p_1}\,|dw|\right)|dz|\,,$$

and

$$\int_{\gamma_{v,w}}\left|f(z)-\frac{1}{w-v}\int_{\gamma_{v,w}} f(y)\,dy\right||dz| \le \frac{L_{p_1}}{|w-v|}\int_{\gamma_{v,w}}\left(\int_{\gamma_{v,w}} |w-z|^{p_1}\,|dw|\right)|dz|\,.$$

By utilising (3.6) we get the desired result (3.8). □

Remark 1. *We remark that if* $g\in\bar{\Delta}_\gamma(\phi,\Phi)$ *where* $\phi,\ \Phi\in\mathbb{C},\ \phi\ne\Phi$ *and* $f\in\mathfrak{Lip}_p(L_p;\gamma)$ *where* $p>0$ *and* $L_p>0$, *then by (3.8) we get*

$$\left|\widetilde{\mathcal{D}}_\gamma(f,g,v)\right| \le \frac{1}{2}L|\Phi-\phi|\left[\frac{1}{|v-u|}\int_{\gamma_{u,v}}\left(\int_{\gamma_{u,v}}|z-w|^p\,|dw|\right)|dz|\right.$$
$$\left.+\frac{1}{|w-v|}\int_{\gamma_{v,w}}\left(\int_{\gamma_{v,w}}|z-w|^p\,|dw|\right)|dz|\right]. \tag{3.9}$$

In particular, if $p=1$ *and* $f\in\mathfrak{Lip}(L;\gamma)$, *then by (3.9) we have*

$$\left|\widetilde{\mathcal{D}}_\gamma(f,g,v)\right| \le \frac{1}{2}L|\Phi-\phi|\left[\frac{1}{|v-u|}\int_{\gamma_{u,v}}\left(\int_{\gamma_{u,v}}|z-w|\,|dw|\right)|dz|\right.$$
$$\left.+\frac{1}{|w-v|}\int_{\gamma_{v,w}}\left(\int_{\gamma_{v,w}}|z-w|\,|dw|\right)|dz|\right]. \tag{3.10}$$

4. Examples for Circular Paths

Let $[a,b]\subseteq[0,2\pi]$ and the circular path $\gamma_{[a,b],R}$ centered in 0 and with radius $R>0$

$$z(t)=R\exp(it)=R(\cos t+i\sin t)\,,\ t\in[a,b]\,.$$

If $[a,b]=[0,\pi]$ then we get a half circle while for $[a,b]=[0,2\pi]$ we get the full circle.

Since

$$\left|e^{is}-e^{it}\right|^2=\left|e^{is}\right|^2-2\operatorname{Re}\left(e^{i(s-t)}\right)+\left|e^{it}\right|^2$$
$$=2-2\cos(s-t)=4\sin^2\left(\frac{s-t}{2}\right)$$

for any $t,\ s\in\mathbb{R}$, then

$$\left|e^{is}-e^{it}\right|^r=2^r\left|\sin\left(\frac{s-t}{2}\right)\right|^r$$

for any t, $s\in\mathbb{R}$ and $r>0$. In particular,

$$\left|e^{is}-e^{it}\right|=2\left|\sin\left(\frac{s-t}{2}\right)\right|$$

for any t, $s\in\mathbb{R}$.

For $s=a$ and $s=b$ we have

$$\left|e^{ia}-e^{it}\right|=2\left|\sin\left(\frac{a-t}{2}\right)\right| \text{ and } \left|e^{ib}-e^{it}\right|=2\left|\sin\left(\frac{b-t}{2}\right)\right|.$$

If $u=R\exp(ia)$ and $w=R\exp(ib)$ then

$$\begin{aligned}w-u&=R\left[\exp(ib)-\exp(ia)\right]=R\left[\cos b+i\sin b-\cos a-i\sin a\right]\\&=R\left[\cos b-\cos a+i\left(\sin b-\sin a\right)\right].\end{aligned}$$

Since

$$\cos b-\cos a=-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

and

$$\sin b-\sin a=2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right),$$

hence

$$\begin{aligned}w-u&=R\left[-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)+2i\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right)\right]\\&=2R\sin\left(\frac{b-a}{2}\right)\left[-\sin\left(\frac{a+b}{2}\right)+i\cos\left(\frac{a+b}{2}\right)\right]\\&=2Ri\sin\left(\frac{b-a}{2}\right)\left[\cos\left(\frac{a+b}{2}\right)+i\sin\left(\frac{a+b}{2}\right)\right]\\&=2Ri\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right].\end{aligned}$$

Moreover,

$$|z-u|=R\left|\exp(it)-\exp(ia)\right|=2R\left|\sin\left(\frac{t-a}{2}\right)\right|$$

and

$$|z-w|=R\left|\exp(it)-\exp(ib)\right|=2R\left|\sin\left(\frac{b-t}{2}\right)\right|$$

for $t\in[a,b]$.

If $[a,b]\subseteq[0,2\pi]$ then $0\le\frac{t-a}{2},\ \frac{b-t}{2}\le\pi$ for $t\in[a,b]$, therefore

$$|z-u|=2R\sin\left(\frac{t-a}{2}\right) \text{ and } |z-w|=2R\sin\left(\frac{b-t}{2}\right).$$

We also have

$$z'(t)=Ri\exp(it) \text{ and } |z'(t)|=R$$

for $t\in[a,b]$.

If $\gamma = \gamma_{[a,b],R}$, then the *circular complex Čebyšev functional* is defined by

$$\begin{aligned}(4.1)\quad \mathcal{C}_{[a,b],R}(f,g) &:= \mathcal{D}_{\gamma_{[a,b],R}}(f,g)\\ &= \frac{1}{2\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right]}\int_a^b f(R\exp(it))\,g(R\exp(it))\exp(it)\,dt\\ &\quad -\frac{1}{4\sin^2\left(\frac{b-a}{2}\right)\exp\left[2\left(\frac{a+b}{2}\right)i\right]}\\ &\quad\times\int_a^b f(R\exp(it))\exp(it)\,dt\int_a^b g(R\exp(it))\exp(it)\,dt.\end{aligned}$$

By making use of the inequality (3.5) for the circular path $\gamma_{[a,b],R}$

$$(4.2)\qquad \left|\mathcal{C}_{[a,b],R}(f,g)\right| \le \frac{1}{2}LR\frac{|\Phi-\phi|}{\sin\left(\frac{b-a}{2}\right)}\int_a^b\left(\int_a^b\left|\sin\left(\frac{s-t}{2}\right)\right|ds\right)dt$$

provided that $g \in \bar{\Delta}_{\gamma_{[a,b],R}}(\phi,\Phi)$ and $f \in \mathfrak{Lip}\left(L;\gamma_{[a,b],R}\right)$.

Observe that

$$\begin{aligned}\int_a^b\left|\sin\left(\frac{s-t}{2}\right)\right|ds &= \int_a^t \sin\left(\frac{t-s}{2}\right)ds + \int_t^b \sin\left(\frac{s-t}{2}\right)ds\\ &= 2\cos\left(\frac{t-s}{2}\right)\Big|_a^t - 2\cos\left(\frac{s-t}{2}\right)\Big|_t^b\\ &= 2\left[1-\cos\left(\frac{t-a}{2}\right)\right] - 2\left[\cos\left(\frac{b-t}{2}\right)-1\right]\\ &= 2\left[2-\cos\left(\frac{t-a}{2}\right)-\cos\left(\frac{b-t}{2}\right)\right]\end{aligned}$$

and

$$\begin{aligned}\int_a^b\left(\int_a^b\left|\sin\left(\frac{s-t}{2}\right)\right|ds\right)dt &= 2\int_a^b\left[2-\cos\left(\frac{t-a}{2}\right)-\cos\left(\frac{b-t}{2}\right)\right]dt\\ &= 2\left[2(b-a)-2\sin\left(\frac{b-a}{2}\right)-2\sin\left(\frac{b-a}{2}\right)\right]\\ &= 4\left[b-a-2\sin\left(\frac{b-a}{2}\right)\right]\\ &= 8\left[\frac{b-a}{2}-\sin\left(\frac{b-a}{2}\right)\right]\end{aligned}$$

and by (4.2) we get

$$(4.3)\qquad \left|\mathcal{C}_{[a,b],R}(f,g)\right| \le 4LR\frac{|\Phi-\phi|}{\sin\left(\frac{b-a}{2}\right)}\left[\frac{b-a}{2}-\sin\left(\frac{b-a}{2}\right)\right]$$

provided that $g \in \bar{\Delta}_{\gamma_{[a,b],R}}(\phi,\Phi)$ and $f \in \mathfrak{Lip}\left(L;\gamma_{[a,b],R}\right)$.

Similar results may be obtained by the use of (3.10), however the details are not presented here.

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