Certain inequalities of Kober and Lazarević type

Yogesh J. Bagul[1], S. K. Panchal[2],
[1]Department of Mathematics, K. K. M. College, Manwath, Dist : Parbhani(M.S.) - 431505, India.
Email : yjbagul@gmail.com
[2]Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad(M. S.) - 431004, India.
Email : drpanchalsk@gmail.com

Abstract : In this work, the authors present new lower and upper bounds for $cosx$ and $coshx$, thus improving some generalized inequalities of Kober and Lazarević type.

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1 Introduction

There has been growing interest among the researchers in generalizing and sharpening the Kober type [12] and Lazarević type [1, 2] inequalities. The famous inequalities are respectively given by

$$1 - \frac{2x}{\pi} \leqslant cosx \leqslant 1 - \frac{x^2}{\pi}; \; x \in [0, \pi/2] \tag{1.1}$$

and

$$coshx < \left(\frac{sinhx}{x}\right)^p; \forall x > 0 \tag{1.2}$$

if and only if $p \geqslant 3$.

In [5, 10, 11, 14] the generalizations and refinements of (1.1) are appeared. B. A. Bhayo and J. Sándor[10] refine the inequality of type (1.1) as follows:

$$1-\frac{x^2/2}{1+x^2/12} < cosx < 1-\frac{24x^2/(5\pi^2)}{1+4x^2/(5\pi^2)};\ x \in (0, \pi/2) \tag{1.3}$$

They further refine the upper bound of $cosx$ in (1.3) as

$$\left(\frac{\pi^2-4x^2}{12}\right)^{3/2} < cosx < \left(1-\frac{x^2}{3}\right)^{3/2};\ x \in (0, \pi/2) \tag{1.4}$$

In [3 - 7], the generalizations and refinements of inequality of type (1.2) i.e. bounds of $coshx$ are appeared. The natural exponential bounds of $coshx$ were established very recently in [9], as follows:

$$e^{ax^2} < coshx < e^{x^2/2};\ x \in (0,1) \tag{1.5}$$

where $a \approx 0.433781$.
In [13] it is given that, for all non-zero real numbers x, the inequality

$$coshx < \left(\frac{sinhx}{x}\right)^3 - \frac{12}{5}\left(1-\frac{x}{sinhx}\right)^2 \tag{1.6}$$

holds.
The main purpose of this paper is to refine the above mentioned bounds and present new improved sharp bounds for $cosx$ and $coshx$.

2 Two Lemmas

Following are the tools to prove our main results.

Lemma 1. *(The Mitrinović - Adamović inequality [2, p.238]): For* $x \in (0, \frac{\pi}{2})$ *one has*

$$cosx < \left(\frac{sinx}{x}\right)^3. \tag{2.1}$$

For the recent refined form of (2.1), we refer reader to [15].

Lemma 2. *(l'Hôpital's Rule of Monotonicity [8, Thm. 1.25]): Let* $f, g : [l, m] \to \mathbb{R}$ *be two continuous functions which are derivable in* (l, m) *and* $g' \neq 0$ *in* (l, m)*. If* f'/g' *is increasing (or decreasing) in* (l, m)*, then the functions* $\frac{f(x)-f(l)}{g(x)-g(l)}$ *and* $\frac{f(x)-f(m)}{g(x)-g(m)}$ *are also increasing (or decreasing) on* (l, m)*. If* f'/g' *is strictly monotone, then the monotonicity in the conclusion is also strict.*

3 Main Results

In our main results we first give more sharp bounds for $cosx$ than the corresponding bounds given in (1.1).

Theorem 1. *If $x \in (0, \pi/2)$ then*

$$1 - \frac{x^2}{2} < cosx < 1 - \frac{4x^2}{\pi^2}. \tag{3.1}$$

Proof. Let, $1 - \frac{x^2}{a} < cosx < 1 - \frac{x^2}{b}$, which implies that, $a < \frac{x^2}{1-cosx} < b$.

Then $f(x) = \frac{x^2}{1-cosx} = \frac{f_1(x)}{f_2(x)}$,

where $f_1(x) = x^2$ and $f_2(x) = 1 - cosx$ with $f_1(0) = f_2(0) = 0$. By Differentiation we get

$$\frac{f_1'(x)}{f_2'(x)} = \frac{2x}{sinx} = \frac{f_3(x)}{f_4(x)}$$

where $f_3(x) = 2x$ and $f_4(x) = sinx$, with $f_3(0) = f_4(0) = 0$. Differentiation gives us
$\frac{f_3'(x)}{f_4'(x)} = \frac{2}{cosx}$, which is clearly strictly increasing in $(0, \pi/2)$. By Lemma 2 , $f(x)$ is strictly increasing in $(0, \pi/2)$. Therefore

$$f(0+) < f(x) < f(\pi/2)$$

Consequently, $a = f(0+) = 2$, by l'Hôpital's rule and $b = f(\pi/2) = \frac{(\pi/2)^2}{1-cos(\pi/2)} = \frac{\pi^2}{4}$. □

Remark 1. *By using lemma 2, we can also obtain that, $cosx < \frac{2}{2+x^2}$ in $(0, \pi/2)$. Thus*

$$\frac{2-x^2}{2} < cosx < \frac{2}{2+x^2}; \; x \in (0, \pi/2). \tag{3.2}$$

The upper bounds of (1.3) and (1.4) are refined in the next theorem.

Theorem 2. *For any $x \in (0, \pi/2)$ one has*

$$1 - \frac{x^2/2}{1 + x^2/12} < cosx < 1 - \frac{x^2/2}{1 + x^2/b} \tag{3.3}$$

where the constants 12 and $b \approx 10.557960$ are best possible.

Proof. Let $1-\frac{x^2/2}{1+x^2/a} < cosx < 1-\frac{x^2/2}{1+x^2/b}$, which implies that, $\frac{1}{a} < f(x) < \frac{1}{b}$; where

$$f(x) = \frac{x^2 - 2(1-cosx)}{2x^2(1-cosx)} = \frac{1}{2(1-cosx)} - \frac{1}{x^2}$$

.

Therefore,

$$f'(x) = \frac{-sinx}{2(1-cosx)^2} + \frac{2}{x^3}$$

.

Using (2.1) we have

$$16\, sin^4\left(\frac{x}{2}\right) > 2x^3\, sin\left(\frac{x}{2}\right)\, cos\left(\frac{x}{2}\right)$$

, which gives

$$4\,(1-cosx)^2 - x^3\, sinx > 0$$

. So that $f'(x) > 0$. Thus, $f(x)$ is increasing in $(0, \pi/2)$. Hence $a = \frac{1}{f(0+)} = 12$ by l'Hôpital's rule and $b = \frac{1}{f(\pi/2)} \approx 10.557960$. $\square$

Note: Though the strict comparison may not be done between the bounds; the bounds in (3.3) are better than the corresponding bounds in (1.3) and (1.4).

In the following theorem, we conclude that the bounds of $coshx$ are more sharp than the corresponding bounds in (1.5).

Theorem 3. *If $x \in (0,1)$ then*

$$1 + \frac{x^2}{2} < coshx < 1 + \frac{x^2}{b} \tag{3.4}$$

with the best possible constants 2 *and* $b \approx 1.841348$.

Proof. Let, $1+\frac{x^2}{a} < coshx < 1+\frac{x^2}{b}$, which implies that, $b < \frac{x^2}{coshx-1} < a$.

Then $f(x) = \frac{x^2}{coshx-1} = \frac{f_1(x)}{f_2(x)}$,

where $f_1(x) = x^2$ and $f_2(x) = coshx - 1$ with $f_1(0) = f_2(0) = 0$. By differentiation

$$\frac{f_1'(x)}{f_2'(x)} = \frac{2x}{sinhx} = \frac{f_3(x)}{f_4(x)}$$

where $f_3(x) = 2x$ and $f_4(x) = sinhx$ with $f_3(0) = f_4(0) = 0$. Differentiation gives

$$\frac{f_3'(x)}{f_4'(x)} = \frac{2}{coshx}$$

which is clearly decreasing in $(0, 1)$. By lemma 2, $f(x)$ is also strictly decreasing in $(0, 1)$. Clearly, then $a = f(0+) = 2$, by l'Hôpital's rule and $b = f(1-) = \frac{1}{cosh1-1} \approx 1.841348$. $\square$

Note : There is no strict comparison between the corresponding bounds of $coshx$ in (1.5) and (3.4).

Corollary 1. *If* $x \in (0, 1)$ *then*

$$2 < cosx + coshx < 2 + c.x^2 \tag{3.5}$$

with the best possible constant $c \approx 0.137796$.

Proof. Combining (3.1) and (3.4), the assertion follows. $\square$

In the following theorem we give more tight bounds of $coshx$ than the corresponding bounds given in (1.5) and (1.6).

Theorem 4. *For* $x \in (0, 1)$ *one has*

$$1 + \frac{ax^2}{\pi^2 - x^2} < coshx < 1 + \frac{(\pi x)^2/2}{\pi^2 - x^2} \tag{3.6}$$

where the constants $a \approx 4.816910$ *and* $\frac{\pi^2}{2}$ *are best possible.*

Proof. Let $1 + \frac{ax^2}{\pi^2-x^2} < coshx < 1 + \frac{bx^2}{\pi^2-x^2}$, which implies that

$a < \frac{(coshx-1)(\pi^2-x^2)}{x^2} < b$. Then $f(x) = \frac{(coshx-1)(\pi^2-x^2)}{x^2}$.

Therefore

$$f'(x) = \frac{\pi^2\, x\, sinhx - 2\pi^2(coshx - 1) - x^3\, sinhx}{x^3}.$$

Now by Taylor's series expansion we have

$$\pi^2\, x\, sinhx = \pi^2 \left(x^2 + \frac{x^4}{6} + \frac{x^6}{120} + \frac{x^8}{5040} + \ldots\ldots\ldots\ldots\ldots \right),$$

$$-2\pi^2\,(coshx - 1) = -\pi^2 \left(x^2 + \frac{x^4}{12} + \frac{x^6}{360} + \frac{x^8}{20160} + \ldots\ldots\ldots\ldots \right),$$

$$-x^3\, sinhx = -x^4 - \frac{x^6}{6} - \frac{x^8}{120} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence

$$f'(x) = \frac{1}{x^3}\left[\frac{(\pi^2-12)}{12}\,x^4 + \frac{(\pi^2-30)}{180}\,x^6 + \frac{(3\pi^2-168)}{20160}\,x^8 + \ldots\ldots\ldots\ldots\right]$$
$$= -\left(\frac{12-\pi^2}{12}\right)x - \left(\frac{30-\pi^2}{180}\right)x^3 - \left(\frac{168-3\pi^2}{20160}\right)x^5$$

Thus, $f'(x) < 0$ in $(0,1)$. So that $f(x)$ is strictly decreasing in $(0,1)$. Consequently, $a = f(1-) = (cosh1-1)(\pi^2-1) \approx 4.816910$ and $b = f(0+) = \frac{\pi^2}{2}$ by l'Hôpital's rule. This completes the proof of theorem. $\square$

4 An Application

R. Klén, M. Visuri and M. Vuorinen [6, Theorem 3.1] proved the following double inequality

$$1 - \frac{x^2}{6} \leqslant \frac{sinx}{x} \leqslant 1 - \frac{2\,x^2}{3\,\pi^2};\ x \in (-\pi/2, \pi/2). \tag{4.1}$$

A reader can see the refined form of (4.1) in the last theorem.

Theorem 5. *For $x \in (-\pi/2, \pi/2)$, it is true that*

$$1 - \frac{x^2}{6} \leqslant \frac{sinx}{x} \leqslant 1 - \frac{4\,x^2}{3\,\pi^2};\ x \in (-\pi/2, \pi/2). \tag{4.2}$$

Proof. Clearly equality holds at $x = 0$. Due to symmetry, it suffices to prove the theorem in $(0, \pi/2)$. On integrating (3.1), we have

$$\int_0^x \left(1 - \frac{t^2}{2}\right) dt < \int_0^x cost\, dt \ < \int_0^x \left(1 - \frac{4t^2}{\pi^2}\right) dt$$

where $x \in (0, \pi/2)$. i. e.

$$x - \frac{x^3}{6} < sinx < x - \frac{4x^3}{3\pi^2}$$

which proves our result. $\square$

References

[1] I. Lazarević, *Neke nejednakosti sa hiperbolickim funkcijama*, Univerzitet u Beogradu. Publikacije Elektrotechničkog Fakulteta. Serija Matematika i Fizika, Vol. **170** (1966), pp. 41-48.

[2] D. S. Mitrinovic, *Analytic Inequalities*, Springer-Verlag, Berlin **1970**.

[3] C. P. Chen, J. W. Zhao and F. Qi, *Three inequalities involving hyperbolically trigonometric functions*, RGMIA Res. Rep. Coll. **6**(3)(2003), Art. 4, pp. 437-443.

[4] R. Redheffer, *problem 5642*, Amer. Math. monthly **76**(1969), 422.

[5] L. Zhu, J.Sun, *Six new Redheffer-type inequalities for circular and hyperbolic functions*, Comput. Math. Appl. Vol. **56** No. 2, pp. 522-529, 2008.

[6] R. Klén, M. Visuri and M. Vuorinen, *On Jordan Type Inequalities for Hyperbolic Functions*, J. Inequal. and Appl. Vol. (**2010**), 14 pages, Art. ID 362548, 2010

[7] Yupei LV, Gendi Wang, Yuming Chu, *A note on Jordan type inequalities for hyperbolic functions*, Appl. Math. Lett. Vol. **25**, pp. 505-508, 2012.

[8] G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen, *Conformal Invarients, Inequalities and Quasiconformal maps*, John Wiley and Sons, New York, **1997**.

[9] Yogesh J. Bagul, *On Exponential Bounds of Hyperbolic Cosine*, Bulletin Of The International Mathematical Virtual Institute, Vol. **8** , No. 2, pp. 365-367, 2018.

[10] Barkat Ali Bhayo and József Sándor, *On Jordan's and Kober's Inequality*, Acta Et Commentationes Universitatis tartuensis de Mathematica, Vol. **20**, No. 2, Dec. 2016.

[11] J. Sándor, *On new refinements of Kober's and Jordan's trigonometric inequalities*, Notes Number Theory Discrete Math. Vol. **19**, No. 1, pp. 73 - 83, (2013).

[12] H. Kober, *Approximation by integral functions in the complex domain*, Trans. Amer. Math. Soc. Vol. **56**, No. 1, pp. 7 - 31, 1944.

[13] Shan-He Wu, Shu-Guang Li, Mihály Bencze, *Sharpened versions of Mitrinović-Adamović, Lazarević and Wilker's inequalities for trigonometric and hyperbolic functions*, J. Nonlinear Sci. Appl. Vol. **9** , pp. 2688-2696, 2016.

[14] X. Zhang, G. Wang and Y. Chu, *Extensions and sharpenings of Jordan's and Kober's inequalities*, J. Inequal. Pure. Appl. Math. Vol. **7** , No. 2, Article 63, 3 pages, 2006.

[15] J. Sándor, *Refinements of the Mitrinović - Adamović inequality, with application*, Notes Number Theory Discrete Math. Vol. **23** , No. 1, pp. 4 - 6, 2017.