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SOME INEQUALITIES OF FEJÉR TYPE FOR HYPERBOLIC p-CONVEX FUNCTIONS

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Abstract. In this paper we establish several Fejér type integral inequalities for hyperbolic p-convex functions.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the *Hermite-Hadamard inequality*.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, for which we would like to refer the reader to the monograph [9], the recent survey paper [8] and the references therein.

In 1906, Fejér [10], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 1 (Fejér's Inequality)**.** *Consider the integral $\int_a^b h(x)\,w(x)\,dx$, where h is a convex function in the interval $[a,b]$ and g is a positive function in the same interval such that*

$$g(x) = g(a+b-x),\ x\in[a,b],$$

i.e., $y = g(x)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b),0\right)$ and is normal to the x-axis. Under those conditions the following inequalities are valid:

$$h\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leq \int_a^b h(x)\,g(x)\,dx \leq \frac{h(a)+h(b)}{2}\int_a^b g(x)\,dx. \tag{1.2}$$

If h is concave on $[a,b]$, then the inequalities reverse in (1.2).

Clearly, for $g(x)\equiv 1$ on $[a,b]$ we get (1.1).

Let I be a finite or infinite open interval of real numbers and $p\in\mathbb{R}$, $p\neq 0$.

In the following we present the basic definitions and results concerning the class of hyperbolic p-convex function, see [3]. For other concepts of modified convex functions see for example [12], [13], [4], [6], [7], [11], [14], [15] and [16].

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We consider the hyperbolic functions of a real argument $x \in \mathbb{R}$ defined by

$$\sinh x := \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x}, \ \cosh x := \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x},$$

$$\tanh x := \frac{\sinh x}{\cosh x} \text{ and } \coth x := \frac{\cosh x}{\sinh x}.$$

We say that a function $f : I \to \mathbb{R}$ is *hyperbolic p-convex* (or sub H-function, according with [3]) on I, if for any closed subinterval $[a, b]$ of I we have

$$f(x) \le \frac{\sinh[p(b-x)]}{\sinh[p(b-a)]} f(a) + \frac{\sinh[p(x-a)]}{\sinh[p(b-a)]} f(b) \tag{1.3}$$

for all $x \in [a, b]$.

If the inequality (1.3) holds with " $\ge$ ", then the function will be called *hyperbolic p-concave* on I.

Geometrically speaking, this means that the graph of f on $[a, b]$ lies nowhere above the p-hyperbolic function determined by the equation

$$H(x) = H(x; a, b, f) := A \cosh(px) + B \sinh(px)$$

where A and B are chosen such that $H(a) = f(a)$ and $H(b) = f(b)$.

If we take $x = (1-t)a + tb \in [a, b]$, $t \in [0, 1]$, then the condition (1.3) becomes

$$f((1-t)a + tb) \le \frac{\sinh[p(1-t)(b-a)]}{\sinh[p(b-a)]} f(a) + \frac{\sinh[pt(b-a)]}{\sinh[p(b-a)]} f(b) \tag{1.4}$$

for any $t \in [0, 1]$.

We have the following properties of hyperbolic p-convex on I, [3].

(i) A hyperbolic p-convex function $f : I \to \mathbb{R}$ has finite right and left derivatives $f'_+(x)$ and $f'_-(x)$ at every point $x \in I$ and $f'_-(x) \le f'_+(x)$. The function f is differentiable on I with the exception of an at most countable set.

(ii) A necessary and sufficient condition for the function $f : I \to \mathbb{R}$ to be hyperbolic p-convex function on I is that it satisfies the *gradient inequality*

$$f(y) \ge f(x) \cosh[p(y-x)] + K_{x,f} \sinh[p(y-x)] \tag{1.5}$$

for any x, $y \in I$ where $K_{x,f} \in \left[f'_-(x), f'_+(x)\right]$. If f is differentiable at the point x then $K_{x,f} = f'(x)$.

(iii) A necessary and sufficient condition for the function f to be a hyperbolic p-convex in I, is that the function

$$\varphi(x) = f'(x) - p^2 \int_a^x f(t)\, dt$$

is nondecreasing on I, where $a \in I$.

(iv) Let $f : I \to \mathbb{R}$ be a two times continuously differentiable function on I. Then f is hyperbolic p-convex on I if and only if for all $x \in I$ we have

$$f''(x) - p^2 f(x) \ge 0. \tag{1.6}$$

For other properties of hyperbolic p-convex functions, see [3].

Consider the function $f_r : (0, \infty) \to (0, \infty)$, $f_r(x) = x^r$ with $p \in \mathbb{R} \setminus \{0\}$. If $r \in (-\infty, 0) \cup [1, \infty)$ the function is convex and if $r \in (0, 1)$ it is concave. We have for $r \in (-\infty, 0) \cup [1, \infty)$

$$f_r''(x) - p^2 f_r(x) = r(r-1)x^{r-2} - p^2 x^r = p^2 x^{r-2}\left(\frac{r(r-1)}{p^2} - x^2\right),\ x > 0.$$

We observe that $f_r''(x) - p^2 f_r(x) > 0$ for $x \in \left(0, \frac{\sqrt{r(r-1)}}{|p|}\right)$ and $f_r''(x) - p^2 f_r(x) < 0$ for $x \in \left(\frac{\sqrt{r(r-1)}}{|p|}, \infty\right)$, which shows that the power function f_r for $r \in (-\infty, 0) \cup [1, \infty)$ is hyperbolic p-convex on $\left(0, \frac{\sqrt{r(r-1)}}{|p|}\right)$ and hyperbolic p-concave on $\left(\frac{\sqrt{r(r-1)}}{|p|}, \infty\right)$.

If $r \in (0, 1)$, then $f_r''(x) - p^2 f_r(x) < 0$ for any $x > 0$, which shows that f_r is hyperbolic p-concave on $(0, \infty)$.

Consider the exponential function $f_\alpha(x) = \exp(\alpha x)$ for $\alpha \neq 0$ and $x \in \mathbb{R}$. Then

$$f_\alpha''(x) - p^2 f_\alpha(x) = \alpha^2 e^{\alpha x} - p^2 e^{\alpha x} = \left(\alpha^2 - p^2\right) e^{\alpha x},\ x > 0.$$

If $|\alpha| > |p|$, then f_α is hyperbolic p-convex on $\mathbb{R}$ and if $|\alpha| < |p|$ then f_α is hyperbolic p-concave on $\mathbb{R}$.

In this paper we establish several Fejér type integral inequalities for hyperbolic p-convex functions.

2. Some Fejér's Type Inequalities

We start with the following lemma of interest in itself:

Lemma 1. *Assume that the function $f : I \to \mathbb{R}$ is hyperbolic p-convex on I. Then for any $a, b \in I$ with $a < b$ and $x \in [a, b]$ we have*

$$f\left(\frac{a+b}{2}\right)\cosh\left[p\left(x - \frac{a+b}{2}\right)\right] \leq \frac{1}{2}\left[f(x) + f(a+b-x)\right] \leq \frac{f(a) + f(b)}{2}\frac{\cosh\left[p\left(x - \frac{a+b}{2}\right)\right]}{\cosh\left[\frac{p(b-a)}{2}\right]}. \tag{2.1}$$

Proof. From (1.3) we have by replacing x with $a + b - x$ that

$$f(a+b-x) \leq \frac{\sinh\left[p(x-a)\right]}{\sinh\left[p(b-a)\right]}f(a) + \frac{\sinh\left[p(b-x)\right]}{\sinh\left[p(b-a)\right]}f(b) \tag{2.2}$$

for any $x \in [a, b]$.

If we add (1.3) with (2.2) we get

$$
\begin{aligned}
(2.3)\qquad & f(x)+f(a+b-x) \\
&\leq \frac{\sinh[p(b-x)]}{\sinh[p(b-a)]} f(a)+\frac{\sinh[p(x-a)]}{\sinh[p(b-a)]} f(b) \\
&+\frac{\sinh[p(x-a)]}{\sinh[p(b-a)]} f(a)+\frac{\sinh[p(b-x)]}{\sinh[p(b-a)]} f(b) \\
&=\frac{\sinh[p(b-x)]+\sinh[p(x-a)]}{\sinh[p(b-a)]} f(a) \\
&+\frac{\sinh[p(b-x)]+\sinh[p(x-a)]}{\sinh[p(b-a)]} f(b) \\
&=\frac{\sinh[p(b-x)]+\sinh[p(x-a)]}{\sinh[p(b-a)]}[f(a)+f(b)]
\end{aligned}
$$

for any $x\in[a,b]$.

Observe that

$$
\begin{aligned}
(2.4)\qquad & \frac{\sinh[p(b-x)]+\sinh[p(x-a)]}{\sinh[p(b-a)]} \\
&=\frac{2\sinh\left[\frac{p(b-a)}{2}\right]\cosh\left[p\left(x-\frac{a+b}{2}\right)\right]}{2\sinh\left[\frac{p(b-a)}{2}\right]\cosh\left[\frac{p(b-a)}{2}\right]}=\frac{\cosh\left[p\left(x-\frac{a+b}{2}\right)\right]}{\cosh\left[\frac{p(b-a)}{2}\right]}
\end{aligned}
$$

for any $x\in[a,b]$.

Using the equality (2.4) and dividing by 2 in (2.3) we get the second inequality in (2.1).

From (1.4) for $t=\frac{1}{2}$ and $a=u$, $b=v$ we get

$$
\begin{aligned}
f\left(\frac{u+v}{2}\right) &\leq \frac{\sinh\left[p\left(\frac{v-u}{2}\right)\right]}{\sinh[p(v-u)]} f(u)+\frac{\sinh\left[p\left(\frac{v-u}{2}\right)\right]}{\sinh[p(v-u)]} f(v) \\
&=\frac{\sinh\left[p\left(\frac{v-u}{2}\right)\right]}{\sinh[p(v-u)]}[f(u)+f(v)] \\
&=\frac{\sinh\left[p\left(\frac{v-u}{2}\right)\right]}{2\sinh\left[p\left(\frac{v-u}{2}\right)\right]\cosh\left[p\left(\frac{v-u}{2}\right)\right]}[f(u)+f(v)] \\
&=\frac{1}{\cosh\left[p\left(\frac{v-u}{2}\right)\right]}\frac{f(u)+f(v)}{2},
\end{aligned}
$$

which implies that

$$
(2.5)\qquad f\left(\frac{u+v}{2}\right)\cosh\left[p\left(\frac{v-u}{2}\right)\right]\leq\frac{f(u)+f(v)}{2}
$$

for any u, $v\in I$.

Now, if we in (2.5) take $v=x$ and $u=a+b-x$, then we get the first inequality in (2.1). □

Remark 1. *By taking* $x = (1-t)a + tb$ *in (2.1) we get the equivalent double inequality*

$$f\left(\frac{a+b}{2}\right)\cosh\left[p\left(t-\frac{1}{2}\right)(b-a)\right] \le \frac{1}{2}\left[f((1-t)a+tb)+f(ta+(1-t)b)\right] \le \frac{f(a)+f(b)}{2}\frac{\cosh\left[p\left(t-\frac{1}{2}\right)(b-a)\right]}{\cosh\left[\frac{p(b-a)}{2}\right]} \tag{2.6}$$

for any $a, b \in I$ *and* $t \in [0,1]$.

We have:

Theorem 2. *Assume that the function* $f : I \to \mathbb{R}$ *is hyperbolic* p*-convex on* I *and* $a, b \in I$. *Assume also that* $w : [a,b] \to \mathbb{R}$ *is a positive, symmetric and integrable function on* $[a,b]$, *then we have*

$$f\left(\frac{a+b}{2}\right)\int_a^b \cosh\left[p\left(x-\frac{a+b}{2}\right)\right]w(x)\,dx \le \int_a^b f(x)\,w(x)\,dx \le \frac{f(a)+f(b)}{2}\operatorname{sech}\left[\frac{p(b-a)}{2}\right]\int_a^b \cosh\left[p\left(x-\frac{a+b}{2}\right)\right]w(x)\,dx. \tag{2.7}$$

Proof. We multiply the inequality (2.1) by $w(x) \ge 0$ and integrate to get

$$f\left(\frac{a+b}{2}\right)\int_a^b \cosh\left[p\left(x-\frac{a+b}{2}\right)\right]w(x)\,dx \le \frac{1}{2}\int_a^b \left[f(x)+f(a+b-x)\right]w(x)\,dx \le \frac{f(a)+f(b)}{2}\frac{\int_a^b \cosh\left[p\left(x-\frac{a+b}{2}\right)\right]w(x)\,dx}{\cosh\left[\frac{p(b-a)}{2}\right]}. \tag{2.8}$$

By using the change of variable $y = a+b-x$ and the symmetry of w we have

$$\int_a^b f(a+b-x)\,w(x)\,dx = \int_a^b f(y)\,w(a+b-y)\,dy = \int_a^b f(y)\,w(y)\,dy,$$

therefore

$$\frac{1}{2}\int_a^b \left[f(x)+f(a+b-x)\right]w(x)\,dx = \int_a^b f(x)\,w(x)\,dx$$

and by (2.8) we get (2.7). □

Corollary 1. *With the assumption of Theorem 2 for* f *we have*

$$(b-a)f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)\operatorname{sech}\left[p\left(x-\frac{a+b}{2}\right)\right]dx \le \frac{f(a)+f(b)}{2}\frac{b-a}{\cosh\left[\frac{p(b-a)}{2}\right]} \tag{2.9}$$

and

$$\text{(2.10)}\quad \frac{1}{2}\left[b-a+\frac{1}{p}\sinh\left[p\left(b-a\right)\right]\right]f\left(\frac{a+b}{2}\right) \leq \int_a^b f\left(x\right)\cosh\left[p\left(x-\frac{a+b}{2}\right)\right]dx \leq \frac{b-a+\frac{1}{p}\sinh\left[p\left(b-a\right)\right]}{2\cosh\left[\frac{p(b-a)}{2}\right]}\frac{f\left(a\right)+f\left(b\right)}{2}.$$

Proof. The inequality (2.9) follows by (2.7) for $w\left(x\right)=\operatorname{sech}\left[p\left(x-\frac{a+b}{2}\right)\right]$, $x\in\left[a,b\right]$.

If we take in (2.7) $w\left(x\right)=\cosh\left[p\left(x-\frac{a+b}{2}\right)\right]$, then we get

$$\text{(2.11)}\quad f\left(\frac{a+b}{2}\right)\int_a^b\cosh^2\left[p\left(x-\frac{a+b}{2}\right)\right]dx \leq \int_a^b f\left(x\right)\cosh\left[p\left(x-\frac{a+b}{2}\right)\right]dx \leq \frac{f\left(a\right)+f\left(b\right)}{2}\frac{\int_a^b\cosh^2\left[p\left(x-\frac{a+b}{2}\right)\right]dx}{\cosh\left[\frac{p(b-a)}{2}\right]}.$$

Since

$$\int_a^b\cosh^2\left[p\left(x-\frac{a+b}{2}\right)\right]dx=\frac{1}{2}\left[\left(b-a\right)+\int_a^b\cosh\left[2p\left(x-\frac{a+b}{2}\right)\right]dx\right]$$

and

$$\begin{aligned}\int_a^b\cosh\left[2p\left(x-\frac{a+b}{2}\right)\right]dx &= \frac{1}{2p}\sinh\left[2p\left(x-\frac{a+b}{2}\right)\right]\Big|_a^b \\ &= \frac{1}{2p}\sinh\left[2p\left(b-\frac{a+b}{2}\right)\right]-\frac{1}{2p}\sinh\left[2p\left(a-\frac{a+b}{2}\right)\right] \\ &= \frac{1}{p}\sinh\left[p\left(b-a\right)\right],\end{aligned}$$

hence

$$\int_a^b\cosh^2\left[p\left(x-\frac{a+b}{2}\right)\right]dx=\frac{1}{2}\left[\left(b-a\right)+\frac{1}{p}\sinh\left[p\left(b-a\right)\right]\right]$$

and by (2.11) we get (2.10). □

Corollary 2. *Assume that the function $f:I\to\mathbb{R}$ is hyperbolic p-convex on I and a, $b\in I$. Assume also that $q:\left[a,b\right]\to\mathbb{R}$ is a positive, symmetric and integrable function on $\left[a,b\right]$, then we have*

$$\text{(2.12)}\quad f\left(\frac{a+b}{2}\right)\leq\frac{1}{\int_a^b q\left(x\right)dx}\int_a^b f\left(x\right)q\left(x\right)\operatorname{sech}\left[p\left(x-\frac{a+b}{2}\right)\right]dx \leq \frac{f\left(a\right)+f\left(b\right)}{2}\operatorname{sech}\left[\frac{p\left(b-a\right)}{2}\right].$$

The proof follows by Theorem 2 for the positive symmetric mean $w(x) = q(x)\operatorname{sech}\left[p\left(x-\frac{a+b}{2}\right)\right]$, $x \in [a,b]$.

3. Related Results

We have:

Theorem 3. *Assume that the function $f : I \to \mathbb{R}$ is hyperbolic p-convex on I and a, $b \in I$. Assume also that $w : [a,b] \to \mathbb{R}$ is a positive, integrable function on $[a,b]$ and*

$$\overline{x}_p := \frac{1}{p}\operatorname{artanh}\left(\frac{\int_a^b \sinh(py)\,w(y)\,dy}{\int_a^b \cosh(py)\,w(y)\,dy}\right) \in [a,b], \tag{3.1}$$

then we have

$$\int_a^b f(t)\,w(t)\,dt \geq f(\overline{x}_p)\int_a^b \cosh[p(t-\overline{x}_p)]\,w(t)\,dt. \tag{3.2}$$

Proof. From the gradient inequality (1.5) we have, by multiplying with $w(y) \geq 0$ and integrating on $[a,b]$, that

$$\int_a^b f(y)\,w(y)\,dy \geq f(x)\int_a^b \cosh[p(y-x)]\,w(y)\,dy + K_{x,f}\int_a^b \sinh[p(y-x)]\,w(y)\,dy \tag{3.3}$$

for any $x \in [a,b]$.

We have

$$\begin{aligned}
&\int_a^b \sinh[p(y-x)]\,w(y)\,dy \\
&= \int_a^b [\sinh(py)\cosh(px) - \sinh(px)\cosh(py)]\,w(y)\,dy \\
&= \cosh(px)\int_a^b \sinh(py)\,w(y)\,dy - \sinh(px)\int_a^b \cosh(py)\,w(y)\,dy \\
&= \cosh(px)\int_a^b \cosh(py)\,w(y)\,dy\left(\frac{\int_a^b \sinh(py)\,w(y)\,dy}{\int_a^b \cosh(py)\,w(y)\,dy} - \tanh(px)\right)
\end{aligned}$$

for any $x \in [a,b]$.

From (3.1) we have

$$\tanh(p\overline{x}_p) = \frac{\int_a^b \sinh(py)\,w(y)\,dy}{\int_a^b \cosh(py)\,w(y)\,dy},$$

which implies that

$$\int_a^b \sinh[p(y-\overline{x}_p)]\,w(y)\,dy = 0.$$

Therefore, by taking $\overline{x}_p$ in (3.3) we get the desired result (3.1). □

The following result also hold.

Theorem 4. *Assume that the function $f : I \to \mathbb{R}$ is hyperbolic p-convex on I and $a, b \in I$. Assume also that $w : [a,b] \to \mathbb{R}$ is a positive, integrable function on $[a,b]$, then*

$$\begin{aligned}(3.4)\quad & \int_a^b f(x)\,w(x)\,dx \\ & \leq \frac{f(a)+f(b)}{2} \operatorname{sech}\left[\frac{p(b-a)}{2}\right] \int_a^b \cosh\left[p\left(x-\frac{a+b}{2}\right)\right] w(x)\,dx \\ & \quad + \frac{f(b)-f(a)}{2} \operatorname{csch}\left[\frac{p(b-a)}{2}\right] \int_a^b \sinh\left[p\left(x-\frac{a+b}{2}\right)\right] w(x)\,dx.\end{aligned}$$

Proof. We have

$$\begin{aligned}& \frac{\sinh[p(b-x)]f(a)+\sinh[p(x-a)]f(b)}{\sinh[p(b-a)]} \\ & - \frac{\sinh[p(b-x)]+\sinh[p(x-a)]}{\sinh[p(b-a)]}\frac{f(a)+f(b)}{2} \\ & = \frac{\sinh[p(b-x)]}{\sinh[p(b-a)]}\left(f(a)-\frac{f(a)+f(b)}{2}\right) \\ & + \frac{\sinh[p(x-a)]}{\sinh[p(b-a)]}\left(f(b)-\frac{f(a)+f(b)}{2}\right) \\ & = \frac{f(b)-f(a)}{2}\left[\frac{\sinh[p(x-a)]-\sinh[p(b-x)]}{\sinh[p(b-a)]}\right] \\ & = \frac{f(b)-f(a)}{2}\frac{2\sinh\left[p\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\frac{p(b-a)}{2}\right]}{2\sinh\left[\frac{p(b-a)}{2}\right]\cosh\left[\frac{p(b-a)}{2}\right]} \\ & = \frac{f(b)-f(a)}{2}\frac{\sinh\left[p\left(x-\frac{a+b}{2}\right)\right]}{\sinh\left[\frac{p(b-a)}{2}\right]}\end{aligned}$$

and

$$\begin{aligned}& \frac{\sinh[p(b-x)]+\sinh[p(x-a)]}{\sinh[p(b-a)]} \\ & = \frac{2\sinh\left[\frac{p(b-a)}{2}\right]\cosh\left[p\left(x-\frac{a+b}{2}\right)\right]}{2\sinh\left[\frac{p(b-a)}{2}\right]\cosh\left[\frac{p(b-a)}{2}\right]} = \frac{\cosh\left[p\left(x-\frac{a+b}{2}\right)\right]}{\cosh\left[\frac{p(b-a)}{2}\right]}\end{aligned}$$

for any $x \in [a,b]$.

Therefore

$$\begin{aligned}(3.5)\quad & \frac{\sinh[p(b-x)]f(a)+\sinh[p(x-a)]f(b)}{\sinh[p(b-a)]} \\ & = \frac{\cosh\left[p\left(x-\frac{a+b}{2}\right)\right]}{\cosh\left[\frac{p(b-a)}{2}\right]}\frac{f(a)+f(b)}{2} + \frac{f(b)-f(a)}{2}\frac{\sinh\left[p\left(x-\frac{a+b}{2}\right)\right]}{\sinh\left[\frac{p(b-a)}{2}\right]},\end{aligned}$$

for any $x \in [a,b]$.

If we multiply (3.5) by $w(x) \geq 0$ and integrate, then we get

$$\frac{f(a)\int_a^b \sinh[p(b-x)]\,w(x)\,dx + f(b)\int_a^b \sinh[p(x-a)]\,w(x)\,dx}{\sinh[p(b-a)]} \tag{3.6}$$
$$= \frac{f(a)+f(b)}{2}\operatorname{sech}\left[\frac{p(b-a)}{2}\right]\int_a^b \cosh\left[p\left(x-\frac{a+b}{2}\right)\right]w(x)\,dx$$
$$+\frac{f(b)-f(a)}{2}\operatorname{csch}\left[\frac{p(b-a)}{2}\right]\int_a^b \sinh\left[p\left(x-\frac{a+b}{2}\right)\right]w(x)\,dx.$$

Now, if we multiply the definition of hyperbolic p-convex functions by $w(x) \geq 0$ and integrate, then we get

$$\int_a^b f(x)\,w(x)\,dx$$
$$\leq \frac{f(a)\int_a^b \sinh[p(b-x)]\,w(x)\,dx + f(b)\int_a^b \sinh[p(x-a)]\,w(x)\,dx}{\sinh[p(b-a)]}$$

and by (3.6) we obtain the desired result (3.4). □

Corollary 3. *With the assumption of Theorem 4 and if*

$$\int_a^b \sinh\left[p\left(x-\frac{a+b}{2}\right)\right]w(x)\,dx = 0, \tag{3.7}$$

then the second inequality in (2.7) holds.

Remark 2. *Since* $g(x) = \sinh\left[p\left(x-\frac{a+b}{2}\right)\right]$ *is antisymmetric on* $[a,b]$, *then, if* w *is symmetric on* $[a,b]$, *the condition (3.7) holds and then the second inequality in (2.7) is valid. The condition (3.7) for the positive weight* w *is a more general condition for the second inequality in (2.7) to hold than the usual symmetry considered in Fejér's type inequalities.*

References

[1] M. S. S. Ali, On certain properties of trigonometrically ρ-convex functions, *Advances in Pure Mathematics*, 2012, 2, 337-340 http://dx.doi.org/10.4236/apm.2012.25047.

[2] M. S. S. Ali, On Hadamard's inequality for trigonometrically ρ-convex functions," accepted to appear in Theoretical Mathematics & Applications., March, 2013.

[3] M. S. S. Ali, On certain properties for two classes of generalized convex functions, Abstract and Applied Analysis Volume 2016, Article ID 4652038, 7 pages http://dx.doi.org/10.1155/2016/4652038

[4] E. F. Beckenbach, Convex functions, *Bulletin of the American Mathematical Society*, Vol. **54**, No. 5, 1948, pp. 439-460. doi:10.1090/S0002-9904-1948-08994-7

[5] M. Bessenyei, Hermite–Hadamard-type inequalities for generalized convex functions, *PhD Thesis*, Univ. Debrecen, Hungary, 2004. RGMIA Monograps, [http://rgmia.org/papers/monographs/dissertatio.pdf']

[6] F. F. Bonsall, The characterization of generalized convex functions, *The Quarterly Journal of Mathematics Oxford Series*, Vol. **1**, 1950, pp. 100-111. doi:10.1093/qmath/1.1.100

[7] A. M. Bruckner and E. Ostrow, Some functions classes related to the class of convex functions," *Pacific Journal of Mathematics*, Vol. **12**, 1962, pp. 1203-1215.

[8] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results, *Australian J. Math. Anal. Appl.*, Volume **14**, Issue 1, Article 1, pp. 1-287, 2017. [Online http://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex].

[9] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000. [Online http://rgmia.org/monographs/hermite_hadamard.html].

[10] L. Fejér, Über die Fourierreihen, II, *Math. Naturwiss, Anz. Ungar. Akad. Wiss.*, **24** (1906), 369-390. (In Hungarian).

[11] J. W. Green, Support, Convergence, and Differentiability Properties of Generalized Convex Functions, *Proceedings of the American Mathematical Society*, Vol. **4**, No. 3, 1953, pp. 391-396. doi:10.1090/S0002-9939-1953-0056039-2

[12] B. Ya. Levin, *Lectures on Entire Functions*, American Mathematical Society, 1996.

[13] L. S. Maergoiz, *Asymptotic Characteristics of Entire Functions and Their Applications in Mathematics and Biophysics*, Kluwer Academic Publishers, New York, 2003.

[14] M. J. Miles, An extremum property of convex functions, *American Mathematical Monthly*, Vol. **76**, 1969, pp. 921-922. doi:10.2307/2317948

[15] M. M. Peixoto, On the existence of derivatives of generalized convex functions, *Summa Brasilian Mathematics,* Vol. 2, No. 3, 1948, pp. 35-42.

[16] M. M. Peixoto, Generalized convex functions and second order differential inequlities, *Bulletin of the American Mathematical Society,* Vol. **55**, No. 6, 1949, pp. 563- 572. doi:10.1090/S0002-9904-1949-09246-7

[17] A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, New York, 1973.

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