Multidimensional Fractional Iyengar type inequalities for radial functions

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Abstract

Here we derive a variety of multivariate fractional Iyengar type inequalities for radial functions defined on the shell and ball. Our approach is based on the polar coordinates in $\mathbb{R}^N$, $N \geq 2$, and the related multivariate polar integration formula. Via this method we transfer author's univariate fractional Iyengar type inequalities into multivariate fractional Iyengar inequalities.

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1 Background

We are motivated by the following famous Iyengar inequality (1938), [10].

Theorem 1 *Let f be a differentiable function on $[a,b]$ and $|f'(x)| \leq M$. Then*

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) \right| \leq \frac{M(b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M}. \quad (1)$$

We need

Definition 2 *([2], p. 394) Let $\nu > 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ the ceiling of the number), $f \in AC^n([a,b])$ (i.e. $f^{(n-1)}$ is absolutely continuous on $[a,b]$). The left Caputo fractional derivative of order ν is defined as*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\,dt, \quad (2)$$

$\forall\, x \in [a,b]$, and it exists almost everywhere over $[a,b]$.

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We need

Definition 3 *([4], p. 336-337) Let* $\nu > 0$, $n = \lceil \nu \rceil$, $f \in AC^n([a,b])$. *The right Caputo fractional derivative of order* ν *is defined as*

$$D_{b-}^{\nu} f(x) = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (z-x)^{n-\nu-1} f^{(n)}(z)\, dz, \tag{3}$$

$\forall\, x \in [a,b]$, *and exists almost everywhere over* $[a,b]$.

In [7] we proved the following Caputo fractional Iyengar type inequalities:

Theorem 4 *([7]) Let* $\nu > 0$, $n = \lceil \nu \rceil$ *($\lceil \cdot \rceil$ is the ceiling of the number), and* $f \in AC^n([a,b])$ *(i.e.* $f^{(n-1)}$ *is absolutely continuous on* $[a,b]$*). We assume that* $D_{*a}^{\nu} f, D_{b-}^{\nu} f \in L_{\infty}([a,b])$. *Then*

i)

$$\left| \int_a^b f(x)\, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_{\infty}([a,b])}, \|D_{b-}^{\nu} f\|_{L_{\infty}([a,b])} \right\}}{\Gamma(\nu+2)} \left[(t-a)^{\nu+1} + (b-t)^{\nu+1} \right], \tag{4}$$

$\forall\, t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$, *the right hand side of (4) is minimized, and we get:*

$$\left| \int_a^b f(x)\, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_{\infty}([a,b])}, \|D_{b-}^{\nu} f\|_{L_{\infty}([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{5}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, *for all* $k = 0, 1, ..., n-1$, *we obtain*

$$\left| \int_a^b f(x)\, dx \right| \leq \frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_{\infty}([a,b])}, \|D_{b-}^{\nu} f\|_{L_{\infty}([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{6}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds*

$$\left| \int_a^b f(x)\, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_{\infty}([a,b])}, \|D_{b-}^{\nu} f\|_{L_{\infty}([a,b])} \right\}}{\Gamma(\nu+2)} \left(\frac{b-a}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \tag{7}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, *from (7) we get:*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N}\right)[jf(a) + (N-j)f(b)] \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_\infty([a,b])}, \|D_{b-}^{\nu} f\|_{L_\infty([a,b])} \right\}}{\Gamma(\nu+2)} \left(\frac{b-a}{N}\right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \tag{8}$$

$j = 0, 1, 2, ..., N$,

vi) when $N = 2$ *and* $j = 1$, *(8) turns to*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)(f(a) + f(b)) \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_\infty([a,b])}, \|D_{b-}^{\nu} f\|_{L_\infty([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{9}$$

vii) when $0 < \nu \leq 1$, *inequality (9) is again valid without any boundary conditions.*

We mention

Theorem 5 *([7]) Let* $\nu \geq 1$, $n = \lceil \nu \rceil$, *and* $f \in AC^n([a,b])$. *We assume that* $D_{*a}^{\nu} f, D_{b-}^{\nu} f \in L_1([a,b])$. *Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_1([a,b])}, \|D_{b-}^{\nu} f\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \left[(t-a)^{\nu} + (b-t)^{\nu} \right], \tag{10}$$

$\forall\, t \in [a,b]$,

ii) when $\nu = 1$, *from (10), we have*

$$\left| \int_a^b f(x)\,dx - [f(a)(t-a) + f(b)(b-t)] \right| \leq$$

$$\|f'\|_{L_1([a,b])}(b-a), \quad \forall\, t \in [a,b], \tag{11}$$

iii) from (11), we obtain ($\nu = 1$ *case)*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)(f(a) + f(b)) \right| \leq \|f'\|_{L_1([a,b])}(b-a), \tag{12}$$

iv) at $t=\frac{a+b}{2}$, $\nu>1$, the right hand side of (10) is minimized, and we get:

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}}, \tag{13}$$

v) if $f^{(k)}(a)=f^{(k)}(b)=0$, for all $k=0,1,...,n-1$; $\nu>1$, from (13), we obtain

$$\left|\int_a^b f(x)\,dx\right|\leq\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}}, \tag{14}$$

which is a sharp inequality,

vi) more generally, for $j=0,1,2,...,$ $N\in\mathbb{N}$, it holds

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{N}\right)^{k+1}\left[j^{k+1}f^{(k)}(a)+(-1)^k(N-j)^{k+1}f^{(k)}(b)\right]\right|$$

$$\leq\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\left(\frac{b-a}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{15}$$

vii) if $f^{(k)}(a)=f^{(k)}(b)=0$, $k=1,...,n-1$, from (15) we get:

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{N}\right)\left[jf(a)+(N-j)f(b)\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\left(\frac{b-a}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{16}$$

$j=0,1,2,...,N,$

viii) when $N=2$ and $j=1$, (16) turns to

$$\left|\int_a^b f(x)\,dx-\frac{(b-a)}{2}(f(a)+f(b))\right|\leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}}. \tag{17}$$

We mention

Theorem 6 *([7]) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$; $f \in AC^n([a,b])$, with $D_{*a}^{\nu} f, D_{b-}^{\nu} f \in L_q([a,b])$. Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_q([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \left[(t-a)^{\nu+\frac{1}{p}} + (b-t)^{\nu+\frac{1}{p}} \right], \tag{18}$$

$\forall\ t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$, the right hand side of (18) is minimized, and we get:

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_q([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{19}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, ..., n-1$, we obtain

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_q([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{20}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_q([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N} \right)^{\nu+\frac{1}{p}} \left[j^{\nu+\frac{1}{p}} + (N-j)^{\nu+\frac{1}{p}} \right], \tag{21}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, from (21) we get:

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_q([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N} \right)^{\nu+\frac{1}{p}} \left[j^{\nu+\frac{1}{p}} + (N-j)^{\nu+\frac{1}{p}} \right], \tag{22}$$

for $j=0,1,2,...,N$,

vi) when $N=2$ *and* $j=1$, *(22) turns to*

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right|\leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}},\tag{23}$$

vii) when $1/q<\nu\leq 1$, *inequality (23) is again valid but without any boundary conditions.*

We need the following different fractional calculus background:

Let $\alpha>0$, $m=[\alpha]$ ($[\cdot]$ is the integral part), $\beta=\alpha-m$, $0<\beta<1$, $f\in C([a,b])$, $[a,b]\subset\mathbb{R}$, $x\in[a,b]$. The gamma function Γ is given by $\Gamma(\alpha)=\int_0^{\infty}e^{-t}t^{\alpha-1}dt$. We define the left Riemann-Liouville integral ([2], p. 24)

$$\left(J_{\alpha}^{a+}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt,\tag{24}$$

$a\leq x\leq b$. We define the subspace $C_{a+}^{\alpha}([a,b])$ of $C^m([a,b])$:

$$C_{a+}^{\alpha}([a,b])=\left\{f\in C^m([a,b]):J_{1-\beta}^{a+}f^{(m)}\in C^1([a,b])\right\}.\tag{25}$$

For $f\in C_{a+}^{\alpha}([a,b])$, we define the left generalized α-fractional derivative of f over $[a,b]$ as

$$D_{a+}^{\alpha}f:=\left(J_{1-\beta}^{a+}f^{(m)}\right)',\tag{26}$$

see [2], p. 24. Canavati first in [9] introduced the above over $[0,1]$.

We have that $D_{a+}^{n}f=f^{(n)}$; $n\in\mathbb{N}$.

Notice that $D_{a+}^{\alpha}f\in C([a,b])$.

Furthermore we need:

Let again $\alpha>0$, $m=[\alpha]$, $\beta=\alpha-m$, $f\in C([a,b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^{\alpha}f\right)(x):=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt,\tag{27}$$

$x\in[a,b]$, see [3]. Define the subspace of functions

$$C_{b-}^{\alpha}([a,b]):=\left\{f\in C^m([a,b]):J_{b-}^{1-\beta}f^{(m)}\in C^1([a,b])\right\}.\tag{28}$$

Define the right generalized α-fractional derivative of f over $[a,b]$ as

$$\overline{D}_{b-}^{\alpha}f=(-1)^{m-1}\left(J_{b-}^{1-\beta}f^{(m)}\right)',\tag{29}$$

see [3]. We set $\overline{D}_{b-}^{0}f=f$. We have $\overline{D}_{b-}^{n}f=(-1)^n f^{(n)}$; $n\in\mathbb{N}$. Notice that $\overline{D}_{b-}^{\alpha}f\in C([a,b])$.

We mention the following Canavati fractional Iyengar type inequalities:

Theorem 7 *([6]) Let $\nu \geq 1$, $n = [\nu]$ and $f \in C_{a+}^{\nu}([a,b]) \cap C_{b-}^{\nu}([a,b])$. Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \left[(t-a)^{\nu+1} + (b-t)^{\nu+1} \right], \quad (30)$$

$\forall\ t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$, the right hand side of (30) is minimized, and we get:

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \quad (31)$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, ..., n-1$, we obtain

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \quad (32)$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ...,\ N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \left(\frac{b-a}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \quad (33)$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, from (33) we get:

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \left(\frac{b-a}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \quad (34)$$

$j = 0, 1, 2, ..., N,$

vi) when $N = 2$ and $j = 1$, (34) turns to

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq$$

$$\frac{\max \left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}. \tag{35}$$

We mention

Theorem 8 *([6]) Let $\nu \geq 1$, $n = [\nu]$, and $f \in C_{a+}^{\nu}([a,b]) \cap C_{b-}^{\nu}([a,b])$. Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\max \left\{ \left\| D_{a+}^{\nu} f \right\|_{L_1([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \left[(t-a)^{\nu} + (b-t)^{\nu} \right], \tag{36}$$

$\forall\, t \in [a,b]$,

ii) when $\nu = 1$, from (36), we have

$$\left| \int_a^b f(x)\,dx - \left[f(a)(t-a) + f(b)(b-t) \right] \right| \leq$$

$$\left\| f' \right\|_{L_1([a,b])} (b-a), \quad \forall\, t \in [a,b], \tag{37}$$

iii) from (37), we obtain ($\nu = 1$ case)

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq \left\| f' \right\|_{L_1([a,b])} (b-a), \tag{38}$$

iv) at $t = \frac{a+b}{2}$, $\nu > 1$, the right hand side of (36) is minimized, and we get:

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max \left\{ \left\| D_{a+}^{\nu} f \right\|_{L_1([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \frac{(b-a)^{\nu}}{2^{\nu-1}}, \tag{39}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, ..., n-1$; $\nu > 1$, from (39), we obtain

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_1([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \frac{(b-a)^{\nu}}{2^{\nu-1}}, \tag{40}$$

which is a sharp inequality,

vi) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_1([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \left(\frac{b-a}{N} \right)^{\nu} \left[j^{\nu} + (N-j)^{\nu} \right], \tag{41}$$

vii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, from (41) we get:

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) \left[j f(a) + (N-j) f(b) \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_1([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \left(\frac{b-a}{N} \right)^{\nu} \left[j^{\nu} + (N-j)^{\nu} \right], \tag{42}$$

$j = 0, 1, 2, ..., N$,

viii) when $N = 2$ and $j = 1$, (42) turns to

$$\left| \int_a^b f(x)\,dx - \frac{(b-a)}{2} \left(f(a) + f(b) \right) \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_1([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_1([a,b])} \right\}}{\Gamma(\nu+1)} \frac{(b-a)^{\nu}}{2^{\nu-1}}, \tag{43}$$

We mention

Theorem 9 *([6]) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu \geq 1$, $n = [\nu]$; $f \in C_{a+}^{\nu}([a,b]) \cap C_{b-}^{\nu}([a,b])$. Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_q([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \left[(t-a)^{\nu+\frac{1}{p}} + (b-t)^{\nu+\frac{1}{p}} \right], \tag{44}$$

$\forall\ t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$, the right hand side of (44) is minimized, and we get:

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_q([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{45}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, ..., n-1$, we obtain

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_q([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{46}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ...,\ N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_q([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N} \right)^{\nu+\frac{1}{p}} \left[j^{\nu+\frac{1}{p}} + (N-j)^{\nu+\frac{1}{p}} \right], \tag{47}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, from (47) we get:

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_q([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N} \right)^{\nu+\frac{1}{p}} \left[j^{\nu+\frac{1}{p}} + (N-j)^{\nu+\frac{1}{p}} \right], \tag{48}$$

for $j = 0, 1, 2, ..., N$,

vi) when $N = 2$ and $j = 1$, (48) turns to

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_q([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}. \tag{49}$$

We need

Definition 10 *([1]) Let $a, b \in \mathbb{R}$. The left conformable fractional derivative starting from a of a function $f : [a, \infty) \to \mathbb{R}$ of order $0 < \alpha \leq 1$ is defined by*

$$\left(T_\alpha^a f\right)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon (t-a)^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{50}$$

If $\left(T_\alpha^a f\right)(t)$ exists on (a, b), then

$$\left(T_\alpha^a f\right)(a) = \lim_{t \to a+} \left(T_\alpha^a f\right)(t). \tag{51}$$

The right conformable fractional derivative of order $0 < \alpha \leq 1$ terminating at b of $f : (-\infty, b] \to \mathbb{R}$ is defined by

$$\left({}_\alpha^b T f\right)(t) = -\lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon (b-t)^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{52}$$

If $\left({}_\alpha^b T f\right)(t)$ exists on (a, b), then

$$\left({}_\alpha^b T f\right)(b) = \lim_{t \to b-} \left({}_\alpha^b T f\right)(t). \tag{53}$$

Note that if f is differentiable then

$$\left(T_\alpha^a f\right)(t) = (t-a)^{1-\alpha} f'(t), \tag{54}$$

and

$$\left({}_\alpha^b T f\right)(t) = -(b-t)^{1-\alpha} f'(t). \tag{55}$$

In the higher order case we can generalize things as follows:

Definition 11 *([1]) Let $\alpha \in (n, n+1]$, and set $\beta = \alpha - n$. Then, the left conformable fractional derivative starting from a of a function $f : [a, \infty) \to \mathbb{R}$ of order α, where $f^{(n)}(t)$ exists, is defined by*

$$\left(\mathbf{T}_\alpha^a f\right)(t) = \left(T_\beta^a f^{(n)}\right)(t), \tag{56}$$

The right conformable fractional derivative of order α terminating at b of $f : (-\infty, b] \to \mathbb{R}$, where $f^{(n)}(t)$ exists, is defined by

$$\left({}_\alpha^b \mathbf{T} f\right)(t) = (-1)^{n+1} \left({}_\beta^b T f^{(n)}\right)(t). \tag{57}$$

If $\alpha = n+1$ then $\beta = 1$ and $\mathbf{T}_{n+1}^a f = f^{(n+1)}$.

If n is odd, then ${}_{n+1}^{b}\mathbf{T} f = -f^{(n+1)}$, and if n is even, then ${}_{n+1}^{b}\mathbf{T} f = f^{(n+1)}$.

When $n = 0$ (or $\alpha \in (0, 1]$), then $\beta = \alpha$, and (56), (57) collapse to (50), (52), respectively.

We need

Remark 12 *([5]) We notice the following: let* $\alpha \in (n, n+1]$ *and* $f \in C^{n+1}([a,b])$*,* $n \in \mathbb{N}$*. Then* $(\beta := \alpha - n,\ 0 < \beta \leq 1)$

$$\left(\mathbf{T}_{\alpha}^{a}(f)\right)(x) = \left(T_{\beta}^{\alpha} f^{(n)}\right)(x) = (x-a)^{1-\beta} f^{(n+1)}(x), \tag{58}$$

and

$$\left({}_{\alpha}^{b}\mathbf{T}(f)\right)(x) = (-1)^{n+1}\left({}_{\beta}^{b}T f^{(n)}\right)(x) =$$
$$(-1)^{n+1}(-1)(b-x)^{1-\beta} f^{(n+1)}(x) = (-1)^{n}(b-x)^{1-\beta} f^{(n+1)}(x). \tag{59}$$

Consequently we get that

$$\left(\mathbf{T}_{\alpha}^{a}(f)\right)(x),\ \left({}_{\alpha}^{b}\mathbf{T}(f)\right)(x) \in C([a,b]).$$

Furthermore it is obvious that

$$\left(\mathbf{T}_{\alpha}^{a}(f)\right)(a) = \left({}_{\alpha}^{b}\mathbf{T}(f)\right)(b) = 0, \tag{60}$$

when $0 < \beta < 1$*, i.e. when* $\alpha \in (n, n+1)$*.*

We mention the following Conformable fractional Iyengar type inequalities:

Theorem 13 *([8]) Let* $\alpha \in (n, n+1]$ *and* $f \in C^{n+1}([a,b])$*,* $n \in \mathbb{N}$*;* $\beta = \alpha - n$*. Then*

i)

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!}\left[f^{(k)}(a)(z-a)^{k+1} + (-1)^k f^{(k)}(b)(b-z)^{k+1}\right]\right| \leq$$
$$\frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{\infty,[a,b]}, \left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{\infty,[a,b]}\right\}}{\Gamma(\alpha+2)}\left[(z-a)^{\alpha+1} + (b-z)^{\alpha+1}\right], \tag{61}$$

$\forall\ z \in [a,b]$,

ii) at $z = \frac{a+b}{2}$*, the right hand side of (61) is minimized, and we get:*

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a) + (-1)^k f^{(k)}(b)\right]\right| \leq$$
$$\frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{\infty,[a,b]}, \left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{\infty,[a,b]}\right\}}{\Gamma(\alpha+2)}\frac{(b-a)^{\alpha+1}}{2^{\alpha}}, \tag{62}$$

iii) assuming $f^{(k)}(a) = f^{(k)}(b) = 0$*, for* $k = 0, 1, ..., n$*, we obtain*

$$\left|\int_a^b f(t)\,dt\right| \leq \frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{\infty,[a,b]}, \left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{\infty,[a,b]}\right\}}{\Gamma(\alpha+2)}\frac{(b-a)^{\alpha+1}}{2^{\alpha}}, \tag{63}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \left(\frac{b-a}{N}\right)^{k+1} \left[f^{(k)}(a)\, j^{k+1} + (-1)^k f^{(k)}(b)\,(N-j)^{k+1} \right] \right|$$

$$\leq \frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_\alpha^a (f) \right\|_{\infty,[a,b]}, \left\| {}_\alpha^b \mathbf{T}(f) \right\|_{\infty,[a,b]} \right\}}{\Gamma(\alpha+2)} \left(\frac{b-a}{N}\right)^{\alpha+1} \left[j^{\alpha+1} + (N-j)^{\alpha+1} \right], \tag{64}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n$, from (64) we get:

$$\left| \int_a^b f(t)\,dt - \left(\frac{b-a}{N}\right) \left[j f(a) + (N-j) f(b) \right] \right| \leq$$

$$\frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_\alpha^a (f) \right\|_{\infty,[a,b]}, \left\| {}_\alpha^b \mathbf{T}(f) \right\|_{\infty,[a,b]} \right\}}{\Gamma(\alpha+2)} \left(\frac{b-a}{N}\right)^{\alpha+1} \left[j^{\alpha+1} + (N-j)^{\alpha+1} \right], \tag{65}$$

$j = 0, 1, 2, ..., N$,

vi) when $N = 2$ and $j = 1$, (65) turns to

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right) (f(a) + f(b)) \right| \leq$$

$$\frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_\alpha^a (f) \right\|_{\infty,[a,b]}, \left\| {}_\alpha^b \mathbf{T}(f) \right\|_{\infty,[a,b]} \right\}}{\Gamma(\alpha+2)} \frac{(b-a)^{\alpha+1}}{2^\alpha}. \tag{66}$$

We mention L_p conformable fractional Iyengar inequalities:

Theorem 14 *([8]) Let $\alpha \in (n, n+1]$ and $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$; $\beta = \alpha - n$. Let also $p_1, p_2, p_3 > 1 : \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, with $\beta > \frac{1}{p_1} + \frac{1}{p_3}$. Then*

i)

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \left[f^{(k)}(a)\,(z-a)^{k+1} + (-1)^k f^{(k)}(b)\,(b-z)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \left\| \mathbf{T}_\alpha^a (f) \right\|_{p_3,[a,b]}, \left\| {}_\alpha^b \mathbf{T}(f) \right\|_{p_3,[a,b]} \right\}}{n!\,(p_1 n + 1)^{\frac{1}{p_1}} \left(p_2(\beta-1)+1\right)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_1} + \frac{1}{p_2}\right)} \tag{67}$$

$$\left[(z-a)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} + (b-z)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} \right],$$

$\forall\ z \in [a,b]$,

ii) at $z=\frac{a+b}{2}$, the right hand side of (67) is minimized, and we get:

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|\leq$$

$$\frac{\max\left\{\left\|\mathbf{T}_\alpha^a(f)\right\|_{p_3,[a,b]},\left\|{}_\alpha^b\mathbf{T}(f)\right\|_{p_3,[a,b]}\right\}}{n!\,(p_1n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}\frac{(b-a)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-\frac{1}{p_3}}},\tag{68}$$

iii) assuming $f^{(k)}(a)=f^{(k)}(b)=0$, for $k=0,1,...,n$, we obtain

$$\left|\int_a^b f(t)\,dt\right|\leq\frac{\max\left\{\left\|\mathbf{T}_\alpha^a(f)\right\|_{p_3,[a,b]},\left\|{}_\alpha^b\mathbf{T}(f)\right\|_{p_3,[a,b]}\right\}}{n!\,(p_1n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}\frac{(b-a)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-\frac{1}{p_3}}},\tag{69}$$

which is a sharp inequality,

iv) more generally, for $j=0,1,2,...,\ N\in\mathbb{N}$, it holds

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n}\frac{1}{(k+1)!}\left(\frac{b-a}{N}\right)^{k+1}\left[f^{(k)}(a)\,j^{k+1}+(-1)^k f^{(k)}(b)\,(N-j)^{k+1}\right]\right|$$

$$\leq\frac{\max\left\{\left\|\mathbf{T}_\alpha^a(f)\right\|_{p_3,[a,b]},\left\|{}_\alpha^b\mathbf{T}(f)\right\|_{p_3,[a,b]}\right\}}{n!\,(p_1n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}$$

$$\left(\frac{b-a}{N}\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\left[j^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}+(N-j)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\right],\tag{70}$$

v) if $f^{(k)}(a)=f^{(k)}(b)=0$, $k=1,...,n$, from (70) we get:

$$\left|\int_a^b f(t)\,dt-\left(\frac{b-a}{N}\right)\left[jf(a)+(N-j)\,f(b)\right]\right|\leq\tag{71}$$

$$\frac{\max\left\{\left\|\mathbf{T}_\alpha^a(f)\right\|_{p_3,[a,b]},\left\|{}_\alpha^b\mathbf{T}(f)\right\|_{p_3,[a,b]}\right\}}{n!\,(p_1n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}$$

$$\left(\frac{b-a}{N}\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\left[j^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}+(N-j)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\right],$$

for $j=0,1,2,...,N$,

vi) when $N=2$ and $j=1$, (71) turns to

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right|\leq$$

$$\frac{\max\left\{\left\|\mathbf{T}_\alpha^a(f)\right\|_{p_3,[a,b]},\left\|{}_\alpha^b\mathbf{T}(f)\right\|_{p_3,[a,b]}\right\}}{n!\,(p_1n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}\frac{(b-a)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-\frac{1}{p_3}}}.\tag{72}$$

We need

Remark 15 *We define the ball* $B(0,R)=\left\{x\in\mathbb{R}^N : |x|<R\right\}\subseteq\mathbb{R}^N$*,* $N\geq 2$*,* $R>0$*, and the sphere*

$$S^{N-1}:=\left\{x\in\mathbb{R}^N : |x|=1\right\},$$

where $|\cdot|$ *is the Euclidean norm. Let* $d\omega$ *be the element of surface measure on* S^{N-1} *and*

$$\omega_N=\int_{S^{N-1}}d\omega=\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$$

is the area of S^{N-1}*.*

For $x\in\mathbb{R}^N-\{0\}$ *we can write uniquely* $x=r\omega$*, where* $r=|x|>0$ *and* $\omega=\frac{x}{r}\in S^{N-1}$*,* $|\omega|=1$*. Note that* $\int_{B(0,R)}dy=\frac{\omega_N R^N}{N}$ *is the Lebesgue measure on the ball, that is the volume of* $B(0,R)$*, which exactly is* $Vol(B(0,R))=\frac{\pi^{\frac{N}{2}}R^N}{\Gamma\left(\frac{N}{2}+1\right)}$*.*

Following [11, pp. 149-150, exercise 6], and [12, pp. 87-88, Theorem 5.2.2] we can write for $F:\overline{B(0,R)}\to\mathbb{R}$ *a Lebesgue integrable function that*

$$\int_{B(0,R)}F(x)\,dx=\int_{S^{N-1}}\left(\int_0^R F(r\omega)\,r^{N-1}dr\right)d\omega, \tag{73}$$

and we use this formula a lot.

Typically here the function $f:\overline{B(0,R)}\to\mathbb{R}$ *is radial; that is, there exists a function* g *such that* $f(x)=g(r)$*, where* $r=|x|$*,* $r\in[0,R]$*,* $\forall\ x\in\overline{B(0,R)}$*.*

We need

Remark 16 *Let the spherical shell* $A:=B(0,R_2)-\overline{B(0,R_1)}$*,* $0<R_1<R_2$*,* $A\subseteq\mathbb{R}^N$*,* $N\geq 2$*,* $x\in\overline{A}$*. Consider that* $f:\overline{A}\to\mathbb{R}$ *is radial; that is, there exists* g *such that* $f(x)=g(r)$*,* $r=|x|$*,* $r\in[R_1,R_2]$*,* $\forall\ x\in\overline{A}$*. Here* x *can be written uniquely as* $x=r\omega$*, where* $r=|x|>0$ *and* $\omega=\frac{x}{r}\in S^{N-1}$*,* $|\omega|=1$*, see ([11], p. 149-150 and [2], p. 421), furthermore for* $F:\overline{A}\to\mathbb{R}$ *a Lebesgue integrable function we have that*

$$\int_A F(x)\,dx=\int_{S^{N-1}}\left(\int_{R_1}^{R_2}F(r\omega)\,r^{N-1}dr\right)d\omega. \tag{74}$$

Here

$$Vol(A)=\frac{\omega_N\left(R_2^N-R_1^N\right)}{N}=\frac{\pi^{\frac{N}{2}}\left(R_2^N-R_1^N\right)}{\Gamma\left(\frac{N}{2}+1\right)}. \tag{75}$$

In this article we derive multivariate fractional Iyengar type inequalities on the shell and ball of $\mathbb{R}^N$, $N\geq 2$, for radial function. Our following results are based on the presented background results.

2 Main Results

In the rest of this article we consider the functions:

i) $f : \overline{A} \to \mathbb{R}$ which is radial, i.e. there exists g such that $f(x) = g(r)$, $r = |x|$, $r \in [R_1, R_2]$, $\forall\, x \in \overline{A}$; where A is the spherical shell $A := B(0, R_2) - \overline{B(0, R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$, also

ii) $f : \overline{B(0, R)} \to \mathbb{R}$ which is radial, i.e. there exists g such that $f(x) = g(r)$, where $r = |x|$, $r \in [0, R]$, $\forall\, x \in \overline{B(0, R)}$; where $B(0, R)$ is the ball, $B(0, R) \subseteq \mathbb{R}^N$, $N \geq 2$, $R > 0$.

We will employ the related function $h(s) := g(s)\, s^{N-1}$, where $s \in [R_1, R_2]$ or $s \in [0, R]$.

We present the following multivariate Caputo fractional Iyengar type inequalities:

Theorem 17 *Let the radial $f : \overline{A} \to \mathbb{R}$. Let $\nu > 0$, $n = \lceil \nu \rceil$, and $h \in AC^n([R_1, R_2])$ (i.e. $h^{(n-1)}$ is absolutely continuous on $[R_1, R_2]$). We assume that $D_{*R_1}^{\nu} h$, $D_{R_2-}^{\nu} h \in L_{\infty}([R_1, R_2])$. Then*

i)

$$\left| \int_A f(y)\, dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(R_1)(t - R_1)^{k+1} + (-1)^k h^{(k)}(R_2)(R_2 - t)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{*R_1}^{\nu} h \right\|_{L_{\infty}([R_1,R_2])}, \left\| D_{R_2-}^{\nu} h \right\|_{L_{\infty}([R_1,R_2])} \right\}}{\Gamma(\nu + 2)} \cdot \left[(t - R_1)^{\nu+1} + (R_2 - t)^{\nu+1} \right], \tag{76}$$

$\forall\, t \in [R_1, R_2]$,

ii) at $t = \frac{R_1 + R_2}{2}$, the right hand side of (76) is minimized, and we get:

$$\left| \int_A f(y)\, dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(R_2 - R_1)^{k+1}}{2^k} \left[h^{(k)}(R_1) + (-1)^k h^{(k)}(R_2) \right] \right\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{*R_1}^{\nu} h \right\|_{L_{\infty}([R_1,R_2])}, \left\| D_{R_2-}^{\nu} h \right\|_{L_{\infty}([R_1,R_2])} \right\}}{\Gamma(\nu + 2)} \frac{(R_2 - R_1)^{\nu+1}}{2^{\nu-1}}, \tag{77}$$

iii) if $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$, for all $k = 0, 1, ..., n-1$, we obtain

$$\left| \int_A f(y)\, dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot$$

$$\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_\infty([R_1,R_2])},\left\|D_{R_2-}^{\nu}h\right\|_{L_\infty([R_1,R_2])}\right\}\frac{(R_2-R_1)^{\nu+1}}{\Gamma(\nu+2)\,2^{\nu-1}}, \tag{78}$$

which is a sharp inequality,

iv) more generally, for $j=0,1,2,...,N\in\mathbb{N}$, it holds

$$\left|\int_A f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{R_2-R_1}{N}\right)^{k+1}\right.\right.$$

$$\left.\left.\left[j^{k+1}h^{(k)}(R_1)+(-1)^k(N-j)^{k+1}h^{(k)}(R_2)\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_\infty([R_1,R_2])},\left\|D_{R_2-}^{\nu}h\right\|_{L_\infty([R_1,R_2])}\right\}}{\Gamma(\nu+2)}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\nu+1}\left[j^{\nu+1}+(N-j)^{\nu+1}\right], \tag{79}$$

v) if $h^{(k)}(R_1)=h^{(k)}(R_2)=0$, $k=1,...,n-1$, from (79) we get:

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{N}\right)\left[jh(R_1)+(N-j)\,h(R_2)\right]\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_\infty([R_1,R_2])},\left\|D_{R_2-}^{\nu}h\right\|_{L_\infty([R_1,R_2])}\right\}}{\Gamma(\nu+2)}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\nu+1}\left[j^{\nu+1}+(N-j)^{\nu+1}\right], \tag{80}$$

$j=0,1,2,...,N,$

vi) when $N=2$ and $j=1$, (80) turns to

$$\left|\int_A f(y)\,dy-(R_2-R_1)\left(h(R_1)+h(R_2)\right)\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_\infty([R_1,R_2])},\left\|D_{R_2-}^{\nu}h\right\|_{L_\infty([R_1,R_2])}\right\}}{\Gamma(\nu+2)}\frac{(R_2-R_1)^{\nu+1}}{2^{\nu-1}}, \tag{81}$$

vii) when $0<\nu\leq 1$, inequality (81) is again valid without any boundary conditions.

Proof. By Theorem 4 and (74). See in the 3. Appendix the general proving method in this article. ■

We give

Corollary 18 *(to Theorem 17) Let the radial* $f : \overline{B(0,R)} \to \mathbb{R}$*. Let* $\nu > 0$*,* $n = \lceil \nu \rceil$*, and* $h \in AC^n([0,R])$*. We assume that* $D_{*0}^{\nu} h$*,* $D_{R-}^{\nu} h \in L_{\infty}([0,R])$*. Then*

i)

$$\left| \int_{B(0,R)} f(y)\, dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(0)\, t^{k+1} + \right.\right.\right.$$

$$\left.\left.\left. (-1)^k h^{(k)}(R)(R-t)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{*0}^{\nu} h \right\|_{L_{\infty}([0,R])}, \left\| D_{R-}^{\nu} h \right\|_{L_{\infty}([0,R])} \right\}}{\Gamma(\nu+2)} \cdot$$

$$\left[t^{\nu+1} + (R-t)^{\nu+1} \right], \tag{82}$$

$\forall\ t \in [0,R]$,

ii) at $t = \frac{R}{2}$*, the right hand side of (82) is minimized, and we get:*

$$\left| \int_{B(0,R)} f(y)\, dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{R^{k+1}}{2^k} \right.\right.$$

$$\left.\left. \left[h^{(k)}(0) + (-1)^k h^{(k)}(R) \right] \right\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{*0}^{\nu} h \right\|_{L_{\infty}([0,R])}, \left\| D_{R-}^{\nu} h \right\|_{L_{\infty}([0,R])} \right\}}{\Gamma(\nu+2)} \frac{R^{\nu+1}}{2^{\nu-1}}, \tag{83}$$

iii) if $h^{(k)}(0) = h^{(k)}(R) = 0$*, for all* $k = 0, 1, ..., n-1$*, we obtain*

$$\left| \int_{B(0,R)} f(y)\, dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot$$

$$\max\left\{ \left\| D_{*0}^{\nu} h \right\|_{L_{\infty}([0,R])}, \left\| D_{R-}^{\nu} h \right\|_{L_{\infty}([0,R])} \right\} \frac{R^{\nu+1}}{\Gamma(\nu+2)\, 2^{\nu-1}}, \tag{84}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$*, it holds*

$$\left| \int_{B(0,R)} f(y)\, dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{R}{N} \right)^{k+1} \right.\right.$$

$$\left.\left. \left[j^{k+1} h^{(k)}(0) + (-1)^k (N-j)^{k+1} h^{(k)}(R) \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_{\infty}([0,R])},\left\|D_{R-}^{\nu}h\right\|_{L_{\infty}([0,R])}\right\}}{\Gamma(\nu+2)}$$

$$\left(\frac{R}{N}\right)^{\nu+1}\left[j^{\nu+1}+(N-j)^{\nu+1}\right], \tag{85}$$

v) if $h^{(k)}(0)=h^{(k)}(R)=0$, $k=1,...,n-1$, *from (85) we get:*

$$\left|\int_{B(0,R)}f(y)\,dy-\left(\frac{R}{N}\right)(N-j)\,h(R)\,\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_{\infty}([0,R])},\left\|D_{R-}^{\nu}h\right\|_{L_{\infty}([0,R])}\right\}}{\Gamma(\nu+2)}$$

$$\left(\frac{R}{N}\right)^{\nu+1}\left[j^{\nu+1}+(N-j)^{\nu+1}\right], \tag{86}$$

$j=0,1,2,...,N$,

vi) when $N=2$ *and* $j=1$, *(86) turns to*

$$\left|\int_{B(0,R)}f(y)\,dy-Rh(R)\,\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_{\infty}([0,R])},\left\|D_{R-}^{\nu}h\right\|_{L_{\infty}([0,R])}\right\}}{\Gamma(\nu+2)}\frac{R^{\nu+1}}{2^{\nu-1}}, \tag{87}$$

vii) when $0<\nu\leq 1$, *inequality (87) is again valid without any boundary conditions.*

Proof. Based on Theorem 17, just set there $R_1=0$, $R_2=R$, the assumptions now are on $B(0,R)$, and use (73). ■

We continue with

Theorem 19 *Let the radial* $f:\overline{A}\to\mathbb{R}$. *Let* $\nu\geq 1$, $n=\lceil\nu\rceil$, *and* $h\in AC^{n}([R_1,R_2])$ *(i.e.* $h^{(n-1)}$ *is absolutely continuous on* $[R_1,R_2]$*). We assume that* $D_{*R_1}^{\nu}h$, $D_{R_2-}^{\nu}h\in L_1([R_1,R_2])$. *Then*

i)

$$\left|\int_{A}f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[h^{(k)}(R_1)(t-R_1)^{k+1}+\right.\right.\right.$$

$$\left.\left.\left.(-1)^{k}h^{(k)}(R_2)(R_2-t)^{k+1}\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_1([R_1,R_2])},\left\|D_{R_2-}^{\nu}h\right\|_{L_1([R_1,R_2])}\right\}}{\Gamma(\nu+1)}.$$

$$\left[(t-R_1)^{\nu}+(R_2-t)^{\nu}\right], \tag{88}$$

$\forall\ t\in[R_1,R_2]$,

ii) when $\nu=1$*, from (88), we have*

$$\left|\int_A f(y)\,dy-\left[h(R_1)(t-R_1)+h(R_2)(R_2-t)\right]\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\|h'\right\|_{L_1([R_1,R_2])}(R_2-R_1), \tag{89}$$

$\forall\ t\in[R_1,R_2]$,

iii) from (89), we obtain ($\nu=1$ *case)*

$$\left|\int_A f(y)\,dy-(R_2-R_1)(h(R_1)+h(R_2))\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\|h'\right\|_{L_1([R_1,R_2])}(R_2-R_1), \tag{90}$$

iv) at $t=\frac{R_1+R_2}{2}$, $\nu>1$*, the right hand side of (88) is minimized, and we get:*

$$\left|\int_A f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(R_2-R_1)^{k+1}}{2^k}\right.\right.$$

$$\left.\left.\left[h^{(k)}(R_1)+(-1)^k h^{(k)}(R_2)\right]\right\}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_1([R_1,R_2])},\left\|D_{R_2-}^{\nu}h\right\|_{L_1([R_1,R_2])}\right\}}{\Gamma(\nu+1)}\frac{(R_2-R_1)^{\nu}}{2^{\nu-2}}, \tag{91}$$

v) if $h^{(k)}(R_1)=h^{(k)}(R_2)=0$*, for all* $k=0,1,...,n-1$, $\nu>1$*, from (91) we obtain*

$$\left|\int_A f(y)\,dy\right|\leq\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\cdot$$

$$\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_1([R_1,R_2])},\left\|D_{R_2-}^{\nu}h\right\|_{L_1([R_1,R_2])}\right\}\frac{(R_2-R_1)^{\nu}}{\Gamma(\nu+1)\,2^{\nu-2}}, \tag{92}$$

which is a sharp inequality,

vi) more generally, for $j=0,1,2,...,N\in\mathbb{N}$*, it holds*

$$\left|\int_A f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{R_2-R_1}{N}\right)^{k+1}\right.\right.$$

$$\left.\left.\left[j^{k+1}h^{(k)}(R_1)+(-1)^k(N-j)^{k+1}h^{(k)}(R_2)\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_1([R_1,R_2])},\left\|D_{R_2-}^{\nu}h\right\|_{L_1([R_1,R_2])}\right\}}{\Gamma(\nu+1)}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{93}$$

vii) if $h^{(k)}(R_1)=h^{(k)}(R_2)=0$, $k=1,...,n-1$, *from (93) we get:*

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{N}\right)\left[jh(R_1)+(N-j)h(R_2)\right]\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_1([R_1,R_2])},\left\|D_{R_2-}^{\nu}h\right\|_{L_1([R_1,R_2])}\right\}}{\Gamma(\nu+1)}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{94}$$

$j=0,1,2,...,N,$

viii) when $N=2$ *and* $j=1$, *(94) turns to*

$$\left|\int_A f(y)\,dy-(R_2-R_1)\left(h(R_1)+h(R_2)\right)\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_1([R_1,R_2])},\left\|D_{R_2-}^{\nu}h\right\|_{L_1([R_1,R_2])}\right\}}{\Gamma(\nu+1)}\frac{(R_2-R_1)^{\nu}}{2^{\nu-2}}. \tag{95}$$

Proof. By Theorem 5 and (74). See in the 3. Appendix the general proving method in this article. ■

We give

Corollary 20 *(to Theorem 19) Let the radial* $f:\overline{B(0,R)}\to\mathbb{R}$. *Let* $\nu\geq 1$, $n=\lceil\nu\rceil$, *and* $h\in AC^n([0,R])$. *We assume that* $D_{*0}^{\nu}h$, $D_{R-}^{\nu}h\in L_1([0,R])$. *Then*

i)

$$\left|\int_{B(0,R)} f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[h^{(k)}(0)\,t^{k+1}+\right.\right.\right.$$

$$\left.\left.\left.(-1)^k h^{(k)}(R)(R-t)^{k+1}\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_1([0,R])},\left\|D_{R-}^{\nu}h\right\|_{L_1([0,R])}\right\}}{\Gamma(\nu+1)}\cdot$$

$$\left[t^{\nu}+(R-t)^{\nu}\right], \tag{96}$$

$\forall\ t\in[0,R],$

ii) when $\nu = 1$*, from (96), we have*

$$\left| \int_{B(0,R)} f(y)\,dy - h(R)(R-t)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\| h' \right\|_{L_1([0,R])} R, \tag{97}$$

$\forall\ t \in [0,R]$,

iii) from (97), we obtain ($\nu = 1$ *case)*

$$\left| \int_{B(0,R)} f(y)\,dy - Rh(R)\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\| h' \right\|_{L_1([0,R])} R, \tag{98}$$

iv) at $t = \frac{R}{2}$*,* $\nu > 1$*, the right hand side of (96) is minimized, and we get:*

$$\left| \int_{B(0,R)} f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{R^{k+1}}{2^k} \right. \right.$$

$$\left. \left. \left[h^{(k)}(0) + (-1)^k h^{(k)}(R) \right] \right\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{*0}^{\nu} h \right\|_{L_1([0,R])}, \left\| D_{R-}^{\nu} h \right\|_{L_1([0,R])} \right\}}{\Gamma(\nu+1)} \frac{R^{\nu}}{2^{\nu-2}}, \tag{99}$$

v) if $h^{(k)}(0) = h^{(k)}(R) = 0$*, for all* $k = 0, 1, ..., n-1$*, from (99) we obtain*

$$\left| \int_{B(0,R)} f(y)\,dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot$$

$$\max\left\{ \left\| D_{*0}^{\nu} h \right\|_{L_1([0,R])}, \left\| D_{R-}^{\nu} h \right\|_{L_1([0,R])} \right\} \frac{R^{\nu}}{\Gamma(\nu+1)\, 2^{\nu-2}}, \tag{100}$$

which is a sharp inequality.

vi) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$*, it holds*

$$\left| \int_{B(0,R)} f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{R}{N} \right)^{k+1} \right. \right.$$

$$\left. \left. \left[j^{k+1} h^{(k)}(0) + (-1)^k (N-j)^{k+1} h^{(k)}(R) \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_1([0,R])}, \left\|D_{R-}^{\nu}h\right\|_{L_1([0,R])}\right\}}{\Gamma(\nu+1)}$$
$$\left(\frac{R}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{101}$$

vii) if $h^{(k)}(0) = h^{(k)}(R) = 0$, $k = 1, ..., n-1$, *from (101) we get:*

$$\left|\int_{B(0,R)} f(y)\,dy - \left(\frac{R}{N}\right)(N-j)\,h(R)\,\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$
$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_1([0,R])}, \left\|D_{R-}^{\nu}h\right\|_{L_1([0,R])}\right\}}{\Gamma(\nu+1)}$$
$$\left(\frac{R}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{102}$$

$j = 0, 1, 2, ..., N,$

viii) when $N = 2$ *and* $j = 1$, *(102) turns to*

$$\left|\int_{B(0,R)} f(y)\,dy - Rh(R)\,\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \leq$$
$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_1([0,R])}, \left\|D_{R-}^{\nu}h\right\|_{L_1([0,R])}\right\}}{\Gamma(\nu+1)} \frac{R^{\nu}}{2^{\nu-2}}. \tag{103}$$

Proof. Based on Theorem 19, just set there $R_1 = 0$, $R_2 = R$, the assumptions now are on $B(0,R)$, and use (73). ∎

We continue with

Theorem 21 *Let the radial* $f : \overline{A} \to \mathbb{R}$, *and* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu > \frac{1}{q}$. *Let* $n = \lceil \nu \rceil$, *and* $h \in AC^n([R_1, R_2])$ *(i.e.* $h^{(n-1)}$ *is absolutely continuous on* $[R_1, R_2]$*). We assume that* $D_{*R_1}^{\nu}h$, $D_{R_2-}^{\nu}h \in L_q([R_1, R_2])$. *Then*

i)

$$\left|\int_A f(y)\,dy - \left\{\sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[h^{(k)}(R_1)(t-R_1)^{k+1} + \right.\right.\right.$$
$$\left.\left.\left.(-1)^k h^{(k)}(R_2)(R_2-t)^{k+1}\right]\right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \leq$$
$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_q([R_1,R_2])}, \left\|D_{R_2-}^{\nu}h\right\|_{L_q([R_1,R_2])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}.$$
$$\left[(t-R_1)^{\nu+\frac{1}{p}} + (R_2-t)^{\nu+\frac{1}{p}}\right], \tag{104}$$

$\forall\ t \in [R_1, R_2]$,

ii) at $t = \frac{R_1+R_2}{2}$, *the right hand side of (104) is minimized, and we get:*

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(R_2-R_1)^{k+1}}{2^k} \right.\right.$$

$$\left.\left. \left[h^{(k)}(R_1) + (-1)^k h^{(k)}(R_2) \right] \right\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{*R_1}^{\nu} h \right\|_{L_q([R_1,R_2])}, \left\| D_{R_2-}^{\nu} h \right\|_{L_q([R_1,R_2])} \right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(R_2-R_1)^{\nu+\frac{1}{p}}}{2^{\nu-1-\frac{1}{q}}}, \quad (105)$$

iii) if $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$, *for all* $k = 0, 1, ..., n-1$, *we obtain*

$$\left| \int_A f(y)\,dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right) 2^{\nu-1-\frac{1}{q}}} \cdot$$

$$\max\left\{ \left\| D_{*R_1}^{\nu} h \right\|_{L_q([R_1,R_2])}, \left\| D_{R_2-}^{\nu} h \right\|_{L_q([R_1,R_2])} \right\} \cdot$$

$$\frac{(R_2-R_1)^{\nu+\frac{1}{p}}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}, \quad (106)$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds*

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{R_2-R_1}{N} \right)^{k+1} \right.\right.$$

$$\left.\left. \left[j^{k+1} h^{(k)}(R_1) + (-1)^k (N-j)^{k+1} h^{(k)}(R_2) \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{*R_1}^{\nu} h \right\|_{L_q([R_1,R_2])}, \left\| D_{R_2-}^{\nu} h \right\|_{L_q([R_1,R_2])} \right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}$$

$$\left(\frac{R_2-R_1}{N} \right)^{\nu+\frac{1}{p}} \left[j^{\nu+\frac{1}{p}} + (N-j)^{\nu+\frac{1}{p}} \right], \quad (107)$$

v) if $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$, $k = 1, ..., n-1$, *from (107) we get:*

$$\left| \int_A f(y)\,dy - \left(\frac{R_2-R_1}{N} \right) [j h(R_1) + (N-j) h(R_2)] \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_q([R_1,R_2])}, \left\|D_{R_2-}^{\nu}h\right\|_{L_q([R_1,R_2])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\nu+\frac{1}{p}} \left[j^{\nu+\frac{1}{p}} + (N-j)^{\nu+\frac{1}{p}}\right], \tag{108}$$

$j = 0, 1, 2, ..., N,$

vi) when $N = 2$ *and* $j = 1$, *(108) turns to*

$$\left|\int_A f(y)\,dy - (R_2 - R_1)(h(R_1) + h(R_2)) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{*R_1}^{\nu}h\right\|_{L_q([R_1,R_2])}, \left\|D_{R_2-}^{\nu}h\right\|_{L_q([R_1,R_2])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(R_2-R_1)^{\nu+\frac{1}{p}}}{2^{\nu-1-\frac{1}{q}}}, \tag{109}$$

vii) when $1/q < \nu \leq 1$, *inequality (109) is again valid without any boundary conditions.*

Proof. By Theorem 6 and (74). See in the 3. Appendix the general proving method in this article. ■

We give

Corollary 22 *(to Theorem 21) Let the radial* $f : \overline{B(0,R)} \to \mathbb{R}$. *Let* $\nu > 0$, $n = \lceil \nu \rceil$, *and* $h \in AC^n([0,R])$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu > \frac{1}{q}$. *We assume that* $D_{*0}^{\nu}h$, $D_{R-}^{\nu}h \in L_q([0,R])$. *Then*

i)

$$\left|\int_{B(0,R)} f(y)\,dy - \left\{\sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[h^{(k)}(0)\,t^{k+1} + \right.\right.\right.$$

$$\left.\left.\left.(-1)^k h^{(k)}(R)(R-t)^{k+1}\right]\right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_q([0,R])}, \left\|D_{R-}^{\nu}h\right\|_{L_q([0,R])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \cdot \tag{110}$$

$$\left[t^{\nu+\frac{1}{p}} + (R-t)^{\nu+\frac{1}{p}}\right],$$

$\forall\ t \in [0, R]$,

ii) at $t = \frac{R}{2}$, *the right hand side of (110) is minimized, and we get:*

$$\left|\int_{B(0,R)} f(y)\,dy - \left\{\sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{R^{k+1}}{2^k}\right.\right.$$

$$\left[h^{(k)}(0)+(-1)^k h^{(k)}(R)\right]\Bigg\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\Bigg| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_q([0,R])}, \left\|D_{R-}^{\nu}h\right\|_{L_q([0,R])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{R^{\nu+\frac{1}{p}}}{2^{\nu-1-\frac{1}{q}}}, \tag{111}$$

iii) if $h^{(k)}(0)=h^{(k)}(R)=0$*, for all* $k=0,1,...,n-1$*, we obtain*

$$\left|\int_{B(0,R)} f(y)\,dy\right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right) 2^{\nu-1-\frac{1}{q}}}\cdot$$

$$\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_q([0,R])}, \left\|D_{R-}^{\nu}h\right\|_{L_q([0,R])}\right\} \frac{R^{\nu+\frac{1}{p}}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}, \tag{112}$$

which is a sharp inequality.

iv) more generally, for $j=0,1,2,...,N\in\mathbb{N}$*, it holds*

$$\Bigg|\int_{B(0,R)} f(y)\,dy - \left\{\sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left(\frac{R}{N}\right)^{k+1}\right.$$

$$\left.\left[j^{k+1}h^{(k)}(0)+(-1)^k (N-j)^{k+1} h^{(k)}(R)\right]\right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\Bigg| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_q([0,R])}, \left\|D_{R-}^{\nu}h\right\|_{L_q([0,R])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}$$

$$\left(\frac{R}{N}\right)^{\nu+\frac{1}{p}} \left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{113}$$

v) if $h^{(k)}(0)=h^{(k)}(R)=0$*,* $k=1,...,n-1$*, from (113) we get:*

$$\left|\int_{B(0,R)} f(y)\,dy - \left(\frac{R}{N}\right)(N-j)\,h(R)\,\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{*0}^{\nu}h\right\|_{L_q([0,R])}, \left\|D_{R-}^{\nu}h\right\|_{L_q([0,R])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}$$

$$\left(\frac{R}{N}\right)^{\nu+\frac{1}{p}} \left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{114}$$

$j=0,1,2,...,N$,

vi) when $N = 2$ and $j = 1$, (114) turns to

$$\left| \int_{B(0,R)} f(y)\, dy - Rh(R) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{*0}^{\nu} h \right\|_{L_q([0,R])}, \left\| D_{R-}^{\nu} h \right\|_{L_q([0,R])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{R^{\nu+\frac{1}{p}}}{2^{\nu-1-\frac{1}{q}}}, \quad (115)$$

vii) when $1/q < \nu \leq 1$, inequality (115) is again valid without any boundary conditions.

Proof. Based on Theorem 21, just set there $R_1 = 0$, $R_2 = R$, the assumptions now are on $B(0,R)$, and use (73). ■

We continue with multivariate Canavati type fractional Iyengar type inequalities:

Theorem 23 *Let the radial $f : \overline{A} \to \mathbb{R}$. Let $\nu \geq 1$, $n = [\nu]$, and $h \in C_{R_1+}^{\nu}([R_1, R_2]) \cap C_{R_2-}^{\nu}([R_1, R_2])$. Then*

i)

$$\left| \int_A f(y)\, dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(R_1)(t-R_1)^{k+1} + \right.\right.\right.$$

$$\left.\left.\left. (-1)^k h^{(k)}(R_2)(R_2-t)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{R_1+}^{\nu} h \right\|_{\infty,[R_1,R_2]}, \left\| \overline{D}_{R_2-}^{\nu} h \right\|_{\infty,[R_1,R_2]} \right\}}{\Gamma(\nu+2)} \cdot$$

$$\left[(t-R_1)^{\nu+1} + (R_2-t)^{\nu+1} \right], \quad (116)$$

$\forall\, t \in [R_1, R_2]$,

ii) at $t = \frac{R_1+R_2}{2}$, the right hand side of (116) is minimized, and we get:

$$\left| \int_A f(y)\, dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(R_2-R_1)^{k+1}}{2^k} \right.\right.$$

$$\left.\left. \left[h^{(k)}(R_1) + (-1)^k h^{(k)}(R_2) \right] \right\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{R_1+}^{\nu} h \right\|_{\infty,[R_1,R_2]}, \left\| \overline{D}_{R_2-}^{\nu} h \right\|_{\infty,[R_1,R_2]} \right\}}{\Gamma(\nu+2)} \frac{(R_2-R_1)^{\nu+1}}{2^{\nu-1}}, \quad (117)$$

iii) if $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$, for all $k = 0, 1, ..., n-1$, we obtain

$$\left|\int_A f(y)\,dy\right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}.$$

$$\max\left\{\left\|D^{\nu}_{R_1+}h\right\|_{\infty,[R_1,R_2]}, \left\|\overline{D}^{\nu}_{R_2-}h\right\|_{\infty,[R_1,R_2]}\right\} \frac{(R_2-R_1)^{\nu+1}}{\Gamma(\nu+2)\,2^{\nu-1}}, \tag{118}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, it holds

$$\left|\int_A f(y)\,dy - \left\{\sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left(\frac{R_2-R_1}{N}\right)^{k+1}\right.\right.$$

$$\left.\left.\left[j^{k+1}h^{(k)}(R_1) + (-1)^k (N-j)^{k+1} h^{(k)}(R_2)\right]\right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D^{\nu}_{R_1+}h\right\|_{\infty,[R_1,R_2]}, \left\|\overline{D}^{\nu}_{R_2-}h\right\|_{\infty,[R_1,R_2]}\right\}}{\Gamma(\nu+2)}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\nu+1}\left[j^{\nu+1} + (N-j)^{\nu+1}\right], \tag{119}$$

v) if $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$, $k = 1, ..., n-1$, from (119) we get:

$$\left|\int_A f(y)\,dy - \left(\frac{R_2-R_1}{N}\right)\left[jh(R_1) + (N-j)\,h(R_2)\right] \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D^{\nu}_{R_1+}h\right\|_{\infty,[R_1,R_2]}, \left\|\overline{D}^{\nu}_{R_2-}h\right\|_{\infty,[R_1,R_2]}\right\}}{\Gamma(\nu+2)}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\nu+1}\left[j^{\nu+1} + (N-j)^{\nu+1}\right], \tag{120}$$

$j = 0, 1, 2, ..., N$,

vi) when $N = 2$ and $j = 1$, (120) turns to

$$\left|\int_A f(y)\,dy - (R_2-R_1)\left(h(R_1) + h(R_2)\right) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D^{\nu}_{R_1+}h\right\|_{\infty,[R_1,R_2]}, \left\|\overline{D}^{\nu}_{R_2-}h\right\|_{\infty,[R_1,R_2]}\right\}}{\Gamma(\nu+2)} \frac{(R_2-R_1)^{\nu+1}}{2^{\nu-1}}. \tag{121}$$

Proof. By Theorem 7 and (74). See in the 3. Appendix the general proving method in this article. ■

We give

Corollary 24 *(to Theorem 23) Let the radial* $f : \overline{B(0,R)} \to \mathbb{R}$. *Let* $\nu \geq 1$, $n = [\nu]$, *and* $h \in C^{\nu}_{0+}([0,R]) \cap C^{\nu}_{R-}([0,R])$. *Then*

i)

$$\left| \int_{B(0,R)} f(y)\, dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(0)\, t^{k+1} + \right.\right.\right.$$

$$\left.\left.\left. (-1)^k h^{(k)}(R)(R-t)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D^{\nu}_{0+}h \right\|_{\infty,[0,R]}, \left\| \overline{D}^{\nu}_{R-}h \right\|_{\infty,[0,R]} \right\}}{\Gamma(\nu+2)} \cdot$$

$$\left[t^{\nu+1} + (R-t)^{\nu+1} \right], \tag{122}$$

$\forall\ t \in [0,R]$,

ii) at $t = \frac{R}{2}$, *the right hand side of (122) is minimized, and we get:*

$$\left| \int_{B(0,R)} f(y)\, dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{R^{k+1}}{2^k} \right.\right.$$

$$\left.\left. \left[h^{(k)}(0) + (-1)^k h^{(k)}(R) \right] \right\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D^{\nu}_{0+}h \right\|_{\infty,[0,R]}, \left\| \overline{D}^{\nu}_{R-}h \right\|_{\infty,[0,R]} \right\}}{\Gamma(\nu+2)} \frac{R^{\nu+1}}{2^{\nu-1}}, \tag{123}$$

iii) if $h^{(k)}(0) = h^{(k)}(R) = 0$, *for all* $k = 0, 1, ..., n-1$, *we obtain*

$$\left| \int_{B(0,R)} f(y)\, dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot$$

$$\max\left\{ \left\| D^{\nu}_{0+}h \right\|_{\infty,[0,R]}, \left\| \overline{D}^{\nu}_{R-}h \right\|_{\infty,[0,R]} \right\} \frac{R^{\nu+1}}{\Gamma(\nu+2)\, 2^{\nu-1}}, \tag{124}$$

which is a sharp inequality.

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds*

$$\left| \int_{B(0,R)} f(y)\, dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{R}{N} \right)^{k+1} \right.\right.$$

$$\left[j^{k+1}h^{(k)}(0)+(-1)^{k}(N-j)^{k+1}h^{(k)}(R)\right]\Bigg\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\Bigg|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{\infty,[0,R]},\left\|\overline{D}_{R-}^{\nu}h\right\|_{\infty,[0,R]}\right\}}{\Gamma(\nu+2)}$$

$$\left(\frac{R}{N}\right)^{\nu+1}\left[j^{\nu+1}+(N-j)^{\nu+1}\right], \tag{125}$$

v) if $h^{(k)}(0)=h^{(k)}(R)=0$, $k=1,...,n-1$, *from (125) we get:*

$$\left|\int_{B(0,R)}f(y)\,dy-\left(\frac{R}{N}\right)(N-j)\,h(R)\,\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{\infty,[0,R]},\left\|\overline{D}_{R-}^{\nu}h\right\|_{\infty,[0,R]}\right\}}{\Gamma(\nu+2)} \tag{126}$$

$$\left(\frac{R}{N}\right)^{\nu+1}\left[j^{\nu+1}+(N-j)^{\nu+1}\right],$$

$j=0,1,2,...,N$,

vi) when $N=2$ *and* $j=1$, *(126) turns to*

$$\left|\int_{B(0,R)}f(y)\,dy-Rh(R)\,\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{\infty,[0,R]},\left\|\overline{D}_{R-}^{\nu}h\right\|_{\infty,[0,R]}\right\}}{\Gamma(\nu+2)}\frac{R^{\nu+1}}{2^{\nu-1}}. \tag{127}$$

Proof. Based on Theorem 23, just set there $R_1=0$, $R_2=R$, the assumptions now are on $B(0,R)$, and use (73). ■

We continue with

Theorem 25 *Let the radial* $f:\overline{A}\to\mathbb{R}$. *Let* $\nu\geq 1$, $n=[\nu]$, *and* $h\in C_{R_1+}^{\nu}([R_1,R_2])\cap C_{R_2-}^{\nu}([R_1,R_2])$. *Then*

i)

$$\left|\int_{A}f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[h^{(k)}(R_1)(t-R_1)^{k+1}+\right.\right.\right.$$

$$\left.\left.\left.(-1)^{k}h^{(k)}(R_2)(R_2-t)^{k+1}\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{R_1+}^{\nu}h\right\|_{L_1([R_1,R_2])},\left\|\overline{D}_{R_2-}^{\nu}h\right\|_{L_1([R_1,R_2])}\right\}}{\Gamma(\nu+1)}. \tag{128}$$

$$\left[(t-R_1)^{\nu}+(R_2-t)^{\nu}\right],$$

$\forall\ t\in[R_1,R_2]$,

ii) when $\nu=1$, from (128), we have

$$\left|\int_A f(y)\,dy-\left[h(R_1)(t-R_1)+h(R_2)(R_2-t)\right]\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\|h'\right\|_{L_1([R_1,R_2])}(R_2-R_1), \tag{129}$$

$\forall\ t\in[R_1,R_2]$,

iii) from (129), we obtain ($\nu=1$ case)

$$\left|\int_A f(y)\,dy-(R_2-R_1)(h(R_1)+h(R_2))\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\|h'\right\|_{L_1([R_1,R_2])}(R_2-R_1), \tag{130}$$

iv) at $t=\frac{R_1+R_2}{2}$, $\nu>1$, the right hand side of (128) is minimized, and we get:

$$\left|\int_A f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(R_2-R_1)^{k+1}}{2^k}\right.\right.$$

$$\left.\left.\left[h^{(k)}(R_1)+(-1)^k h^{(k)}(R_2)\right]\right\}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{R_1+}^{\nu}h\right\|_{L_1([R_1,R_2])},\left\|\overline{D}_{R_2-}^{\nu}h\right\|_{L_1([R_1,R_2])}\right\}}{\Gamma(\nu+1)}\frac{(R_2-R_1)^{\nu}}{2^{\nu-2}}, \tag{131}$$

v) if $h^{(k)}(R_1)=h^{(k)}(R_2)=0$, for all $k=0,1,...,n-1$, $\nu>1$, from (131) we obtain

$$\left|\int_A f(y)\,dy\right|\leq\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\cdot$$

$$\max\left\{\left\|D_{R_1+}^{\nu}h\right\|_{L_1([R_1,R_2])},\left\|\overline{D}_{R_2-}^{\nu}h\right\|_{L_1([R_1,R_2])}\right\}\frac{(R_2-R_1)^{\nu}}{\Gamma(\nu+1)\,2^{\nu-2}}, \tag{132}$$

which is a sharp inequality,

vi) more generally, for $j=0,1,2,...,N\in\mathbb{N}$, it holds

$$\left|\int_A f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{R_2-R_1}{N}\right)^{k+1}\right.\right.$$

$$\left[j^{k+1}h^{(k)}(R_1)+(-1)^k(N-j)^{k+1}h^{(k)}(R_2)\right]\Bigg\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\Bigg|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D^{\nu}_{R_1+}h\right\|_{L_1([R_1,R_2])},\left\|\overline{D}^{\nu}_{R_2-}h\right\|_{L_1([R_1,R_2])}\right\}}{\Gamma(\nu+1)}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{133}$$

vii) if $h^{(k)}(R_1)=h^{(k)}(R_2)=0$, $k=1,...,n-1$, *from (133) we get:*

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{N}\right)\left[jh(R_1)+(N-j)h(R_2)\right]\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D^{\nu}_{R_1+}h\right\|_{L_1([R_1,R_2])},\left\|\overline{D}^{\nu}_{R_2-}h\right\|_{L_1([R_1,R_2])}\right\}}{\Gamma(\nu+1)}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{134}$$

$j=0,1,2,...,N$,

viii) when $N=2$ *and* $j=1$, *(134) turns to*

$$\left|\int_A f(y)\,dy-(R_2-R_1)\left(h(R_1)+h(R_2)\right)\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D^{\nu}_{R_1+}h\right\|_{L_1([R_1,R_2])},\left\|\overline{D}^{\nu}_{R_2-}h\right\|_{L_1([R_1,R_2])}\right\}}{\Gamma(\nu+1)}\frac{(R_2-R_1)^{\nu}}{2^{\nu-2}}. \tag{135}$$

Proof. By Theorem 8 and (74). See in the 3. Appendix the general proving method in this article. ■

We give

Corollary 26 *(to Theorem 25) Let the radial* $f:\overline{B(0,R)}\to\mathbb{R}$. *Let* $\nu\geq 1$, $n=[\nu]$, *and* $h\in C^{\nu}_{0+}([0,R])\cap C^{\nu}_{R-}([0,R])$. *Then*

i)

$$\left|\int_{B(0,R)}f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[h^{(k)}(0)\,t^{k+1}+\right.\right.\right.$$

$$\left.\left.\left.(-1)^k h^{(k)}(R)(R-t)^{k+1}\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D^{\nu}_{0+}h\right\|_{L_1([0,R])},\left\|\overline{D}^{\nu}_{R-}h\right\|_{L_1([0,R])}\right\}}{\Gamma(\nu+1)}.$$

$$\left[t^{\nu}+(R-t)^{\nu}\right], \tag{136}$$

$\forall\ t\in[0,R]$,

ii) when $\nu=1$, from (136), we have

$$\left|\int_{B(0,R)} f(y)\,dy - h(R)(R-t)\,\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\|h'\right\|_{L_1([0,R])} R, \tag{137}$$

$\forall\ t\in[0,R]$,

iii) from (137), we obtain ($\nu=1$ case)

$$\left|\int_{B(0,R)} f(y)\,dy - Rh(R)\,\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\|h'\right\|_{L_1([0,R])} R, \tag{138}$$

iv) at $t=\frac{R}{2}$, $\nu>1$, the right hand side of (136) is minimized, and we get:

$$\left|\int_{B(0,R)} f(y)\,dy - \left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{R^{k+1}}{2^k}\right.\right.$$

$$\left.\left.\left[h^{(k)}(0)+(-1)^k h^{(k)}(R)\right]\right\}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{L_1([0,R])},\left\|\overline{D}_{R-}^{\nu}h\right\|_{L_1([0,R])}\right\}}{\Gamma(\nu+1)}\frac{R^{\nu}}{2^{\nu-2}}, \tag{139}$$

v) if $h^{(k)}(0)=h^{(k)}(R)=0$, for all $k=0,1,...,n-1$, $\nu>1$, from (139) we obtain

$$\left|\int_{B(0,R)} f(y)\,dy\right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\cdot$$

$$\max\left\{\left\|D_{0+}^{\nu}h\right\|_{L_1([0,R])},\left\|\overline{D}_{R-}^{\nu}\right\|_{L_1([0,R])}\right\}\frac{R^{\nu}}{\Gamma(\nu+1)\,2^{\nu-2}}, \tag{140}$$

which is a sharp inequality.

vi) more generally, for $j=0,1,2,...,N\in\mathbb{N}$, it holds

$$\left|\int_{B(0,R)} f(y)\,dy - \left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{R}{N}\right)^{k+1}\right.\right.$$

$$\left[j^{k+1}h^{(k)}(0)+(-1)^k(N-j)^{k+1}h^{(k)}(R)\right]\Bigg\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\Bigg| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{L_1([0,R])},\left\|\overline{D}_{R-}^{\nu}h\right\|_{L_1([0,R])}\right\}}{\Gamma(\nu+1)}$$

$$\left(\frac{R}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{141}$$

vii) if $h^{(k)}(0)=h^{(k)}(R)=0$*,* $k=1,...,n-1$*, from (141) we get:*

$$\left|\int_{B(0,R)}f(y)\,dy-\left(\frac{R}{N}\right)(N-j)\,h(R)\,\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{L_1([0,R])},\left\|\overline{D}_{R-}^{\nu}h\right\|_{L_1([0,R])}\right\}}{\Gamma(\nu+1)}$$

$$\left(\frac{R}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{142}$$

$j=0,1,2,...,N$,

viii) when $N=2$ *and* $j=1$*, (142) turns to*

$$\left|\int_{B(0,R)}f(y)\,dy-Rh(R)\,\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{0+}^{\nu}h\right\|_{L_1([0,R])},\left\|\overline{D}_{R-}^{\nu}h\right\|_{L_1([0,R])}\right\}}{\Gamma(\nu+1)}\frac{R^{\nu}}{2^{\nu-2}}. \tag{143}$$

Proof. Based on Theorem 25, just set there $R_1=0$, $R_2=R$, the assumptions now are on $B(0,R)$, and use (73). ■

We continue with

Theorem 27 *Let the radial* $f:\overline{A}\to\mathbb{R}$*. Let* $\nu\geq 1$*,* $n=[\nu]$*, and* $h\in C_{R_1+}^{\nu}([R_1,R_2])\cap C_{R_2-}^{\nu}([R_1,R_2])$*. Here* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$*. Then*

i)

$$\left|\int_A f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[h^{(k)}(R_1)(t-R_1)^{k+1}+\right.\right.\right.$$

$$\left.\left.\left.(-1)^k h^{(k)}(R_2)(R_2-t)^{k+1}\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{R_1+}^{\nu}h\right\|_{L_q([R_1,R_2])},\left\|\overline{D}_{R_2-}^{\nu}h\right\|_{L_q([R_1,R_2])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)\left(p(\nu-1)+1\right)^{\frac{1}{p}}}\cdot$$

$$\left[(t-R_1)^{\nu+\frac{1}{p}}+(R_2-t)^{\nu+\frac{1}{p}}\right], \tag{144}$$

$\forall\ t\in[R_1,R_2]$,

ii) at $t=\frac{R_1+R_2}{2}$*, the right hand side of (144) is minimized, and we get:*

$$\left|\int_A f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(R_2-R_1)^{k+1}}{2^k}\right.\right.$$

$$\left.\left.\left[h^{(k)}(R_1)+(-1)^k h^{(k)}(R_2)\right]\right\}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{R_1+}^{\nu}h\right\|_{L_q([R_1,R_2])},\left\|\overline{D}_{R_2-}^{\nu}h\right\|_{L_q([R_1,R_2])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)\left(p(\nu-1)+1\right)^{\frac{1}{p}}}\frac{(R_2-R_1)^{\nu+\frac{1}{p}}}{2^{\nu-1-\frac{1}{q}}}, \tag{145}$$

iii) if $h^{(k)}(R_1)=h^{(k)}(R_2)=0$*, for all* $k=0,1,...,n-1$*, we obtain*

$$\left|\int_A f(y)\,dy\right|\leq\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)2^{\nu-1-\frac{1}{q}}}\cdot$$

$$\max\left\{\left\|D_{R_1+}^{\nu}h\right\|_{L_q([R_1,R_2])},\left\|\overline{D}_{R_2-}^{\nu}h\right\|_{L_q([R_1,R_2])}\right\}\cdot$$

$$\frac{(R_2-R_1)^{\nu+\frac{1}{p}}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)\left(p(\nu-1)+1\right)^{\frac{1}{p}}}, \tag{146}$$

which is a sharp inequality,

iv) more generally, for $j=0,1,2,...,N\in\mathbb{N}$*, it holds*

$$\left|\int_A f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{R_2-R_1}{N}\right)^{k+1}\right.\right.$$

$$\left.\left.\left[j^{k+1}h^{(k)}(R_1)+(-1)^k(N-j)^{k+1}h^{(k)}(R_2)\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|D_{R_1+}^{\nu}h\right\|_{L_q([R_1,R_2])},\left\|\overline{D}_{R_2-}^{\nu}h\right\|_{L_q([R_1,R_2])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)\left(p(\nu-1)+1\right)^{\frac{1}{p}}}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{147}$$

v) if $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$, $k = 1, ..., n-1$, from (147) we get:

$$\left| \int_A f(y)\,dy - \left(\frac{R_2 - R_1}{N} \right) [j h(R_1) + (N-j) h(R_2)] \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{R_1+}^{\nu} h \right\|_{L_q([R_1,R_2])}, \left\| \overline{D}_{R_2-}^{\nu} h \right\|_{L_q([R_1,R_2])} \right\}}{\Gamma(\nu) \left(\nu + \frac{1}{p} \right) (p(\nu-1)+1)^{\frac{1}{p}}}$$

$$\left(\frac{R_2 - R_1}{N} \right)^{\nu + \frac{1}{p}} \left[j^{\nu + \frac{1}{p}} + (N-j)^{\nu + \frac{1}{p}} \right], \tag{148}$$

$j = 0, 1, 2, ..., N$,

vi) when $N = 2$ and $j = 1$, (148) turns to

$$\left| \int_A f(y)\,dy - (R_2 - R_1)(h(R_1) + h(R_2)) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{R_1+}^{\nu} h \right\|_{L_q([R_1,R_2])}, \left\| \overline{D}_{R_2-}^{\nu} h \right\|_{L_q([R_1,R_2])} \right\}}{\Gamma(\nu) \left(\nu + \frac{1}{p} \right) (p(\nu-1)+1)^{\frac{1}{p}}} \frac{(R_2 - R_1)^{\nu + \frac{1}{p}}}{2^{\nu - 1 - \frac{1}{q}}}. \tag{149}$$

Proof. By Theorem 9 and (74). See in the 3. Appendix the general proving method in this article. ■

We give

Corollary 28 *(to Theorem 27) Let the radial $f : \overline{B(0,R)} \to \mathbb{R}$. Let $\nu \geq 1$, $n = [\nu]$, and $h \in C_{0+}^{\nu}([0,R]) \cap C_{R-}^{\nu}([0,R])$. Here $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

i)

$$\left| \int_{B(0,R)} f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(0)\, t^{k+1} + \right.\right.\right.$$

$$\left.\left.\left. (-1)^k h^{(k)}(R)(R-t)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{0+}^{\nu} h \right\|_{L_q([0,R])}, \left\| \overline{D}_{R-}^{\nu} h \right\|_{L_q([0,R])} \right\}}{\Gamma(\nu) \left(\nu + \frac{1}{p} \right) (p(\nu-1)+1)^{\frac{1}{p}}}. \tag{150}$$

$$\left[t^{\nu + \frac{1}{p}} + (R-t)^{\nu + \frac{1}{p}} \right],$$

$\forall\ t \in [0, R]$,

ii) at $t = \frac{R}{2}$, the right hand side of (150) is minimized, and we get:

$$\left| \int_{B(0,R)} f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{R^{k+1}}{2^k} \right.\right.$$

$$\left[h^{(k)}(0)+(-1)^k h^{(k)}(R)\right]\Bigg\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\Bigg| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{0+}^{\nu} h\right\|_{L_q([0,R])}, \left\|\overline{D}_{R-}^{\nu} h\right\|_{L_q([0,R])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{R^{\nu+\frac{1}{p}}}{2^{\nu-1-\frac{1}{q}}}, \tag{151}$$

iii) if $h^{(k)}(0)=h^{(k)}(R)=0$*, for all* $k=0,1,...,n-1$*, we obtain*

$$\left|\int_{B(0,R)} f(y)\,dy\right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right) 2^{\nu-1-\frac{1}{q}}}\cdot$$

$$\max\left\{\left\|D_{0+}^{\nu} h\right\|_{L_q([0,R])}, \left\|\overline{D}_{R-}^{\nu} h\right\|_{L_q([0,R])}\right\} \frac{R^{\nu+\frac{1}{p}}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}, \tag{152}$$

which is a sharp inequality,

iv) more generally, for $j=0,1,2,...,N\in\mathbb{N}$*, it holds*

$$\Bigg|\int_{B(0,R)} f(y)\,dy - \Bigg\{\sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left(\frac{R}{N}\right)^{k+1}$$

$$\left[j^{k+1} h^{(k)}(0)+(-1)^k (N-j)^{k+1} h^{(k)}(R)\right]\Bigg\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\Bigg| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{0+}^{\nu} h\right\|_{L_q([0,R])}, \left\|\overline{D}_{R-}^{\nu} h\right\|_{L_q([0,R])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}$$

$$\left(\frac{R}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{153}$$

v) if $h^{(k)}(0)=h^{(k)}(R)=0$*,* $k=1,...,n-1$*, from (153) we get:*

$$\left|\int_{B(0,R)} f(y)\,dy - \left(\frac{R}{N}\right)(N-j)\,h(R)\,\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D_{0+}^{\nu} h\right\|_{L_q([0,R])}, \left\|\overline{D}_{R-}^{\nu} h\right\|_{L_q([0,R])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}$$

$$\left(\frac{R}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{154}$$

$j=0,1,2,...,N,$

vi) when $N = 2$ and $j = 1$, (154) turns to

$$\left| \int_{B(0,R)} f(y)\,dy - Rh(R) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| D_{0+}^{\nu} h \right\|_{L_q([0,R])}, \left\| \overline{D}_{R-}^{\nu} h \right\|_{L_q([0,R])} \right\}}{\Gamma(\nu) \left(\nu + \frac{1}{p}\right) (p(\nu - 1) + 1)^{\frac{1}{p}}} \frac{R^{\nu + \frac{1}{p}}}{2^{\nu - 1 - \frac{1}{q}}}. \tag{155}$$

Proof. Based on Theorem 27, just set there $R_1 = 0$, $R_2 = R$, the assumptions now are on $B(0,R)$, and use (73). ∎

If $g \in C^{n+1}([R_1, R_2])$, $0 \leq R_1 < R_2$, then $h(s) = g(s)\, s^{N-1} \in C^{n+1}([R_1, R_2])$, $n \in \mathbb{N}$, $N \geq 2$.

Next we present multivariate Conformable fractional Iyengar type inequalities:

Theorem 29 *Let $\alpha \in (n, n+1]$ and $g \in C^{n+1}([R_1, R_2])$, $0 < R_1 < R_2$, $n \in \mathbb{N}$; $\beta = \alpha - n$. Then*

i)

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n} \frac{1}{(k+1)!} \left[h^{(k)}(R_1)(z - R_1)^{k+1} + \right.\right.\right.$$

$$\left.\left.\left. (-1)^k h^{(k)}(R_2)(R_2 - z)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_{\alpha}^{R_1}(h) \right\|_{\infty,[R_1,R_2]}, \left\| {}_{\alpha}^{R_2}\mathbf{T}(h) \right\|_{\infty,[R_1,R_2]} \right\}}{\Gamma(\alpha + 2)} \cdot$$

$$\left[(z - R_1)^{\alpha+1} + (R_2 - z)^{\alpha+1} \right], \tag{156}$$

$\forall\, z \in [R_1, R_2]$,

ii) at $z = \frac{R_1 + R_2}{2}$, the right hand side of (156) is minimized, and we get:

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n} \frac{1}{(k+1)!} \frac{(R_2 - R_1)^{k+1}}{2^k} \right.\right.$$

$$\left.\left. \left[h^{(k)}(R_1) + (-1)^k h^{(k)}(R_2) \right] \right\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_{\alpha}^{R_1}(h) \right\|_{\infty,[R_1,R_2]}, \left\| {}_{\alpha}^{R_2}\mathbf{T}(h) \right\|_{\infty,[R_1,R_2]} \right\}}{\Gamma(\alpha + 2)} \frac{(R_2 - R_1)^{\alpha+1}}{2^{\alpha-1}}, \tag{157}$$

iii) if $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$*, for all* $k = 0, 1, ..., n$*, we obtain*

$$\left| \int_A f(y)\, dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot$$

$$\frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_{\alpha}^{R_1}(h) \right\|_{\infty,[R_1,R_2]}, \left\| {}_{\alpha}^{R_2}\mathbf{T}(h) \right\|_{\infty,[R_1,R_2]} \right\}}{\Gamma(\alpha+2)} \frac{(R_2 - R_1)^{\alpha+1}}{2^{\alpha-1}}, \tag{158}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$*, it holds*

$$\left| \int_A f(y)\, dy - \left\{ \sum_{k=0}^{n} \frac{1}{(k+1)!} \left(\frac{R_2 - R_1}{N} \right)^{k+1} \right. \right.$$

$$\left. \left. \left[h^{(k)}(R_1)\, j^{k+1} + (-1)^k h^{(k)}(R_2)(N-j)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_{\alpha}^{R_1}(h) \right\|_{\infty,[R_1,R_2]}, \left\| {}_{\alpha}^{R_2}\mathbf{T}(h) \right\|_{\infty,[R_1,R_2]} \right\}}{\Gamma(\alpha+2)}$$

$$\left(\frac{R_2 - R_1}{N} \right)^{\alpha+1} \left[j^{\alpha+1} + (N-j)^{\alpha+1} \right], \tag{159}$$

v) if $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$*,* $k = 1, ..., n$*, from (159) we get:*

$$\left| \int_A f(y)\, dy - \left(\frac{R_2 - R_1}{N} \right) \left[j h(R_1) + (N-j) h(R_2) \right] \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_{\alpha}^{R_1}(h) \right\|_{\infty,[R_1,R_2]}, \left\| {}_{\alpha}^{R_2}\mathbf{T}(h) \right\|_{\infty,[R_1,R_2]} \right\}}{\Gamma(\alpha+2)}$$

$$\left(\frac{R_2 - R_1}{N} \right)^{\alpha+1} \left[j^{\alpha+1} + (N-j)^{\alpha+1} \right], \tag{160}$$

$j = 0, 1, 2, ..., N,$

vi) when $N = 2$ *and* $j = 1$*, (160) turns to*

$$\left| \int_A f(y)\, dy - (R_2 - R_1)(h(R_1) + h(R_2)) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_{\alpha}^{R_1}(h) \right\|_{\infty,[R_1,R_2]}, \left\| {}_{\alpha}^{R_2}\mathbf{T}(h) \right\|_{\infty,[R_1,R_2]} \right\}}{\Gamma(\alpha+2)} \frac{(R_2 - R_1)^{\alpha+1}}{2^{\alpha-1}}, \tag{161}$$

Proof. By Theorem 13 and as in our other multivariate results. ■

We continue with

Corollary 30 *Let* $\alpha \in (n, n+1]$ *and* $g \in C^{n+1}([0,R])$, $R > 0$, $n \in \mathbb{N}$; $\beta = \alpha - n$. *Then*

i)

$$\left| \int_{B(0,R)} f(y)\,dy - \left\{ \sum_{k=0}^{n} \frac{1}{(k+1)!} \left[h^{(k)}(0)\, z^{k+1} + \right. \right. \right.$$

$$\left. \left. \left. (-1)^k h^{(k)}(R)(R-z)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_{\alpha}^{0}(h) \right\|_{\infty,[0,R]}, \left\| {}_{\alpha}^{R}\mathbf{T}(h) \right\|_{\infty,[0,R]} \right\}}{\Gamma(\alpha+2)} \cdot$$

$$\left[z^{\alpha+1} + (R-z)^{\alpha+1} \right], \tag{162}$$

$\forall\ z \in [0, R]$,

ii) at $z = \frac{R}{2}$, *the right hand side of (162) is minimized, and we get:*

$$\left| \int_{B(0,R)} f(y)\,dy - \left\{ \sum_{k=0}^{n} \frac{1}{(k+1)!} \frac{R^{k+1}}{2^k} \right. \right.$$

$$\left. \left. \left[h^{(k)}(0) + (-1)^k h^{(k)}(R) \right] \right\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_{\alpha}^{0}(h) \right\|_{\infty,[0,R]}, \left\| {}_{\alpha}^{R}\mathbf{T}(h) \right\|_{\infty,[0,R]} \right\}}{\Gamma(\alpha+2)} \frac{R^{\alpha+1}}{2^{\alpha-1}}, \tag{163}$$

iii) assuming $h^{(k)}(0) = h^{(k)}(R) = 0$, *for all* $k = 0, 1, ..., n$, *we obtain*

$$\left| \int_{B(0,R)} f(y)\,dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot$$

$$\frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_{\alpha}^{0}(h) \right\|_{\infty,[0,R]}, \left\| {}_{\alpha}^{R}\mathbf{T}(h) \right\|_{\infty,[0,R]} \right\}}{\Gamma(\alpha+2)} \frac{R^{\alpha+1}}{2^{\alpha-1}}, \tag{164}$$

which is a sharp inequality.

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds*

$$\left| \int_{B(0,R)} f(y)\,dy - \left\{ \sum_{k=0}^{n} \frac{1}{(k+1)!} \left(\frac{R}{N} \right)^{k+1} \right. \right.$$

$$\left. \left. \left[h^{(k)}(0)\, j^{k+1} + (-1)^k h^{(k)}(R)(N-j)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{0}(h)\right\|_{\infty,[0,R]}, \left\|{}_{\alpha}^{R}\mathbf{T}(h)\right\|_{\infty,[0,R]}\right\}}{\Gamma(\alpha+2)}$$

$$\left(\frac{R}{N}\right)^{\alpha+1}\left[j^{\alpha+1}+(N-j)^{\alpha+1}\right], \tag{165}$$

v) if $h^{(k)}(0)=h^{(k)}(R)=0$, $k=1,...,n$, *from (165) we get:*

$$\left|\int_{B(0,R)} f(y)\,dy - \left(\frac{R}{N}\right)(N-j)\,h(R)\,\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{0}(h)\right\|_{\infty,[0,R]}, \left\|{}_{\alpha}^{R}\mathbf{T}(h)\right\|_{\infty,[0,R]}\right\}}{\Gamma(\alpha+2)}$$

$$\left(\frac{R}{N}\right)^{\alpha+1}\left[j^{\alpha+1}+(N-j)^{\alpha+1}\right], \tag{166}$$

$j=0,1,2,...,N$,

vi) when $N=2$ *and* $j=1$, *(166) turns to*

$$\left|\int_{B(0,R)} f(y)\,dy - Rh(R)\,\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{0}(h)\right\|_{\infty,[0,R]}, \left\|{}_{\alpha}^{R}\mathbf{T}(h)\right\|_{\infty,[0,R]}\right\}}{\Gamma(\alpha+2)} \frac{R^{\alpha+1}}{2^{\alpha-1}}, \tag{167}$$

Proof. By Theorem 29, just set there $R_1=0$, $R_2=R$, the assumptions now are on $B(0,R)$, and use (73). ■

We continue with L_p results.

Theorem 31 *Let* $\alpha\in(n,n+1]$ *and* $g\in C^{n+1}([R_1,R_2])$, $0<R_1<R_2$, $n\in\mathbb{N}$; $\beta=\alpha-n$. *Let also* $p_1,p_2,p_3>1:\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$, *with* $\beta>\frac{1}{p_1}+\frac{1}{p_3}$. *Then*

i)

$$\left|\int_{A} f(y)\,dy - \left\{\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[h^{(k)}(R_1)(z-R_1)^{k+1}+\right.\right.\right.$$

$$\left.\left.\left.(-1)^{k}h^{(k)}(R_2)(R_2-z)^{k+1}\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|\mathbf{T}_{\alpha}^{R_1}(h)\right\|_{p_3,[R_1,R_2]}, \left\|{}_{\alpha}^{R_2}\mathbf{T}(h)\right\|_{p_3,[R_1,R_2]}\right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}\cdot$$

$$\left[(z-R_1)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}+(R_2-z)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\right], \tag{168}$$

$\forall\ z \in [R_1, R_2]$,

ii) at $z = \frac{R_1+R_2}{2}$, *the right hand side of (168) is minimized, and we get:*

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n} \frac{1}{(k+1)!} \frac{(R_2 - R_1)^{k+1}}{2^k} \right.\right.$$

$$\left.\left. \left[h^{(k)}(R_1) + (-1)^k h^{(k)}(R_2) \right] \right\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{R_1}(h) \right\|_{p_3,[R_1,R_2]}, \left\| {}_{\alpha}^{R_2}\mathbf{T}(h) \right\|_{p_3,[R_1,R_2]} \right\}}{n!\,(p_1 n + 1)^{\frac{1}{p_1}} (p_2(\beta - 1) + 1)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_1} + \frac{1}{p_2} \right)} \frac{(R_2 - R_1)^{\alpha + \frac{1}{p_1} + \frac{1}{p_2}}}{2^{\alpha - 1 - \frac{1}{p_3}}}, \tag{169}$$

iii) assuming $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$, *for all* $k = 0, 1, ..., n$, *we obtain*

$$\left| \int_A f(y)\,dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot$$

$$\frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{R_1}(h) \right\|_{p_3,[R_1,R_2]}, \left\| {}_{\alpha}^{R_2}\mathbf{T}(h) \right\|_{p_3,[R_1,R_2]} \right\}}{n!\,(p_1 n + 1)^{\frac{1}{p_1}} (p_2(\beta - 1) + 1)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_1} + \frac{1}{p_2} \right)} \frac{(R_2 - R_1)^{\alpha + \frac{1}{p_1} + \frac{1}{p_2}}}{2^{\alpha - 1 - \frac{1}{p_3}}}, \tag{170}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds*

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n} \frac{1}{(k+1)!} \left(\frac{R_2 - R_1}{N} \right)^{k+1} \right.\right.$$

$$\left.\left. \left[h^{(k)}(R_1)\, j^{k+1} + (-1)^k h^{(k)}(R_2)\,(N - j)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{R_1}(h) \right\|_{p_3,[R_1,R_2]}, \left\| {}_{\alpha}^{R_2}\mathbf{T}(h) \right\|_{p_3,[R_1,R_2]} \right\}}{n!\,(p_1 n + 1)^{\frac{1}{p_1}} (p_2(\beta - 1) + 1)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_1} + \frac{1}{p_2} \right)}$$

$$\left(\frac{R_2 - R_1}{N} \right)^{\alpha + \frac{1}{p_1} + \frac{1}{p_2}} \left[j^{\alpha + \frac{1}{p_1} + \frac{1}{p_2}} + (N - j)^{\alpha + \frac{1}{p_1} + \frac{1}{p_2}} \right], \tag{171}$$

v) if $h^{(k)}(R_1) = h^{(k)}(R_2) = 0$, $k = 1, ..., n$, *from (171) we get:*

$$\left| \int_A f(y)\,dy - \left(\frac{R_2 - R_1}{N} \right) [j h(R_1) + (N - j) h(R_2)] \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{R_1}(h) \right\|_{p_3,[R_1,R_2]}, \left\| {}_{\alpha}^{R_2}\mathbf{T}(h) \right\|_{p_3,[R_1,R_2]} \right\}}{n!\,(p_1 n + 1)^{\frac{1}{p_1}} (p_2(\beta - 1) + 1)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_1} + \frac{1}{p_2} \right)}$$

$$\left(\frac{R_2-R_1}{N}\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\left[j^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}+(N-j)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\right], \tag{172}$$

$j=0,1,2,...,N$,

vi) when $N=2$ and $j=1$, (172) turns to

$$\left|\int_A f(y)\,dy-(R_2-R_1)(h(R_1)+h(R_2))\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|\mathbf{T}_{\alpha}^{R_1}(h)\right\|_{p_3,[R_1,R_2]},\left\|{}_{\alpha}^{R_2}\mathbf{T}(h)\right\|_{p_3,[R_1,R_2]}\right\}}{n!\,(p_1n+1)^{\frac{1}{p_1}}(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}$$

$$\frac{(R_2-R_1)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-1-\frac{1}{p_3}}}, \tag{173}$$

Proof. By Theorem 14 and as in our other multivariate results. ■

We continue with

Corollary 32 *Let $\alpha\in(n,n+1]$ and $g\in C^{n+1}([0,R])$, $R>0$, $n\in\mathbb{N}$; $\beta=\alpha-n$. Let also $p_1,p_2,p_3>1:\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$, with $\beta>\frac{1}{p_1}+\frac{1}{p_3}$. Then*

i)

$$\left|\int_{B(0,R)}f(y)\,dy-\left\{\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[h^{(k)}(0)\,z^{k+1}+\right.\right.\right.$$

$$\left.\left.\left.(-1)^k h^{(k)}(R)(R-z)^{k+1}\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|\mathbf{T}_{\alpha}^{0}(h)\right\|_{p_3,[0,R]},\left\|{}_{\alpha}^{R}\mathbf{T}(h)\right\|_{p_3,[0,R]}\right\}}{n!\,(p_1n+1)^{\frac{1}{p_1}}(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}\cdot$$

$$\left[t^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}+(R-t)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\right], \tag{174}$$

$\forall\ z\in[0,R]$,

ii) at $z=\frac{R}{2}$, the right hand side of (174) is minimized, and we get:

$$\left|\int_{B(0,R)}f(y)\,dy-\left\{\sum_{k=0}^{n}\frac{1}{(k+1)!}\frac{R^{k+1}}{2^k}\right.\right.$$

$$\left.\left.\left[h^{(k)}(0)+(-1)^k h^{(k)}(R)\right]\right\}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\max\left\{\left\|\mathbf{T}_{\alpha}^{0}(h)\right\|_{p_3,[0,R]},\left\|{}_{\alpha}^{R}\mathbf{T}(h)\right\|_{p_3,[0,R]}\right\}}{n!\,(p_1n+1)^{\frac{1}{p_1}}(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}$$

$$\frac{R^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-1-\frac{1}{p_3}}}, \tag{175}$$

iii) assuming $h^{(k)}(0) = h^{(k)}(R) = 0$, for all $k = 0, 1, ..., n$, we obtain

$$\left| \int_{B(0,R)} f(y)\, dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot$$

$$\frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{0}(h) \right\|_{p_3,[0,R]}, \left\| {}_{\alpha}^{R}\mathbf{T}(h) \right\|_{p_3,[0,R]} \right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)} \frac{R^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-1-\frac{1}{p_3}}}, \tag{176}$$

which is a sharp inequality.

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, it holds

$$\left| \int_{B(0,R)} f(y)\, dy - \left\{ \sum_{k=0}^{n} \frac{1}{(k+1)!} \left(\frac{R}{N}\right)^{k+1} \right.\right.$$

$$\left.\left. \left[h^{(k)}(0)\, j^{k+1} + (-1)^k h^{(k)}(R)\,(N-j)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{0}(h) \right\|_{p_3,[0,R]}, \left\| {}_{\alpha}^{R}\mathbf{T}(h) \right\|_{p_3,[0,R]} \right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}$$

$$\left(\frac{R}{N}\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} \left[j^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} + (N-j)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} \right], \tag{177}$$

v) if $h^{(k)}(0) = h^{(k)}(R) = 0$, $k = 1, ..., n$, from (177) we get:

$$\left| \int_{B(0,R)} f(y)\, dy - \left(\frac{R}{N}\right)(N-j)\, h(R) \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{0}(h) \right\|_{p_3,[0,R]}, \left\| {}_{\alpha}^{R}\mathbf{T}(h) \right\|_{p_3,[0,R]} \right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}$$

$$\left(\frac{R}{N}\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} \left[j^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} + (N-j)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} \right], \tag{178}$$

$j = 0, 1, 2, ..., N,$

viii) when $N = 2$ and $j = 1$, (178) turns to

$$\left| \int_{B(0,R)} f(y)\, dy - Rh(R) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{0}(h) \right\|_{p_3,[0,R]}, \left\| {}_{\alpha}^{R}\mathbf{T}(h) \right\|_{p_3,[0,R]} \right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)} \frac{R^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-1-\frac{1}{p_3}}}. \tag{179}$$

Proof. By Theorem 31, just set there $R_1 = 0$, $R_2 = R$, the assumptions now are on $B(0, R)$, and use (73). ■

Our proving method follows next.

3 Appendix

Proof. Detailed proof of Theorem 17 (serving as a model proof for this article).

We apply Theorem 4 (i) for h:

$$\left|\int_{R_1}^{R_2} h(s)\,ds - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[h^{(k)}(R_1)(t-R_1)^{k+1} + \right.\right.$$

$$\left.\left.(-1)^k h^{(k)}(R_2)(R_2-t)^{k+1}\right]\right| \leq$$

$$\frac{\max\left\{\left\|D_{*R_1}^{\nu} h\right\|_{L_\infty([R_1,R_2])}, \left\|D_{R_2-}^{\nu}\right\|_{L_\infty([R_1,R_2])}\right\}}{\Gamma(\nu+2)}\cdot$$

$$\left[(t-R_1)^{\nu+1} + (R_2-t)^{\nu+1}\right] =: \psi(t), \tag{180}$$

$\forall\, t \in [R_1, R_2]$.

Equivalently, we have that

$$-\psi(t) \leq \int_{R_1}^{R_2} h(s)\,ds - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[h^{(k)}(R_1)(t-R_1)^{k+1} + \right.$$

$$\left.(-1)^k h^{(k)}(R_2)(R_2-t)^{k+1}\right] \leq \psi(t), \tag{181}$$

$\forall\, t \in [R_1, R_2]$.

That is

$$-\psi(t) \leq \int_{R_1}^{R_2} f(s\omega)\, s^{N-1} ds - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[h^{(k)}(R_1)(t-R_1)^{k+1}\right.$$

$$\left. + (-1)^k h^{(k)}(R_2)(R_2-t)^{k+1}\right] \leq \psi(t), \tag{182}$$

$\forall\, t \in [R_1, R_2]$, and $\forall\, \omega \in S^{N-1}$.

Therefore it holds

$$-\psi(t)\int_{S^{N-1}} d\omega \leq \int_{S^{N-1}}\left(\int_{R_1}^{R_2} f(s\omega)\, s^{N-1} ds\right) d\omega -$$

$$\left\{\sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[h^{(k)}(R_1)(t-R_1)^{k+1} + (-1)^k h^{(k)}(R_2)(R_2-t)^{k+1}\right]\right\}\int_{S^{N-1}} d\omega$$

$$\leq \psi(t) \int_{S^{N-1}} d\omega, \quad \forall\, t \in [R_1, R_2], \tag{183}$$

which is (by (74))

$$-\psi(t) \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \leq \int_A f(y)\, dy -$$

$$\left\{\sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(R_1)(t-R_1)^{k+1} + (-1)^k h^{(k)}(R_2)(R_2-t)^{k+1}\right]\right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$$

$$\leq \psi(t) \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \quad \forall\, t \in [R_1, R_2]. \tag{184}$$

Consequently, we derive

$$\left| \int_A f(y)\, dy - \left\{\sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[h^{(k)}(R_1)(t-R_1)^{k+1} + \right.\right.\right.$$

$$\left.\left.\left.(-1)^k h^{(k)}(R_2)(R_2-t)^{k+1}\right]\right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq \psi(t) \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D^{\nu}_{*R_1} h\right\|_{L_\infty([R_1,R_2])}, \left\|D^{\nu}_{R_2-}\right\|_{L_\infty([R_1,R_2])}\right\}}{\Gamma(\nu+2)} \cdot$$

$$\left[(t-R_1)^{\nu+1} + (R_2-t)^{\nu+1}\right], \quad \forall\, t \in [R_1, R_2], \tag{185}$$

proving Theorem 17 (i).

Next consider

$$\varphi(t) := (t-R_1)^{\nu+1} + (R_2-t)^{\nu+1}, \quad \forall\, t \in [R_1, R_2].$$

Then

$$\varphi'(t) = (\nu+1)\left[(t-R_1)^{\nu} - (R_2-t)^{\nu}\right] = 0,$$

and φ has the only critical number $t = \frac{R_1+R_2}{2}$. Hence $\varphi(t)$ has a minimum over $[R_1, R_2]$ which is $\varphi\left(\frac{R_1+R_2}{2}\right) = \frac{(R_2-R_1)^{\nu+1}}{2^{\nu}}$.

Consequently, it holds (by (185))

$$\left| \int_A f(y)\, dy - \left\{\sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(R_2-R_1)^{k+1}}{2^k} \right.\right.$$

$$\left.\left.\left[h^{(k)}(R_1) + (-1)^k h^{(k)}(R_2)\right]\right\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\max\left\{\left\|D^{\nu}_{*R_1} h\right\|_{L_\infty([R_1,R_2])}, \left\|D^{\nu}_{R_2-}\right\|_{L_\infty([R_1,R_2])}\right\}}{\Gamma(\nu+2)} \frac{(R_2-R_1)^{\nu+1}}{2^{\nu-1}}, \tag{186}$$

proving Theorem 17 (ii).

The rest of Theorem 17 is obvious or follows the same way as above. ■

The rest of the proofs of this article as similar are omitted.

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