TWO POINTS TAYLOR'S TYPE REPRESENTATIONS FOR ANALYTIC COMPLEX FUNCTIONS WITH INTEGRAL REMAINDERS

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Abstract. In this paper we establish some two point weighted Taylor's expansions for analytic functions $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ defined on a convex domain D. Some error bounds for these expansions are also provided. Examples for the complex logarithm and the complex exponential are also given.

1. Introduction

Let $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function on the convex domain D and z, $v \in D$, then we have the following Taylor's expansion with integral remainder

$$f(z) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(v)(z-v)^k + \frac{1}{n!}(z-v)^{n+1} \int_0^1 f^{(n+1)}[(1-s)v + sz](1-s)^n ds \tag{1.1}$$

for $n \geq 0$, see for instance [13].

In this paper we establish some two point weighted Taylor's expansions for analytic functions $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ defined on a convex domain D. Some error bounds for these expansions are also provided. Examples for the complex logarithm and the complex exponential are also given.

Consider the function $f(z) = \operatorname{Log}(z)$ where $\operatorname{Log}(z) = \ln|z| + i \operatorname{Arg}(z)$ and $\operatorname{Arg}(z)$ is such that $-\pi < \operatorname{Arg}(z) \leq \pi$. Log is called the *"principal branch"* of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_\ell := \mathbb{C} \setminus \{x + iy : x \leq 0,\ y = 0\}$ and

$$f^{(k)}(z) = \frac{(-1)^{k-1}(k-1)!}{z^k},\ k \geq 1,\ z \in \mathbb{C}_\ell.$$

Using the representation (1.1) we then have

$$\operatorname{Log}(z) = \operatorname{Log}(x) + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \left(\frac{z-v}{v}\right)^k + (-1)^n (z-v)^{n+1} \int_0^1 \frac{(1-s)^n ds}{[(1-s)v + sz]^{n+1}} \tag{1.2}$$

for all z, $v \in \mathbb{C}_\ell$ with $(1-s)v + sz \in \mathbb{C}_\ell$ for $s \in [0,1]$.

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Consider the complex exponential function $f(z)=\exp(z)$, then by (1.1) we get

$$\exp(z)=\sum_{k=0}^{n}\frac{1}{k!}(z-v)^{k}\exp(v)+\frac{1}{n!}(z-v)^{n+1}\int_{0}^{1}(1-s)^{n}\exp[(1-s)v+sz]\,ds \tag{1.3}$$

for all $z,\ v\in\mathbb{C}$.

For various inequalities related to Taylor's expansions for real functions see [1]-[12].

In this paper we establish some two point weighted Taylor's expansions for analytic functions $f:D\subseteq\mathbb{C}\to\mathbb{C}$ defined on a convex domain D. Some error bounds for these expansions are also provided. Examples for the complex logarithm and the complex exponential are also given.

2. Two Points Taylor's Expansions

We have:

Theorem 1. *Let $f:D\subseteq\mathbb{C}\to\mathbb{C}$ be an analytic function on the convex domain D and $z,\ v,\ w\in D$, then for all $\lambda\in\mathbb{C}$ we have*

$$\begin{aligned}f(z)&=(1-\lambda)f(v)+\lambda f(w)\\&+\sum_{k=1}^{n}\frac{1}{k!}\left[(1-\lambda)f^{(k)}(v)(z-v)^{k}+(-1)^{k}\lambda f^{(k)}(w)(w-z)^{k}\right]\\&+S_{n,\lambda}(z,v,w),\end{aligned} \tag{2.1}$$

where the remainder $S_{n,\lambda}(z,v,w)$ is given by

$$\begin{aligned}&S_{n,\lambda}(z,v,w)\\&:=\frac{1}{n!}\left[(1-\lambda)(z-v)^{n+1}\int_{0}^{1}f^{(n+1)}[(1-s)v+sz](1-s)^{n}\,ds\right.\\&\left.+(-1)^{n+1}\lambda(w-z)^{n+1}\int_{0}^{1}f^{(n+1)}[(1-s)z+sw]s^{n}ds\right].\end{aligned} \tag{2.2}$$

Proof. If we replace in (1.1) v by w, then we get

$$f(z)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(v)(z-w)^{k}+\frac{1}{n!}(z-w)^{n+1}\int_{0}^{1}f^{(n+1)}[(1-s)w+sz](1-s)^{n}\,ds \tag{2.3}$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(v)(w-z)^k + \frac{(-1)^{n+1}}{n!}(w-z)^{n+1} \int_0^1 f^{(n+1)}[(1-s)w+sz](1-s)^n ds$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(v)(w-z)^k + \frac{(-1)^{n+1}}{n!}(w-z)^{n+1} \int_0^1 f^{(n+1)}[(1-s)z+sw] s^n ds.$$

Assume that $\lambda \neq 1, 0$. If we multiply (1.1) by $1-\lambda$ and (2.3) by λ we get the desired representation (2.1) with the remainder $S_{n,\lambda}(z,v,w)$ given by (2.2).

If either $\lambda = 1$ or $\lambda = 0$, then the theorem also holds by the use of Taylor's usual expansion. □

Remark 1. *We observe that for* $n = 0$ *the representation from Theorem 1 becomes*

$$f(z) = (1-\lambda) f(v) + \lambda f(w) + S_\lambda(z,v,w), \tag{2.4}$$

where the remainder $S_\lambda(z,v,w)$ *is given by*

$$S_\lambda(z,v,w) := (1-\lambda)(z-v)\int_0^1 f'((1-s)v+sz)\,ds - \lambda(w-z)\int_0^1 f'((1-s)z+sw)\,ds. \tag{2.5}$$

Remark 2. *If we take in (2.3)* $z = \frac{v+w}{2}$*, with* $v, w \in D$*, then we have for any* $\lambda \in \mathbb{C}$ *that*

$$f\left(\frac{v+w}{2}\right) = (1-\lambda) f(v) + \lambda f(w) + \sum_{k=1}^{n} \frac{1}{2^k k!}\left[(1-\lambda) f^{(k)}(v) + (-1)^k \lambda f^{(k)}(w)\right](w-v)^k + \tilde{S}_{n,\lambda}(v,w), \tag{2.6}$$

where the remainder $\tilde{S}_{n,\lambda}(v,w)$ *is given by*

$$\tilde{S}_{n,\lambda}(v,w) := \frac{1}{2^{n+1} n!}(w-v)^{n+1}\left[(1-\lambda)\int_0^1 f^{(n+1)}\left((1-s)v + s\frac{v+w}{2}\right)(1-s)^n ds + (-1)^{n+1}\lambda \int_0^1 f^{(n+1)}\left((1-s)\frac{v+w}{2} + sw\right) s^n ds\right]. \tag{2.7}$$

In particular, for $\lambda = \frac{1}{2}$ *in (2.6) we have*

$$f\left(\frac{v+w}{2}\right) = \frac{f(v)+f(w)}{2} + \sum_{k=1}^{n} \frac{1}{2^{k+1} k!}\left[f^{(k)}(v) + (-1)^k f^{(k)}(w)\right](w-v)^k + \tilde{S}_n(v,w), \tag{2.8}$$

where the remainder $\tilde{S}_n(v,w)$ is given by

$$\tilde{S}_n(v,w) := \frac{1}{2^{n+2}n!}(w-v)^{n+1}\left[\int_0^1 f^{(n+1)}\left((1-s)v+s\frac{v+w}{2}\right)(1-s)^n ds + (-1)^{n+1}\int_0^1 f^{(n+1)}\left((1-s)\frac{v+w}{2}+sw\right)s^n ds\right]. \tag{2.9}$$

Now, by the change of variable in (2.9) we also get the following representation for the remainder $\tilde{S}_n(v,w)$ as a single integral

$$\tilde{S}_n(v,w) := \frac{1}{2^{n+2}n!}(w-v)^{n+1} \times\int_0^1\left[f^{(n+1)}\left(sv+(1-s)\frac{v+w}{2}\right)+(-1)^{n+1}f^{(n+1)}\left((1-s)\frac{v+w}{2}+sw\right)\right]s^n ds, \tag{2.10}$$

for $n\geq 0$.

Corollary 1. *With the assumptions in Theorem 1 we have for each distinct z, v, $w\in D$ with $w\neq v$*

$$f(z) = \frac{1}{w-v}[(w-z)f(v)+(z-v)f(w)] + \frac{(w-z)(z-v)}{w-v} \times\sum_{k=1}^{n}\frac{1}{k!}\left\{(z-v)^{k-1}f^{(k)}(v)+(-1)^k(w-z)^{k-1}f^{(k)}(w)\right\} + L_n(z,v,w), \tag{2.11}$$

where

$$L_n(z,v,w) := \frac{(w-z)(z-v)}{n!(w-v)}\left[(z-v)^n\int_0^1 f^{(n+1)}((1-s)v+sz)(1-s)^n ds + (-1)^{n+1}(w-z)^n\int_0^1 f^{(n+1)}((1-s)z+sw)s^n ds\right]$$

and

$$f(z) = \frac{1}{w-v}[(z-v)f(v)+(w-z)f(w)] + \frac{1}{w-v}\sum_{k=1}^{n}\frac{1}{k!}\left\{(z-v)^{k+1}f^{(k)}(v)+(-1)^k(w-z)^{k+1}f^{(k)}(w)\right\} + P_n(z,v,w), \tag{2.12}$$

where

$$P_n(z,v,w) := \frac{1}{n!(w-v)}\left[(z-v)^{n+2}\int_0^1 f^{(n+1)}((1-s)v+sz)(1-s)^n ds + (-1)^{n+1}(w-z)^{n+2}\int_0^1 f^{(n+1)}((1-s)z+sw)s^n ds\right],$$

respectively.

The proof is obvious, by choosing $\lambda=(z-v)/(w-v)$ and $\lambda=(w-z)/(w-v)$, respectively, in Theorem 1. The details are omitted.

Corollary 2. *With the assumption in Theorem 1 we have for each $\lambda \in [0,1]$ and any distinct v, $w \in D$ that*

$$\begin{aligned} &f\left((1-\lambda)v+\lambda w\right)=(1-\lambda)f(v)+\lambda f(w)+\lambda(1-\lambda) \\ &\quad\times\sum_{k=1}^{n}\frac{1}{k!}\left[\lambda^{k-1}f^{(k)}(v)+(-1)^{k}(1-\lambda)^{k-1}f^{(k)}(w)\right](w-v)^{k}+S_{n,\lambda}(v,w), \end{aligned} \tag{2.13}$$

where the remainder $S_{n,\lambda}(v,w)$ is given by

$$\begin{aligned} &S_{n,\lambda}(v,w) \\ &:=\frac{1}{n!}(1-\lambda)\lambda(w-v)^{n+1}\left[\lambda^{n}\int_{0}^{1}f^{(n+1)}\left((1-s\lambda)v+s\lambda w\right)(1-s)^{n}ds\right. \\ &\left.+(-1)^{n+1}(1-\lambda)^{n}\int_{0}^{1}f^{(n+1)}\left((1-s-\lambda+s\lambda)v+(\lambda+s-s\lambda)w\right)s^{n}ds\right]. \end{aligned} \tag{2.14}$$

We also have

$$\begin{aligned} &f\left((1-\lambda)w+\lambda v\right)=(1-\lambda)f(v)+\lambda f(w) \\ &+\sum_{k=1}^{n}\frac{1}{k!}\left[(1-\lambda)^{k+1}f^{(k)}(v)+(-1)^{k}\lambda^{k+1}f^{(k)}(w)\right](w-v)^{k}+P_{n,\lambda}(v,w), \end{aligned} \tag{2.15}$$

where the remainder $P_{n,\lambda}(v,w)$ is given by

$$\begin{aligned} &P_{n,\lambda}(v,w) \\ &:=\frac{1}{n!}(w-v)^{n+1}\left[(1-\lambda)^{n+2}\int_{0}^{1}f^{(n+1)}\left((1-s+\lambda s)v+(1-\lambda)sw\right)(1-s)^{n}ds\right. \\ &\qquad\left.+(-1)^{n+1}\lambda^{n+2}\int_{0}^{1}f^{(n+1)}\left((1-s)\lambda v+(1-\lambda+\lambda s)w\right)s^{n}ds\right]. \end{aligned} \tag{2.16}$$

The case $n=0$ produces the following simple identities for each distinct z, v, $w \in D$ and $\lambda \in \mathbb{C}$

$$f(z)=(1-\lambda)f(v)+\lambda f(w)+S_{\lambda}(z,v,w), \tag{2.17}$$

where the remainder $S_{\lambda}(z,v,w)$ is given by

$$\begin{aligned} S_{\lambda}(z,v,w)&:=(1-\lambda)(z-v)\int_{0}^{1}f'\left((1-s)v+sz\right)ds \\ &\qquad-\lambda(w-z)\int_{0}^{1}f'\left((1-s)z+sw\right)ds. \end{aligned} \tag{2.18}$$

We then have for each distinct z, v, $w \in D$

$$f(z)=\frac{1}{w-v}\left[(w-z)f(v)+(z-v)f(w)\right]+L(z,v,w), \tag{2.19}$$

where

$$\begin{aligned} &L(z,v,w) \\ &:=\frac{(w-z)(z-v)}{w-v}\left[\int_{0}^{1}f'\left((1-s)v+sz\right)ds-\int_{0}^{1}f'\left((1-s)z+sw\right)ds\right] \end{aligned} \tag{2.20}$$

and

$$f(z)=\frac{1}{w-v}\left[(z-v)f(v)+(w-z)f(w)\right]+P(z,v,w), \tag{2.21}$$

where

$$\begin{aligned} &P(z,v,w) \\ &:=\frac{1}{w-v}\left[(z-v)^2\int_0^1 f'((1-s)v+sz)\,ds-(w-z)^2\int_0^1 f'((1-s)z+sw)\,ds\right]. \end{aligned} \tag{2.22}$$

We also have for $\lambda\in[0,1]$

$$f((1-\lambda)v+\lambda w)=(1-\lambda)f(v)+\lambda f(w)+S_\lambda(v,w), \tag{2.23}$$

where the remainder $S_\lambda(v,w)$ is given by

$$\begin{aligned} S_\lambda(v,w):=(1-\lambda)\lambda(w-v)\Bigg[&\int_0^1 f'((1-s\lambda)v+s\lambda w)\,ds \\ &-\int_0^1 f'((1-s-\lambda+s\lambda)v+(\lambda+s-s\lambda)w)\,ds\Bigg] \end{aligned} \tag{2.24}$$

and

$$f((1-\lambda)w+\lambda v)=(1-\lambda)f(v)+\lambda f(w)+P_\lambda(v,w), \tag{2.25}$$

where the remainder $P_\lambda(v,w)$ is given by

$$\begin{aligned} P_\lambda(v,w):=(w-v)\Bigg[&(1-\lambda)^2\int_0^1 f'((1-s+\lambda s)v+(1-\lambda)sw)\,ds \\ &-\lambda^2\int_0^1 f'((1-s)\lambda v+(1-\lambda+\lambda s)w)\,ds\Bigg]. \end{aligned} \tag{2.26}$$

Moreover, if we take in (2.17) $z=\frac{v+w}{2}$ for each distinct v, $w\in D$ and $\lambda\in\mathbb{C}$, then we have

$$f\left(\frac{v+w}{2}\right)=(1-\lambda)f(v)+\lambda f(w)+S_\lambda(v,w), \tag{2.27}$$

where the remainder $S_\lambda(v,w)$ is given by

$$\begin{aligned} &S_\lambda(v,w):=\frac{1}{2}(w-v) \\ &\times\left[(1-\lambda)\int_0^1 f'\left((1-s)v+s\frac{v+w}{2}\right)ds-\lambda\int_0^1 f'\left((1-s)\frac{v+w}{2}+sw\right)ds\right]. \end{aligned} \tag{2.28}$$

In particular, for $\lambda=\frac{1}{2}$ we have

$$f\left(\frac{v+w}{2}\right)=\frac{f(v)+f(w)}{2}+S(v,w), \tag{2.29}$$

where

$$\begin{aligned} S(v,w)&:=\frac{1}{4}(w-v) \\ &\times\left[\int_0^1 f'\left((1-s)v+s\frac{v+w}{2}\right)ds-\int_0^1 f'\left((1-s)\frac{v+w}{2}+sw\right)ds\right]. \end{aligned} \tag{2.30}$$

Now, assume that z, v, $w \in D \subset \mathbb{C}_\ell$, with D a convex set, then for all $\lambda \in \mathbb{C}$ we have by Theorem 1 for the function $f(z) = \operatorname{Log}(z)$ that

$$\begin{aligned}\operatorname{Log}(z) &= (1-\lambda)\operatorname{Log}(v) + \lambda \operatorname{Log}(w) \\ &+ \sum_{k=1}^{n} \frac{1}{k}\left[(1-\lambda)(-1)^{k-1}\frac{(z-v)^k}{v^k} - \lambda \frac{(w-z)^k}{w^k}\right] \\ &+ S_{n,\lambda}(z,v,w),\end{aligned} \tag{2.31}$$

where the remainder $\Lambda_{n,\lambda}(z,v,w)$ is given by

$$\begin{aligned}\Lambda_{n,\lambda}(z,v,w) &:= (1-\lambda)(z-v)^{n+1}(-1)^n \int_0^1 \frac{(1-s)^n}{((1-s)v+sz)^{n+1}}ds \\ &- \lambda (w-z)^{n+1} \int_0^1 \frac{s^n}{((1-s)z+sw)^{n+1}}ds\end{aligned} \tag{2.32}$$

for $n \geq 0$.

Consider the function $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = \exp z$, then for z, v, $w \in \mathbb{C}$ we have by Theorem 1 that

$$\begin{aligned}\exp z &= (1-\lambda)\exp v + \lambda \exp w \\ &+ \sum_{k=1}^{n} \frac{1}{k!}\left[(1-\lambda)(z-v)^k \exp v + (-1)^k \lambda (w-z)^k \exp w\right] \\ &+ \Theta_{n,\lambda}(z,v,w),\end{aligned} \tag{2.33}$$

where the remainder $\Theta_{n,\lambda}(z,v,w)$ is given by

$$\begin{aligned}&\Theta_{n,\lambda}(z,v,w) \\ &:= \frac{1}{n!}\left[(1-\lambda)(z-v)^{n+1}\int_0^1 \exp((1-s)v+sz)(1-s)^n ds\right. \\ &\left.+ (-1)^{n+1}\lambda(w-z)^{n+1}\int_0^1 \exp((1-s)z+sw)s^n ds\right].\end{aligned} \tag{2.34}$$

for $n \geq 0$ and for all $\lambda \in \mathbb{C}\setminus\{0,1\}$.

3. Some Inequalities

We can state now some results concerning error bounds in approximating an analytic function by two points Taylor's expansion:

Theorem 2. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and z, v, $w \in D$, then for all $\lambda \in \mathbb{C}$ we have*

$$\begin{aligned}f(z) &= (1-\lambda) f(v) + \lambda f(w) \\ &+ \sum_{k=1}^{n} \frac{1}{k!}\left[(1-\lambda) f^{(k)}(v)(z-v)^k + (-1)^k \lambda f^{(k)}(w)(w-z)^k\right] \\ &+ S_{n,\lambda}(z,v,w),\end{aligned} \tag{3.1}$$

and the remainder $S_{n,\lambda}(z,v,w)$ *satisfies the inequalities*

$$
\begin{aligned}
&|S_{n,\lambda}(z,v,w)| \\
&\leq \frac{1}{n!}|1-\lambda||z-v|^{n+1}\int_0^1 \left|f^{(n+1)}((1-s)v+sz)\right|(1-s)^n ds \\
&\quad + \frac{1}{n!}|\lambda||w-z|^{n+1}\int_0^1 \left|f^{(n+1)}((1-s)z+sw)\right| s^n ds \\
&\leq \frac{1}{n!}|1-\lambda||z-v|^{n+1}\begin{cases} \frac{1}{n+1}\sup_{s\in[0,1]}\left|f^{(n+1)}((1-s)v+sz)\right| \\ \\ \frac{1}{(qn+1)^{1/q}}\left(\int_0^1\left|f^{(n+1)}((1-s)v+sz)\right|^p ds\right)^{1/p} \\ \text{where } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \int_0^1\left|f^{(n+1)}((1-s)v+sz)\right| ds \end{cases} \\
&\quad + \frac{1}{n!}|\lambda||w-z|^{n+1}\begin{cases} \frac{1}{n+1}\sup_{s\in[0,1]}\left|f^{(n+1)}((1-s)z+sw)\right| \\ \\ \frac{1}{(qn+1)^{1/q}}\left(\int_0^1\left|f^{(n+1)}((1-s)z+sw)\right|^p ds\right)^{1/p} \\ \text{where } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \int_0^1\left|f^{(n+1)}((1-s)z+sw)\right| ds \end{cases}
\end{aligned} \tag{3.2}
$$

for $n \geq 0$.

Proof. Taking the modulus in the representation (2.2), we get

$$
\begin{aligned}
&|S_{n,\lambda}(z,v,w)| \\
&\leq \frac{1}{n!}\left[\left|(1-\lambda)(z-v)^{n+1}\int_0^1 f^{(n+1)}((1-s)v+sz)(1-s)^n ds\right|\right. \\
&\left. + \left|\lambda(w-z)^{n+1}\int_0^1 f^{(n+1)}((1-s)z+sw)s^n ds\right|\right] \\
&\leq |1-\lambda||z-v|^{n+1}\int_0^1\left|f^{(n+1)}((1-s)v+sz)\right|(1-s)^n ds \\
&+ |\lambda||w-z|^{n+1}\int_0^1\left|f^{(n+1)}((1-s)z+sw)\right| s^n ds.
\end{aligned} \tag{3.3}
$$

By Hölder's integral inequality we have

$$\int_0^1\left|f^{(n+1)}((1-s)v+sz)\right|(1-s)^n ds$$

$$\leq \begin{cases} \sup_{s\in[0,1]} \left|f^{(n+1)}\left((1-s)v+sz\right)\right| \int_0^1 (1-s)^n\,ds \\ \\ \left(\int_0^1 \left|f^{(n+1)}\left((1-s)v+sz\right)\right|^p ds\right)^{1/p} \left(\int_0^1 (1-s)^{qn}\,ds\right)^{1/q} \\ \text{where } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \sup_{s\in[0,1]} \left\{(1-s)^n\right\} \int_0^1 \left|f^{(n+1)}\left((1-s)v+sz\right)\right| ds \end{cases}$$

$$= \begin{cases} \frac{1}{n+1} \sup_{s\in[0,1]} \left|f^{(n+1)}\left((1-s)v+sz\right)\right| \\ \\ \frac{1}{(qn+1)^{1/q}} \left(\int_0^1 \left|f^{(n+1)}\left((1-s)v+sz\right)\right|^p ds\right)^{1/p} \\ \text{where } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \int_0^1 \left|f^{(n+1)}\left((1-s)v+sz\right)\right| ds \end{cases}$$

and

$$\int_0^1 \left|f^{(n+1)}\left((1-s)z+sw\right)\right| s^n ds$$

$$\leq \begin{cases} \sup_{s\in[0,1]} \left|f^{(n+1)}\left((1-s)z+sw\right)\right| \int_0^1 s^n ds \\ \\ \left(\int_0^1 \left|f^{(n+1)}\left((1-s)z+sw\right)\right|^p ds\right)^{1/p} \left(\int_0^1 s^{qn} ds\right)^{1/q} \\ \text{where } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \sup_{s\in[0,1]} \left\{s^n\right\} \int_0^1 \left|f^{(n+1)}\left((1-s)z+sw\right)\right| ds \end{cases}$$

$$= \begin{cases} \frac{1}{n+1} \sup_{s\in[0,1]} \left|f^{(n+1)}\left((1-s)z+sw\right)\right| \\ \\ \frac{1}{(qn+1)^{1/q}} \left(\int_0^1 \left|f^{(n+1)}\left((1-s)z+sw\right)\right|^p ds\right)^{1/p} \\ \text{where } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \int_0^1 \left|f^{(n+1)}\left((1-s)z+sw\right)\right| ds, \end{cases}$$

which proves the second inequality in (3.2). $\square$

Corollary 3. *With the assumptions of Theorem 2 and if*

$$\left\|f^{(n+1)}\right\|_{D,\infty} := \sup_{y\in D} \left|f^{(n+1)}(y)\right| < \infty,$$

then we have the simple bound

$$|S_{n,\lambda}(z,v,w)| \leq \frac{1}{(n+1)!} \left\|f^{(n+1)}\right\|_{D,\infty} \left(|1-\lambda|\,|z-v|^{n+1} + |\lambda|\,|w-z|^{n+1}\right) \tag{3.4}$$

for $n \geq 0$.

Remark 3. *If we take $z=\frac{v+w}{2}$, with $v,\ w\in D$, then we have for any $\lambda\in\mathbb{C}$ that*

$$f\left(\frac{v+w}{2}\right)=(1-\lambda)\,f(v)+\lambda f(w)+\sum_{k=1}^{n}\frac{1}{2^{k}k!}\left[(1-\lambda)\,f^{(k)}(v)+(-1)^{k}\lambda f^{(k)}(w)\right](w-v)^{k}+\tilde{S}_{n,\lambda}(v,w)\,, \tag{3.5}$$

and if $\left\|f^{(n+1)}\right\|_{D,\infty}:=\sup_{y\in D}\left|f^{(n+1)}(y)\right|<\infty$, then by (3.4) we get

$$\left|\tilde{S}_{n,\lambda}(v,w)\right|\leq\frac{1}{2^{n+1}(n+1)!}\left\|f^{(n+1)}\right\|_{D,\infty}(|1-\lambda|+|\lambda|)\,|w-v|^{n+1}$$

or any $\lambda\in\mathbb{C}$.

In particular, if $\lambda=\frac{1}{2}$, then we have

$$f\left(\frac{v+w}{2}\right)=\frac{f(v)+f(w)}{2}+\sum_{k=1}^{n}\frac{1}{2^{k+1}k!}\left[f^{(k)}(v)+(-1)^{k}f^{(k)}(w)\right](w-v)^{k}+\tilde{S}_{n}(v,w)\,, \tag{3.6}$$

and the remainder $\tilde{S}_{n}(v,w)$ satisfies the bound

$$\left|\tilde{S}_{n}(v,w)\right|\leq\frac{1}{2^{n+1}(n+1)!}\left\|f^{(n+1)}\right\|_{D,\infty}|w-v|^{n+1}$$

for $n\geq 0$.

Remark 4. *The case $n=0$ provides some simple inequalities as follows*

$$\begin{aligned}
&|f(z)-(1-\lambda)\,f(v)-\lambda f(w)| \tag{3.7}\\
&\leq|1-\lambda|\,|z-v|\int_{0}^{1}|f'((1-s)\,v+sz)|\,ds++|\lambda|\,|w-z|\int_{0}^{1}|f'((1-s)\,z+sw)|\,ds\\
&\leq|1-\lambda|\,|z-v|\begin{cases}\sup_{s\in[0,1]}|f'((1-s)\,v+sz)|\\[2mm] \left(\int_{0}^{1}|f'((1-s)\,v+sz)|^{p}\,ds\right)^{1/p}\\ \text{where } p>1\\[2mm] \int_{0}^{1}|f'((1-s)\,v+sz)|\,ds\end{cases}\\
&\quad+|\lambda|\,|w-z|\begin{cases}\sup_{s\in[0,1]}|f'((1-s)\,z+sw)|\\[2mm] \left(\int_{0}^{1}|f'((1-s)\,z+sw)|^{p}\,ds\right)^{1/p}\\ \text{where } p>1\\[2mm] \int_{0}^{1}|f'((1-s)\,z+sw)|\,ds\end{cases}
\end{aligned}$$

where $f:D\subseteq\mathbb{C}\to\mathbb{C}$ is an analytic function on the convex domain D, $z,\ v,\ w\in D$ and $\lambda\in\mathbb{C}$.

If

$$\|f'\|_{D,\infty} := \sup_{y\in D} |f'(y)| < \infty,$$

then we have the simple bound

$$|f(z) - (1-\lambda) f(v) - \lambda f(w)| \le \|f'\|_{D,\infty} (|1-\lambda| |z-v| + |\lambda| |w-z|) \tag{3.8}$$

for $z,\ v,\ w \in D$ and $\lambda \in \mathbb{C}$.

If we take $z = \frac{v+w}{2}$, with $v,\ w \in D$, then we have for any $\lambda \in \mathbb{C}$ that

$$\left| f\left(\frac{v+w}{2}\right) - (1-\lambda) f(v) - \lambda f(w) \right| \le \frac{1}{2} \|f'\|_{D,\infty} (|1-\lambda| + |\lambda|) |w-v|, \tag{3.9}$$

which for $\lambda = \frac{1}{2}$ gives the simple inequality

$$\left| f\left(\frac{v+w}{2}\right) - \frac{f(v)+f(w)}{2} \right| \le \frac{1}{2} \|f'\|_{D,\infty} |w-v|. \tag{3.10}$$

If n is even, namely $n = 2m$, $m \ge 0$, then by (2.8) we have the representation

$$\begin{aligned} f\left(\frac{v+w}{2}\right) &= \frac{f(v)+f(w)}{2} \\ &+ \sum_{k=1}^{2m} \frac{1}{2^{k+1} k!} \left[f^{(k)}(v) + (-1)^k f^{(k)}(w) \right] (w-v)^k \\ &+ \tilde{S}_{2m}(v,w), \end{aligned} \tag{3.11}$$

where the remainder $\tilde{S}_{2m}(v,w)$ is given by (2.10) as

$$\begin{aligned} \tilde{S}_{2m}(v,w) &:= \frac{1}{2^{2m+2}(2m)!} (w-v)^{2m+1} \\ \times \int_0^1 &\left[f^{(2m+1)}\left(sv + (1-s)\frac{v+w}{2}\right) - f^{(2m+1)}\left((1-s)\frac{v+w}{2} + sw\right) \right] s^{2m} ds, \end{aligned} \tag{3.12}$$

We also have the following result:

Theorem 3. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and $v,\ w \in D$. Let $m \ge 0$ and assume that*

$$\left| f^{(2m+1)}(z) - f^{(2m+1)}(y) \right| \le L_{2m+1} |z-y| \text{ for all } z,\ y \in D \tag{3.13}$$

for some $L_{2m+1} > 0$, namely that $f^{(2m+1)}$ is Lipschitzian on D. Then we have the representation (3.11) and the remainder $\tilde{S}_{2m}(v,w)$ satisfies the bound

$$\left| \tilde{S}_{2m}(v,w) \right| \le \frac{1}{2^{2m+2}(2m+2)(2m)!} |w-v|^{2m+2} L_{2m+1} \tag{3.14}$$

Proof. By taking the modulus in (3.12), we have

$$\begin{aligned}
\left|\tilde{S}_{2m}(v,w)\right| &\leq \frac{1}{2^{2m+2}(2m)!}|w-v|^{2m+1} \\
&\times \int_0^1 \left|f^{(2m+1)}\left(sv+(1-s)\frac{v+w}{2}\right)-f^{(2m+1)}\left((1-s)\frac{v+w}{2}+sw\right)\right|s^{2m}ds \\
&\leq \frac{1}{2^{2m+2}(2m)!}|w-v|^{2m+1} \\
&\times L_{2m+1}\int_0^1 \left|sv+(1-s)\frac{v+w}{2}-(1-s)\frac{v+w}{2}-sw\right|s^{2m}ds \\
&= \frac{1}{2^{2m+2}(2m)!}|w-v|^{2m+2}L_{2m+1}\int_0^1 s^{2m+1}ds \\
&= \frac{1}{2^{2m+2}(2m+2)(2m)!}|w-v|^{2m+2}L_{2m+1},
\end{aligned}$$

which proves the desired result (3.14). □

Corollary 4. *Let $f:D\subseteq\mathbb{C}\to\mathbb{C}$ be an analytic function on the convex domain D and $v,\ w\in D$. Assume that*

$$|f'(z)-f'(y)|\leq L|z-y| \text{ for all } z,\ y\in D \tag{3.15}$$

for some $L>0$. Then we have the inequality

$$\left|f\left(\frac{v+w}{2}\right)-\frac{f(v)+f(w)}{2}\right|\leq\frac{1}{8}L|w-v|^2. \tag{3.16}$$

4. Inequalities for Convex Derivatives in Absolute Value

We have:

Theorem 4. *Let $f:D\subseteq\mathbb{C}\to\mathbb{C}$ be an analytic function on the convex domain D and such that for a given $n\geq0$, $\left|f^{(n+1)}\right|$ is convex on D. If $z,\ v,\ w\in D$, then for all $\lambda\in\mathbb{C}$ we have*

$$\begin{aligned}
f(z) &= (1-\lambda)f(v)+\lambda f(w) \\
&+\sum_{k=1}^{n}\frac{1}{k!}\left[(1-\lambda)f^{(k)}(v)(z-v)^k+(-1)^k\lambda f^{(k)}(w)(w-z)^k\right] \\
&+S_{n,\lambda}(z,v,w),
\end{aligned} \tag{4.1}$$

and the remainder $S_{n,\lambda}(z,v,w)$ satisfies the inequality

$$\begin{aligned}
|S_{n,\lambda}(z,v,w)| &\leq \frac{1}{n!(n+2)}\left[|1-\lambda||z-v|^{n+1}\left|f^{(n+1)}(v)\right|\right. \\
&+\frac{1}{(n+1)}\left[|1-\lambda||z-v|^{n+1}+|\lambda||w-z|^{n+1}\right]\left|f^{(n+1)}(z)\right| \\
&\left.+|\lambda||w-z|^{n+1}\left|f^{(n+1)}(w)\right|\right].
\end{aligned} \tag{4.2}$$

Proof. Using the representation (2.2), we get

$$
\begin{aligned}
&\left|S_{n,\lambda}(z,v,w)\right| \\
&\leq \frac{1}{n!}\left[\left|1-\lambda\right|\left|z-v\right|^{n+1}\left|\int_0^1 f^{(n+1)}\left((1-s)v+sz\right)(1-s)^n\,ds\right|\right. \\
&\left.+\left|\lambda\right|\left|w-z\right|^{n+1}\left|\int_0^1 f^{(n+1)}\left((1-s)z+sw\right)s^n ds\right|\right] \\
&\leq \frac{1}{n!}\left[\left|1-\lambda\right|\left|z-v\right|^{n+1}\int_0^1 \left|f^{(n+1)}\left((1-s)v+sz\right)\right|(1-s)^n\,ds\right. \\
&\left.+\left|\lambda\right|\left|w-z\right|^{n+1}\int_0^1 \left|f^{(n+1)}\left((1-s)z+sw\right)\right|s^n ds\right] \\
&=: A_n(\lambda,w)
\end{aligned} \tag{4.3}
$$

By the convexity of $\left|f^{(n+1)}\right|$ we have

$$
\begin{aligned}
&\int_0^1 \left|f^{(n+1)}\left((1-s)v+sz\right)\right|(1-s)^n\,ds \\
&\leq \int_0^1 \left[(1-s)\left|f^{(n+1)}(v)\right|+s\left|f^{(n+1)}(z)\right|\right](1-s)^n\,ds \\
&= \left|f^{(n+1)}(v)\right|\int_0^1 (1-s)^{n+1}\,ds+\left|f^{(n+1)}(z)\right|\int_0^1 s(1-s)^n\,ds \\
&= \frac{1}{n+2}\left|f^{(n+1)}(v)\right|+\left|f^{(n+1)}(z)\right|\int_0^1 (1-s)s^n ds \\
&= \frac{1}{n+2}\left|f^{(n+1)}(v)\right|+\left|f^{(n+1)}(z)\right|\left(\frac{1}{n+1}-\frac{1}{n+2}\right) \\
&= \frac{1}{n+2}\left|f^{(n+1)}(v)\right|+\frac{1}{(n+1)(n+2)}\left|f^{(n+1)}(z)\right|
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 \left|f^{(n+1)}\left((1-s)z+sw\right)\right|s^n ds \\
&\leq \int_0^1 \left[(1-s)\left|f^{(n+1)}(z)\right|+s\left|f^{(n+1)}(w)\right|\right]s^n ds \\
&= \left|f^{(n+1)}(z)\right|\int_0^1 (1-s)s^n ds+\left|f^{(n+1)}(w)\right|\int_0^1 s^{n+1}ds \\
&= \frac{1}{(n+1)(n+2)}\left|f^{(n+1)}(z)\right|+\frac{1}{n+2}\left|f^{(n+1)}(w)\right|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&A_n(\lambda,w) \\
&\leq \frac{1}{n!}\left[\left|1-\lambda\right|\left|z-v\right|^{n+1}\left[\frac{1}{n+2}\left|f^{(n+1)}(v)\right|+\frac{1}{(n+1)(n+2)}\left|f^{(n+1)}(z)\right|\right]\right. \\
&\left.+\left|\lambda\right|\left|w-z\right|^{n+1}\left[\frac{1}{(n+1)(n+2)}\left|f^{(n+1)}(z)\right|+\frac{1}{n+2}\left|f^{(n+1)}(w)\right|\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{n!\,(n+2)}\left[|1-\lambda|\,|z-v|^{n+1}\left[\left|f^{(n+1)}(v)\right| + \frac{1}{(n+1)}\left|f^{(n+1)}(z)\right|\right]\right. \\
&\left. + |\lambda|\,|w-z|^{n+1}\left[\frac{1}{(n+1)}\left|f^{(n+1)}(z)\right| + \left|f^{(n+1)}(w)\right|\right]\right] \\
&= \frac{1}{n!\,(n+2)}\left[|1-\lambda|\,|z-v|^{n+1}\left|f^{(n+1)}(v)\right|\right. \\
&+ \frac{1}{(n+1)}\left[|1-\lambda|\,|z-v|^{n+1} + |\lambda|\,|w-z|^{n+1}\right]\left|f^{(n+1)}(z)\right| \\
&\left. + |\lambda|\,|w-z|^{n+1}\left|f^{(n+1)}(w)\right|\right],
\end{aligned}
$$

which together with (4.3) produce the desired result (4.3). □

Remark 5. *Assume that for a given $n \geq 0$, $\left|f^{(n+1)}\right|$ is convex on D. If we take in (4.1) $z = \frac{v+w}{2}$, with v, $w \in D$, then we have for any $\lambda \in \mathbb{C}$ that*

$$
\begin{aligned}
f\left(\frac{v+w}{2}\right) &= (1-\lambda) f(v) + \lambda f(w) \\
&+ \sum_{k=1}^{n} \frac{1}{2^k k!}\left[(1-\lambda) f^{(k)}(v) + (-1)^k \lambda f^{(k)}(w)\right](w-v)^k \\
&+ \tilde{S}_{n,\lambda}(v,w),
\end{aligned}
\tag{4.4}
$$

where the remainder $\tilde{S}_{n,\lambda}(v,w)$ satisfies the bound

$$
\begin{aligned}
\left|\tilde{S}_{n,\lambda}(v,w)\right| &\leq \frac{1}{2^{n+1} n!\,(n+2)}\,|w-v|^{n+1}\left[|1-\lambda|\left|f^{(n+1)}(v)\right|\right. \\
&\left. + \frac{1}{(n+1)}\left[|1-\lambda| + |\lambda|\right]\left|f^{(n+1)}\left(\frac{v+w}{2}\right)\right| + |\lambda|\left|f^{(n+1)}(w)\right|\right].
\end{aligned}
\tag{4.5}
$$

In particular, for $\lambda = \frac{1}{2}$ in (4.5) we have

$$
\begin{aligned}
f\left(\frac{v+w}{2}\right) &= \frac{f(v) + f(w)}{2} \\
&+ \sum_{k=1}^{n} \frac{1}{2^{k+1} k!}\left[f^{(k)}(v) + (-1)^k f^{(k)}(w)\right](w-v)^k \\
&+ \tilde{S}_n(v,w),
\end{aligned}
\tag{4.6}
$$

where the remainder $\tilde{S}_n(v,w)$ satisfies the bound

$$
\begin{aligned}
\left|\tilde{S}_n(v,w)\right| &\leq \frac{1}{2^{n+1} n!\,(n+2)}\,|w-v|^{n+1}\left[\frac{1}{2}\left|f^{(n+1)}(v)\right|\right. \\
&\left. + \frac{1}{(n+1)}\left|f^{(n+1)}\left(\frac{v+w}{2}\right)\right| + \frac{1}{2}\left|f^{(n+1)}(w)\right|\right].
\end{aligned}
\tag{4.7}
$$

Corollary 5. *With the assumption in Theorem 4 we have for each* $\lambda \in [0,1]$ *and any distinct* v, $w \in D$ *that*

$$f((1-\lambda)v+\lambda w) = (1-\lambda) f(v) + \lambda f(w) + \lambda(1-\lambda) \times \sum_{k=1}^{n} \frac{1}{k!}\left[\lambda^{k-1} f^{(k)}(v) + (-1)^k (1-\lambda)^{k-1} f^{(k)}(w)\right](w-v)^k + S_{n,\lambda}(v,w), \tag{4.8}$$

where the remainder $S_{n,\lambda}(v,w)$ *satisfies the bound*

$$|S_{n,\lambda}(v,w)| \le \frac{1}{n!\,(n+2)}(1-\lambda)\lambda |w-v|^{n+1}\left[\lambda^n \left|f^{(n+1)}(v)\right| + \frac{1}{(n+1)}[\lambda^n + (1-\lambda)^n]\left|f^{(n+1)}((1-\lambda)v+\lambda w)\right| + (1-\lambda)^n \left|f^{(n+1)}(w)\right|\right]. \tag{4.9}$$

We also have

$$f((1-\lambda)w+\lambda v) = (1-\lambda) f(v) + \lambda f(w) + \sum_{k=1}^{n} \frac{1}{k!}\left[(1-\lambda)^{k+1} f^{(k)}(v) + (-1)^k \lambda^{k+1} f^{(k)}(w)\right](w-v)^k + P_{n,\lambda}(v,w), \tag{4.10}$$

where the remainder $P_{n,\lambda}(v,w)$ *satisfies the bound*

$$|P_{n,\lambda}(v,w)| \le \frac{1}{n!\,(n+2)}|w-v|^{n+1}\left[(1-\lambda)^{n+2}\left|f^{(n+1)}(v)\right| + \frac{1}{(n+1)}\left[(1-\lambda)^{n+2} + \lambda^{n+2}\right]\left|f^{(n+1)}((1-\lambda)w+\lambda v)\right| + \lambda^{n+2}\left|f^{(n+1)}(w)\right|\right]. \tag{4.11}$$

For $n=0$, namely if $|f'|$ is convex on D, then by (4.2) we get

$$|f(z) - (1-\lambda) f(v) - \lambda f(w)| \le \frac{1}{2}\left[|1-\lambda||z-v||f'(v)| + [|1-\lambda||z-v| + |\lambda||w-z|]|f'(z)| + |\lambda||w-z||f'(w)|\right], \tag{4.12}$$

for z, v, $w \in D$ and for all $\lambda \in \mathbb{C}$.

From (4.5) we get

$$\left|f\left(\frac{v+w}{2}\right) - (1-\lambda) f(v) - \lambda f(w)\right| \le \frac{1}{4}|w-v|\left[|1-\lambda||f'(v)| + [|1-\lambda| + |\lambda|]\left|f'\left(\frac{v+w}{2}\right)\right| + |\lambda||f'(w)|\right] \tag{4.13}$$

for v, $w \in D$ and for all $\lambda \in \mathbb{C}$.

In particular, for $\lambda = \frac{1}{2}$ we get

$$\left|f\left(\frac{v+w}{2}\right) - \frac{f(v)+f(w)}{2}\right| \le \frac{1}{8}|w-v|\left[|f'(v)| + 2\left|f'\left(\frac{v+w}{2}\right)\right| + |f'(w)|\right] \tag{4.14}$$

for v, $w \in D$.

5. Examples for Logarithm and Exponential

Consider the function $f(z) = \operatorname{Log}(z)$ where $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$ and $\operatorname{Arg}(z)$ is such that $-\pi < \operatorname{Arg}(z) \leq \pi$. Log is called the *"principal branch"* of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_\ell := \mathbb{C}\setminus\{x+iy : x \leq 0,\ y = 0\}$ and

$$f^{(k)}(z) = \frac{(-1)^{k-1}(k-1)!}{z^k},\ k \geq 1,\ z \in \mathbb{C}_\ell.$$

Let D be a convex domain in $\mathbb{C}_\ell$ an assume that $d_D := \inf_{z\in D}|z|$ is a positive and finite number. If $z,\ v,\ w \in D \subset \mathbb{C}_\ell$, then by the representation (2.14) and the inequality (3.4)

$$\begin{aligned}(5.1)\quad & \Big|\operatorname{Log}(z) - (1-\lambda)\operatorname{Log}(v) - \lambda\operatorname{Log}(w) \\ & - \sum_{k=1}^{n}\frac{1}{k}\left[(1-\lambda)(-1)^{k-1}\frac{(z-v)^k}{v^k} - \lambda\frac{(w-z)^k}{w^k}\right]\Bigg| \\ & \leq \frac{1}{(n+1)\,d_D^{n+1}}\left(|1-\lambda|\,|z-v|^{n+1} + |\lambda|\,|w-z|^{n+1}\right)\end{aligned}$$

for $n \geq 1$ and for $n = 0$ we have

$$(5.2)\quad |\operatorname{Log}(z) - (1-\lambda)\operatorname{Log}(v) - \lambda\operatorname{Log}(w)| \leq \frac{1}{d_D}\left(|1-\lambda|\,|z-v| + |\lambda|\,|w-z|\right),$$

for all $\lambda \in \mathbb{C}$.

If $\lambda = \tau \in [0,1]$, then by (5.1) we get

$$\begin{aligned}(5.3)\quad & \Big|\operatorname{Log}(z) - (1-\tau)\operatorname{Log}(v) - \tau\operatorname{Log}(w) \\ & - \sum_{k=1}^{n}\frac{1}{k}\left[(1-\tau)(-1)^{k-1}\frac{(z-v)^k}{v^k} - \tau\frac{(w-z)^k}{w^k}\right]\Bigg| \\ & \leq \frac{1}{(n+1)\,d_D^{n+1}}\left((1-\tau)\,|z-v|^{n+1} + \tau\,|w-z|^{n+1}\right) \\ & \leq \frac{1}{(n+1)\,d_D^{n+1}}\max\left\{|z-v|^{n+1}, |w-z|^{n+1}\right\}\end{aligned}$$

and for $\tau = \frac{1}{2}$ we get

$$\begin{aligned}(5.4)\quad & \Bigg|\operatorname{Log}(z) - \frac{\operatorname{Log}(v) + \operatorname{Log}(w)}{2} \\ & - \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}\left[(-1)^{k-1}\frac{(z-v)^k}{v^k} - \frac{(w-z)^k}{w^k}\right]\Bigg| \\ & \leq \frac{1}{2(n+1)\,d_D^{n+1}}\left(|z-v|^{n+1} + |w-z|^{n+1}\right)\end{aligned}$$

for $z,\ v,\ w \in D \subset \mathbb{C}_\ell$.

Moreover, if we take $z = \frac{v+w}{2}$ in (5.4), then we get

$$\left|\mathrm{Log}\left(\frac{v+w}{2}\right) - \frac{\mathrm{Log}(v) + \mathrm{Log}(w)}{2} - \frac{1}{2}\sum_{k=1}^{n} \frac{1}{2^k k}\left[\frac{(-1)^{k-1}}{v^k} - \frac{1}{w^k}\right](w-v)^k\right| \leq \frac{1}{2^{n+1}(n+1)\, d_D^{n+1}} |w-v|^{n+1} \tag{5.5}$$

for $v,\ w \in D \subset \mathbb{C}_\ell$.

The case $n = 0$ gives that

$$\left|\mathrm{Log}\left(\frac{v+w}{2}\right) - \frac{\mathrm{Log}(v) + \mathrm{Log}(w)}{2}\right| \leq \frac{1}{2d_D} |w-v| \tag{5.6}$$

for $v,\ w \in D \subset \mathbb{C}_\ell$.

For $f(z) = \mathrm{Log}(z)$, $z \in D \subset \mathbb{C}_\ell$, we have

$$|f'(z) - f'(w)| = \left|\frac{1}{z} - \frac{1}{w}\right| = \frac{|w-v|}{|z|\,|w|} \leq \frac{1}{d_D^2} |w-v|$$

showing that f is Lipschitzian on D with the constant $L = \frac{1}{d_D^2}$.

By the inequality (3.16) we then get

$$\left|\mathrm{Log}\left(\frac{v+w}{2}\right) - \frac{\mathrm{Log}(v) + \mathrm{Log}(w)}{2}\right| \leq \frac{1}{8d_D^2} |w-v|^2, \tag{5.7}$$

for $v,\ w \in D \subset \mathbb{C}_\ell$.

Now consider the exponential function $f(z) = \exp z$. Then

$$|\exp z| = \exp(\mathrm{Re}\, z)$$

and

$$|\exp((1-t)z + tw)| \leq (1-t)|\exp z| + t|\exp w|$$

for any $z,\ w \in \mathbb{C}$ and $t \in [0,1]$, showing that f is convex in absolute value.

Now let D be a convex domain in C and assume that $E_D := \sup_{z\in D}[\exp(\mathrm{Re}\, z)] < \infty$. If we use the representation (2.33) and the inequality (3.4), we have

$$\left|\exp z - (1-\lambda)\exp v - \lambda \exp w - \sum_{k=1}^{n} \frac{1}{k!}\left[(1-\lambda)(z-v)^k \exp v + (-1)^k \lambda (w-z)^k \exp w\right]\right| \leq \frac{1}{(n+1)!} E_D \left(|1-\lambda|\,|z-v|^{n+1} + |\lambda|\,|w-z|^{n+1}\right) \tag{5.8}$$

for all $z,\ v,\ w \in D \subset \mathbb{C}$ and $n \geq 1$.

For $n = 0$ we have the simpler inequality

$$|\exp z - (1-\lambda)\exp v - \lambda \exp w| \leq E_D(|1-\lambda|\,|z-v| + |\lambda|\,|w-z|) \tag{5.9}$$

for all $z,\ v,\ w \in D \subset \mathbb{C}$.

If $\lambda = \tau \in [0,1]$, then by (5.8) we get

$$\begin{aligned} &\Big|\exp z - (1-\tau)\exp v - \tau \exp w \\ &\quad - \sum_{k=1}^{n} \frac{1}{k!}\left[(1-\tau)(z-v)^k \exp v + (-1)^k \tau (w-z)^k \exp w\right]\Big| \\ &\quad \le \frac{1}{(n+1)!} E_D \left((1-\tau)|z-v|^{n+1} + \tau |w-z|^{n+1}\right) \\ &\quad\quad \le \frac{1}{(n+1)!} E_D \max\left\{|z-v|^{n+1}, |w-z|^{n+1}\right\} \end{aligned} \tag{5.10}$$

for all z, v, $w \in D \subset \mathbb{C}$ and for $\tau = \frac{1}{2}$ we get

$$\begin{aligned} &\Big|\exp z - \frac{\exp v + \exp w}{2} \\ &\quad -\frac{1}{2}\sum_{k=1}^{n} \frac{1}{k!}\left[(z-v)^k \exp v + (-1)^k (w-z)^k \exp w\right]\Big| \\ &\quad\quad \le \frac{1}{2(n+1)!} E_D \left(|z-v|^{n+1} + |w-z|^{n+1}\right) \end{aligned} \tag{5.11}$$

for all z, v, $w \in D \subset \mathbb{C}$.

Moreover, if we take $z = \frac{v+w}{2}$ in (5.11), then we get

$$\begin{aligned} &\Big|\exp\left(\frac{v+w}{2}\right) - \frac{\exp v + \exp w}{2} \\ &\quad -\frac{1}{2}\sum_{k=1}^{n} \frac{1}{2^k k!}\left[\exp v + (-1)^k \exp w\right](w-v)^k\Big| \\ &\quad\quad \le \frac{1}{2^{n+1}(n+1)!} E_D |w-v|^{n+1} \end{aligned} \tag{5.12}$$

for all v, $w \in D \subset \mathbb{C}$.

The case $n = 0$ gives that

$$\left|\exp\left(\frac{v+w}{2}\right) - \frac{\exp v + \exp w}{2}\right| \le \frac{1}{2} E_D |w-v| \tag{5.13}$$

for all v, $w \in D \subset \mathbb{C}$.

The function $f(z) = \exp z$ is Lipschitzian on D with the constant $L = E_D$, then by (3.16) we get

$$\left|\exp\left(\frac{v+w}{2}\right) - \frac{\exp v + \exp w}{2}\right| \le \frac{1}{8} E_D |w-v|^2 . \tag{5.14}$$

for all v, $w \in D \subset \mathbb{C}$.

By the convexity in modulus of the complex function and by (4.14) we also have

$$\begin{aligned} &\left|\exp\left(\frac{v+w}{2}\right) - \frac{\exp v + \exp w}{2}\right| \\ &\quad \le \frac{1}{8}|w-v|\left[\exp(\operatorname{Re} v) + 2\exp\left(\operatorname{Re}\left(\frac{v+w}{2}\right)\right) + \exp(\operatorname{Re} w)\right] \end{aligned} \tag{5.15}$$

for all v, $w \in \mathbb{C}$.

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