General Multidimensional Fractional Iyengar type inequalities

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Abstract

Here we derive a variety of general multivariate fractional Iyengar type inequalities for not necessarily radial functions defined on the shell and ball. Our approach is based on the polar coordinates in $\mathbb{R}^N$, $N \geq 2$, and the related multivariate polar integration formula. Via this method we transfer author's univariate fractional Iyengar type inequalities into general multivariate fractional Iyengar inequalities.

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1 Background

We are motivated by the following famous Iyengar inequality (1938), [10].

Theorem 1 *Let f be a differentiable function on $[a,b]$ and $|f'(x)| \leq M$. Then*

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) \right| \leq \frac{M(b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M}. \quad (1)$$

We need

Definition 2 *([2], p. 394) Let $\nu > 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ the ceiling of the number), $f \in AC^n([a,b])$ (i.e. $f^{(n-1)}$ is absolutely continuous on $[a,b]$). The left Caputo fractional derivative of order ν is defined as*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\,dt, \quad (2)$$

$\forall\, x \in [a,b]$, and it exists almost everywhere over $[a,b]$.

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We need

Definition 3 *([4], p. 336-337) Let $\nu > 0$, $n = \lceil \nu \rceil$, $f \in AC^n([a,b])$. The right Caputo fractional derivative of order ν is defined as*

$$D_{b-}^{\nu} f(x) = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (z-x)^{n-\nu-1} f^{(n)}(z)\, dz, \tag{3}$$

$\forall\, x \in [a,b]$, and exists almost everywhere over $[a,b]$.

In [7] we proved the following Caputo fractional Iyengar type inequalities:

Theorem 4 *([7]) Let $\nu > 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), and $f \in AC^n([a,b])$ (i.e. $f^{(n-1)}$ is absolutely continuous on $[a,b]$). We assume that $D_{*a}^{\nu} f, D_{b-}^{\nu} f \in L_{\infty}([a,b])$. Then*

i)

$$\left| \int_a^b f(x)\, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_{\infty}([a,b])}, \|D_{b-}^{\nu} f\|_{L_{\infty}([a,b])} \right\}}{\Gamma(\nu+2)} \left[(t-a)^{\nu+1} + (b-t)^{\nu+1} \right], \tag{4}$$

$\forall\, t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$, the right hand side of (4) is minimized, and we get:

$$\left| \int_a^b f(x)\, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_{\infty}([a,b])}, \|D_{b-}^{\nu} f\|_{L_{\infty}([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{5}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, ..., n-1$, we obtain

$$\left| \int_a^b f(x)\, dx \right| \leq \frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_{\infty}([a,b])}, \|D_{b-}^{\nu} f\|_{L_{\infty}([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{6}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(x)\, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_{\infty}([a,b])}, \|D_{b-}^{\nu} f\|_{L_{\infty}([a,b])} \right\}}{\Gamma(\nu+2)} \left(\frac{b-a}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \tag{7}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, from (7) we get:

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N}\right)\left[jf(a) + (N-j)f(b)\right]\right| \leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_\infty([a,b])}, \left\|D_{b-}^{\nu}f\right\|_{L_\infty([a,b])}\right\}}{\Gamma(\nu+2)}\left(\frac{b-a}{N}\right)^{\nu+1}\left[j^{\nu+1} + (N-j)^{\nu+1}\right], \tag{8}$$

$j = 0, 1, 2, ..., N,$

vi) when $N = 2$ and $j = 1$, (8) turns to

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)(f(a) + f(b))\right| \leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_\infty([a,b])}, \left\|D_{b-}^{\nu}f\right\|_{L_\infty([a,b])}\right\}}{\Gamma(\nu+2)}\frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{9}$$

vii) when $0 < \nu \leq 1$, inequality (9) is again valid without any boundary conditions.

We mention

Theorem 5 *([7]) Let $\nu \geq 1$, $n = \lceil \nu \rceil$, and $f \in AC^n([a,b])$. We assume that $D_{*a}^{\nu}f, D_{b-}^{\nu}f \in L_1([a,b])$. Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1}\right]\right| \leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])}, \left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\left[(t-a)^{\nu} + (b-t)^{\nu}\right], \tag{10}$$

$\forall\ t \in [a,b]$,

ii) when $\nu = 1$, from (10), we have

$$\left| \int_a^b f(x)\,dx - \left[f(a)(t-a) + f(b)(b-t)\right]\right| \leq$$

$$\left\|f'\right\|_{L_1([a,b])}(b-a), \quad \forall\ t \in [a,b], \tag{11}$$

iii) from (11), we obtain ($\nu = 1$ case)

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)(f(a) + f(b))\right| \leq \left\|f'\right\|_{L_1([a,b])}(b-a), \tag{12}$$

iv) at $t=\frac{a+b}{2}$, $\nu>1$, the right hand side of (10) is minimized, and we get:

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}},\tag{13}$$

v) if $f^{(k)}(a)=f^{(k)}(b)=0$, for all $k=0,1,...,n-1$; $\nu>1$, from (13), we obtain

$$\left|\int_a^b f(x)\,dx\right|\leq\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}},\tag{14}$$

which is a sharp inequality,

vi) more generally, for $j=0,1,2,...,$ $N\in\mathbb{N}$, it holds

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{N}\right)^{k+1}\left[j^{k+1}f^{(k)}(a)+(-1)^k(N-j)^{k+1}f^{(k)}(b)\right]\right|$$

$$\leq\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\left(\frac{b-a}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right],\tag{15}$$

vii) if $f^{(k)}(a)=f^{(k)}(b)=0$, $k=1,...,n-1$, from (15) we get:

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{N}\right)\left[jf(a)+(N-j)f(b)\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\left(\frac{b-a}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right],\tag{16}$$

$j=0,1,2,...,N,$

viii) when $N=2$ and $j=1$, (16) turns to

$$\left|\int_a^b f(x)\,dx-\frac{(b-a)}{2}(f(a)+f(b))\right|\leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}}.\tag{17}$$

We mention

Theorem 6 *([7]) Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$; $f \in AC^n([a,b])$, *with* $D_{*a}^{\nu} f, D_{b-}^{\nu} f \in L_q([a,b])$. *Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_q([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \left[(t-a)^{\nu+\frac{1}{p}} + (b-t)^{\nu+\frac{1}{p}} \right], \quad (18)$$

$\forall\ t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$, *the right hand side of (18) is minimized, and we get:*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_q([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \quad (19)$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, *for all* $k = 0, 1, ..., n-1$, *we obtain*

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_q([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \quad (20)$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_q([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N} \right)^{\nu+\frac{1}{p}} \left[j^{\nu+\frac{1}{p}} + (N-j)^{\nu+\frac{1}{p}} \right], \quad (21)$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, *from (21) we get:*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_q([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N} \right)^{\nu+\frac{1}{p}} \left[j^{\nu+\frac{1}{p}} + (N-j)^{\nu+\frac{1}{p}} \right], \quad (22)$$

for $j = 0, 1, 2, ..., N$,

vi) when $N = 2$ *and* $j = 1$, *(22) turns to*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)(f(a) + f(b)) \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_q([a,b])}, \|D_{b-}^{\nu} f\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{23}$$

vii) when $\frac{1}{q} < \nu \leq 1$, *inequality (23) is again valid but without any boundary conditions.*

We need the following different fractional calculus background:

Let $\alpha > 0$, $m = [\alpha]$ ($[\cdot]$ is the integral part), $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a,b])$, $[a,b] \subset \mathbb{R}$, $x \in [a,b]$. The gamma function Γ is given by $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$. We define the left Riemann-Liouville integral ([2], p. 24)

$$\left(J_{\alpha}^{a+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt, \tag{24}$$

$a \leq x \leq b$. We define the subspace $C_{a+}^{\alpha}([a,b])$ of $C^m([a,b])$:

$$C_{a+}^{\alpha}([a,b]) = \left\{ f \in C^m([a,b]) : J_{1-\beta}^{a+} f^{(m)} \in C^1([a,b]) \right\}. \tag{25}$$

For $f \in C_{a+}^{\alpha}([a,b])$, we define the left generalized α-fractional derivative of f over $[a,b]$ as

$$D_{a+}^{\alpha} f := \left(J_{1-\beta}^{a+} f^{(m)}\right)', \tag{26}$$

see [2], p. 24. Canavati first in [9] introduced the above over $[0,1]$.

We have that $D_{a+}^{n} f = f^{(n)}$; $n \in \mathbb{N}$.

Notice that $D_{a+}^{\alpha} f \in C([a,b])$.

Furthermore we need:

Let again $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $f \in C([a,b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt, \tag{27}$$

$x \in [a,b]$, see [3]. Define the subspace of functions

$$C_{b-}^{\alpha}([a,b]) := \left\{ f \in C^m([a,b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a,b]) \right\}. \tag{28}$$

Define the right generalized α-fractional derivative of f over $[a,b]$ as

$$\overline{D}_{b-}^{\alpha} f = (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)}\right)', \tag{29}$$

see [3]. We set $\overline{D}_{b-}^{0} f = f$. We have $\overline{D}_{b-}^{n} f = (-1)^n f^{(n)}$; $n \in \mathbb{N}$. Notice that $\overline{D}_{b-}^{\alpha} f \in C([a,b])$.

We mention the following Canavati fractional Iyengar type inequalities:

Theorem 7 *([6]) Let $\nu \geq 1$, $n = [\nu]$ and $f \in C_{a+}^{\nu}([a,b]) \cap C_{b-}^{\nu}([a,b])$. Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \left[(t-a)^{\nu+1} + (b-t)^{\nu+1} \right], \quad (30)$$

$\forall\ t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$, the right hand side of (30) is minimized, and we get:

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \quad (31)$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, ..., n-1$, we obtain

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \quad (32)$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ...,\ N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \left(\frac{b-a}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \quad (33)$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, from (33) we get:

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \left(\frac{b-a}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \quad (34)$$

$j=0,1,2,...,N,$

vi) when $N=2$ and $j=1$, (34) turns to

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right|\leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{\infty,([a,b])},\left\|\overline{D}_{b-}^{\nu}f\right\|_{\infty,([a,b])}\right\}}{\Gamma(\nu+2)}\frac{(b-a)^{\nu+1}}{2^{\nu}}. \tag{35}$$

We mention

Theorem 8 *([6]) Let $\nu\geq 1$, $n=[\nu]$, and $f\in C_{a+}^{\nu}([a,b])\cap C_{b-}^{\nu}([a,b])$. Then*

i)

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1}+(-1)^k f^{(k)}(b)(b-t)^{k+1}\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_1([a,b])},\left\|\overline{D}_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\left[(t-a)^{\nu}+(b-t)^{\nu}\right], \tag{36}$$

$\forall\ t\in[a,b],$

ii) when $\nu=1$, from (36), we have

$$\left|\int_a^b f(x)\,dx-[f(a)(t-a)+f(b)(b-t)]\right|\leq$$

$$\|f'\|_{L_1([a,b])}(b-a),\quad\forall\ t\in[a,b], \tag{37}$$

iii) from (37), we obtain ($\nu=1$ case)

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right|\leq\|f'\|_{L_1([a,b])}(b-a), \tag{38}$$

iv) at $t=\frac{a+b}{2}$, $\nu>1$, the right hand side of (36) is minimized, and we get:

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_1([a,b])},\left\|\overline{D}_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}}, \tag{39}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, ..., n-1$; $\nu > 1$, from (39), we obtain

$$\left|\int_a^b f(x)\,dx\right| \leq \frac{\max\left\{\left\|D_{a+}^{\nu} f\right\|_{L_1([a,b])}, \left\|\overline{D}_{b-}^{\nu} f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)} \frac{(b-a)^{\nu}}{2^{\nu-1}}, \qquad (40)$$

which is a sharp inequality,

vi) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, it holds

$$\left|\int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N}\right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b)\right]\right|$$

$$\leq \frac{\max\left\{\left\|D_{a+}^{\nu} f\right\|_{L_1([a,b])}, \left\|\overline{D}_{b-}^{\nu} f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)} \left(\frac{b-a}{N}\right)^{\nu} \left[j^{\nu} + (N-j)^{\nu}\right], \qquad (41)$$

vii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, from (41) we get:

$$\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{N}\right)\left[jf(a) + (N-j)f(b)\right]\right| \leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu} f\right\|_{L_1([a,b])}, \left\|\overline{D}_{b-}^{\nu} f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)} \left(\frac{b-a}{N}\right)^{\nu} \left[j^{\nu} + (N-j)^{\nu}\right], \qquad (42)$$

$j = 0, 1, 2, ..., N$,

viii) when $N = 2$ and $j = 1$, (42) turns to

$$\left|\int_a^b f(x)\,dx - \frac{(b-a)}{2}\left(f(a) + f(b)\right)\right| \leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu} f\right\|_{L_1([a,b])}, \left\|\overline{D}_{b-}^{\nu} f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)} \frac{(b-a)^{\nu}}{2^{\nu-1}}. \qquad (43)$$

We mention

Theorem 9 *([6]) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu \geq 1$, $n = [\nu]$; $f \in C_{a+}^{\nu}([a,b]) \cap C_{b-}^{\nu}([a,b])$. Then*

i)

$$\left|\int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1}\right]\right| \leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu} f\right\|_{L_q([a,b])}, \left\|\overline{D}_{b-}^{\nu} f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)\left(p(\nu-1)+1\right)^{\frac{1}{p}}} \left[(t-a)^{\nu+\frac{1}{p}} + (b-t)^{\nu+\frac{1}{p}}\right], \qquad (44)$$

$\forall\ t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$*, the right hand side of (44) is minimized, and we get:*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_q([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{45}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$*, for all* $k = 0, 1, ..., n-1$*, we obtain*

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_q([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{46}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ...,\ N \in \mathbb{N}$*, it holds*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N}\right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_q([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N}\right)^{\nu+\frac{1}{p}} \left[j^{\nu+\frac{1}{p}} + (N-j)^{\nu+\frac{1}{p}} \right], \tag{47}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$*,* $k = 1, ..., n-1$*, from (47) we get:*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N}\right)[j f(a) + (N-j) f(b)] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_q([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N}\right)^{\nu+\frac{1}{p}} \left[j^{\nu+\frac{1}{p}} + (N-j)^{\nu+\frac{1}{p}} \right], \tag{48}$$

for $j = 0, 1, 2, ..., N$*,*

vi) when $N = 2$ *and* $j = 1$*, (48) turns to*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)(f(a) + f(b)) \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{L_q([a,b])}, \left\| \overline{D}_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}. \tag{49}$$

We need

Definition 10 *([1]) Let $a, b \in \mathbb{R}$. The left conformable fractional derivative starting from a of a function $f : [a, \infty) \to \mathbb{R}$ of order $0 < \alpha \leq 1$ is defined by*

$$\left(T_{\alpha}^{a} f\right)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon (t-a)^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{50}$$

If $\left(T_{\alpha}^{a} f\right)(t)$ exists on (a, b), then

$$\left(T_{\alpha}^{a} f\right)(a) = \lim_{t \to a+} \left(T_{\alpha}^{a} f\right)(t). \tag{51}$$

The right conformable fractional derivative of order $0 < \alpha \leq 1$ terminating at b of $f : (-\infty, b] \to \mathbb{R}$ is defined by

$$\left({}_{\alpha}^{b} T f\right)(t) = -\lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon (b-t)^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{52}$$

If $\left({}_{\alpha}^{b} T f\right)(t)$ exists on (a, b), then

$$\left({}_{\alpha}^{b} T f\right)(b) = \lim_{t \to b-} \left({}_{\alpha}^{b} T f\right)(t). \tag{53}$$

Note that if f is differentiable then

$$\left(T_{\alpha}^{a} f\right)(t) = (t-a)^{1-\alpha} f'(t), \tag{54}$$

and

$$\left({}_{\alpha}^{b} T f\right)(t) = -(b-t)^{1-\alpha} f'(t). \tag{55}$$

In the higher order case we can generalize things as follows:

Definition 11 *([1]) Let $\alpha \in (n, n+1]$, and set $\beta = \alpha - n$. Then, the left conformable fractional derivative starting from a of a function $f : [a, \infty) \to \mathbb{R}$ of order α, where $f^{(n)}(t)$ exists, is defined by*

$$\left(\mathbf{T}_{\alpha}^{a} f\right)(t) = \left(T_{\beta}^{a} f^{(n)}\right)(t), \tag{56}$$

The right conformable fractional derivative of order α terminating at b of $f : (-\infty, b] \to \mathbb{R}$, where $f^{(n)}(t)$ exists, is defined by

$$\left({}_{\alpha}^{b} \mathbf{T} f\right)(t) = (-1)^{n+1} \left({}_{\beta}^{b} T f^{(n)}\right)(t). \tag{57}$$

If $\alpha = n+1$ then $\beta = 1$ and $\mathbf{T}_{n+1}^{a} f = f^{(n+1)}$.

If n is odd, then ${}_{n+1}^{b} \mathbf{T} f = -f^{(n+1)}$, and if n is even, then ${}_{n+1}^{b} \mathbf{T} f = f^{(n+1)}$.

When $n = 0$ (or $\alpha \in (0, 1]$), then $\beta = \alpha$, and (56), (57) collapse to (50), (52), respectively.

We need

Remark 12 *([5]) We notice the following: let* $\alpha \in (n, n+1]$ *and* $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$. *Then* ($\beta := \alpha - n$, $0 < \beta \leq 1$)

$$\left(\mathbf{T}_{\alpha}^{a}(f)\right)(x) = \left(T_{\beta}^{\alpha} f^{(n)}\right)(x) = (x-a)^{1-\beta} f^{(n+1)}(x), \tag{58}$$

and

$$\left({}_{\alpha}^{b}\mathbf{T}(f)\right)(x) = (-1)^{n+1}\left({}_{\beta}^{b}T f^{(n)}\right)(x) =$$

$$(-1)^{n+1}(-1)(b-x)^{1-\beta} f^{(n+1)}(x) = (-1)^{n}(b-x)^{1-\beta} f^{(n+1)}(x). \tag{59}$$

Consequently we get that

$$\left(\mathbf{T}_{\alpha}^{a}(f)\right)(x), \ \left({}_{\alpha}^{b}\mathbf{T}(f)\right)(x) \in C([a,b]).$$

Furthermore it is obvious that

$$\left(\mathbf{T}_{\alpha}^{a}(f)\right)(a) = \left({}_{\alpha}^{b}\mathbf{T}(f)\right)(b) = 0, \tag{60}$$

when $0 < \beta < 1$, *i.e. when* $\alpha \in (n, n+1)$.

We mention the following Conformable fractional Iyengar type inequalities:

Theorem 13 *([8]) Let* $\alpha \in (n, n+1]$ *and* $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$; $\beta = \alpha - n$. *Then*

i)

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!}\left[f^{(k)}(a)(z-a)^{k+1} + (-1)^k f^{(k)}(b)(b-z)^{k+1}\right]\right| \leq$$

$$\frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{\infty,[a,b]}, \left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{\infty,[a,b]}\right\}}{\Gamma(\alpha+2)}\left[(z-a)^{\alpha+1} + (b-z)^{\alpha+1}\right], \tag{61}$$

$\forall\ z \in [a,b]$,

ii) at $z = \frac{a+b}{2}$, *the right hand side of (61) is minimized, and we get:*

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a) + (-1)^k f^{(k)}(b)\right]\right| \leq$$

$$\frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{\infty,[a,b]}, \left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{\infty,[a,b]}\right\}}{\Gamma(\alpha+2)}\frac{(b-a)^{\alpha+1}}{2^{\alpha}}, \tag{62}$$

iii) assuming $f^{(k)}(a) = f^{(k)}(b) = 0$, *for* $k = 0, 1, ..., n$, *we obtain*

$$\left|\int_a^b f(t)\,dt\right| \leq \frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{\infty,[a,b]}, \left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{\infty,[a,b]}\right\}}{\Gamma(\alpha+2)}\frac{(b-a)^{\alpha+1}}{2^{\alpha}}, \tag{63}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds*

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[f^{(k)}(a)\, j^{k+1} + (-1)^k f^{(k)}(b)\,(N-j)^{k+1} \right] \right|$$

$$\leq \frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_\alpha^a (f) \right\|_{\infty,[a,b]}, \left\| {}_\alpha^b \mathbf{T}(f) \right\|_{\infty,[a,b]} \right\}}{\Gamma(\alpha+2)} \left(\frac{b-a}{N} \right)^{\alpha+1} \left[j^{\alpha+1} + (N-j)^{\alpha+1} \right], \tag{64}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n$, *from (64) we get:*

$$\left| \int_a^b f(t)\,dt - \left(\frac{b-a}{N} \right) \left[j f(a) + (N-j) f(b) \right] \right| \leq$$

$$\frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_\alpha^a (f) \right\|_{\infty,[a,b]}, \left\| {}_\alpha^b \mathbf{T}(f) \right\|_{\infty,[a,b]} \right\}}{\Gamma(\alpha+2)} \left(\frac{b-a}{N} \right)^{\alpha+1} \left[j^{\alpha+1} + (N-j)^{\alpha+1} \right], \tag{65}$$

$j = 0, 1, 2, ..., N$,

vi) when $N = 2$ *and* $j = 1$, *(65) turns to*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq$$

$$\frac{\Gamma(\beta) \max\left\{ \left\| \mathbf{T}_\alpha^a (f) \right\|_{\infty,[a,b]}, \left\| {}_\alpha^b \mathbf{T}(f) \right\|_{\infty,[a,b]} \right\}}{\Gamma(\alpha+2)} \frac{(b-a)^{\alpha+1}}{2^\alpha}. \tag{66}$$

We mention L_p conformable fractional Iyengar inequalities:

Theorem 14 *([8]) Let* $\alpha \in (n, n+1]$ *and* $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$; $\beta = \alpha - n$. *Let also* $p_1, p_2, p_3 > 1 : \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, *with* $\beta > \frac{1}{p_1} + \frac{1}{p_3}$. *Then*

i)

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \left[f^{(k)}(a)\,(z-a)^{k+1} + (-1)^k f^{(k)}(b)\,(b-z)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \left\| \mathbf{T}_\alpha^a (f) \right\|_{p_3,[a,b]}, \left\| {}_\alpha^b \mathbf{T}(f) \right\|_{p_3,[a,b]} \right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_1} + \frac{1}{p_2} \right)} \tag{67}$$

$$\left[(z-a)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} + (b-z)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} \right],$$

$\forall\ z \in [a,b]$,

ii) at $z=\frac{a+b}{2}$, the right hand side of (67) is minimized, and we get:

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|\leq$$

$$\frac{\max\left\{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{p_3,[a,b]},\left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{p_3,[a,b]}\right\}}{n!\,(p_1n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}\frac{(b-a)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-\frac{1}{p_3}}}, \tag{68}$$

iii) assuming $f^{(k)}(a)=f^{(k)}(b)=0$, for $k=0,1,...,n$, we obtain

$$\left|\int_a^b f(t)\,dt\right|\leq\frac{\max\left\{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{p_3,[a,b]},\left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{p_3,[a,b]}\right\}}{n!\,(p_1n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}\frac{(b-a)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-\frac{1}{p_3}}}, \tag{69}$$

which is a sharp inequality,

iv) more generally, for $j=0,1,2,...,\ N\in\mathbb{N}$, it holds

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n}\frac{1}{(k+1)!}\left(\frac{b-a}{N}\right)^{k+1}\left[f^{(k)}(a)\,j^{k+1}+(-1)^k f^{(k)}(b)\,(N-j)^{k+1}\right]\right|$$

$$\leq\frac{\max\left\{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{p_3,[a,b]},\left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{p_3,[a,b]}\right\}}{n!\,(p_1n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}$$

$$\left(\frac{b-a}{N}\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\left[j^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}+(N-j)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\right], \tag{70}$$

v) if $f^{(k)}(a)=f^{(k)}(b)=0$, $k=1,...,n$, from (70) we get:

$$\left|\int_a^b f(t)\,dt-\left(\frac{b-a}{N}\right)[jf(a)+(N-j)\,f(b)]\right|\leq \tag{71}$$

$$\frac{\max\left\{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{p_3,[a,b]},\left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{p_3,[a,b]}\right\}}{n!\,(p_1n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}$$

$$\left(\frac{b-a}{N}\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\left[j^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}+(N-j)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\right],$$

for $j=0,1,2,...,N$,

vi) when $N=2$ and $j=1$, (71) turns to

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right|\leq$$

$$\frac{\max\left\{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{p_3,[a,b]},\left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{p_3,[a,b]}\right\}}{n!\,(p_1n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}\frac{(b-a)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-\frac{1}{p_3}}}. \tag{72}$$

We need

Remark 15 *We define the ball* $B(0,R) = \left\{x \in \mathbb{R}^N : |x| < R\right\} \subseteq \mathbb{R}^N$*,* $N \geq 2$*,* $R > 0$*, and the sphere*

$$S^{N-1} := \left\{x \in \mathbb{R}^N : |x| = 1\right\},$$

where $|\cdot|$ *is the Euclidean norm. Let* $d\omega$ *be the element of surface measure on* S^{N-1} *and*

$$\omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$$

is the area of S^{N-1}*.*

For $x \in \mathbb{R}^N - \{0\}$ *we can write uniquely* $x = r\omega$*, where* $r = |x| > 0$ *and* $\omega = \frac{x}{r} \in S^{N-1}$*,* $|\omega| = 1$*. Note that* $\int_{B(0,R)} dy = \frac{\omega_N R^N}{N}$ *is the Lebesgue measure on the ball, that is the volume of* $B(0,R)$*, which exactly is* $Vol(B(0,R)) = \frac{\pi^{\frac{N}{2}} R^N}{\Gamma\left(\frac{N}{2}+1\right)}$*.*

Following [11, pp. 149-150, exercise 6], and [12, pp. 87-88, Theorem 5.2.2] we can write for $F : \overline{B(0,R)} \to \mathbb{R}$ *a Lebesgue integrable function that*

$$\int_{B(0,R)} F(x)\,dx = \int_{S^{N-1}} \left(\int_0^R F(r\omega)\, r^{N-1} dr\right) d\omega, \tag{73}$$

and we use this formula a lot.

Typically here the function $f : \overline{B(0,R)} \to \mathbb{R}$ *is not radial. A radial function* f *is such that there exists a function* g *such that* $f(x) = g(r)$*, where* $r = |x|$*,* $r \in [0,R]$*,* $\forall\ x \in \overline{B(0,R)}$*.*

We need

Remark 16 *Let the spherical shell* $A := B(0,R_2) - \overline{B(0,R_1)}$*,* $0 < R_1 < R_2$*,* $A \subseteq \mathbb{R}^N$*,* $N \geq 2$*,* $x \in \overline{A}$*. Consider that* $f : \overline{A} \to \mathbb{R}$ *is not necessarily radial. A radial function* f *is such that there exists a function* g *such that* $f(x) = g(r)$*,* $r = |x|$*,* $r \in [R_1, R_2]$*,* $\forall\ x \in \overline{A}$*. Here* x *can be written uniquely as* $x = r\omega$*, where* $r = |x| > 0$ *and* $\omega = \frac{x}{r} \in S^{N-1}$*,* $|\omega| = 1$*, see ([11], p. 149-150 and [2], p. 421), furthermore for* $F : \overline{A} \to \mathbb{R}$ *a Lebesgue integrable function we have that*

$$\int_A F(x)\,dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega)\, r^{N-1} dr\right) d\omega. \tag{74}$$

Here

$$Vol(A) = \frac{\omega_N \left(R_2^N - R_1^N\right)}{N} = \frac{\pi^{\frac{N}{2}} \left(R_2^N - R_1^N\right)}{\Gamma\left(\frac{N}{2}+1\right)}. \tag{75}$$

In this article we derive general multivariate fractional Iyengar type inequalities on the shell and ball of $\mathbb{R}^N$, $N \geq 2$, for not necessarily radial functions. Our following results are based on the described background fractional results.

2 Main Results

From now on the fractional derivatives of $f(s\omega) s^{N-1}$ are in $s \in [R_1, R_2]$ or $s \in [0, R]$.

We present Caputo type results on the shell:

Theorem 17 *Let $\nu > 0$, $n = \lceil \nu \rceil$. Consider $f : \overline{A} \to \mathbb{R}$ be Lebesgue integrable, which is not necessarily radial. Assume that $f(s\omega) s^{N-1} \in AC^n([R_1, R_2])$ (i.e. $\left(f(s\omega) s^{N-1}\right)^{(n-1)} \in AC([R_1, R_2])$ - absolutely continuous functions), $\forall\ \omega \in S^{N-1}$, $N \geq 2$. We assume that $D^{\nu}_{*R_1}\left(f(s\omega) s^{N-1}\right)$, $D^{\nu}_{R_2-}\left(f(s\omega) s^{N-1}\right) \in L_\infty([R_1, R_2])$, $\forall\ \omega \in S^{N-1}$, and that $\max\{\left\|D^{\nu}_{*R_1}\left(f(s\omega) s^{N-1}\right)\right\|_{L_\infty([R_1,R_2])}$, $\left\|D^{\nu}_{R_2-}\left(f(s\omega) s^{N-1}\right)\right\|_{L_\infty([R_1,R_2])}\} \leq K_1$, where $K_1 > 0$, $s \in [R_1, R_2]$, $\forall\ \omega \in S^{N-1}$. Then*

i)

$$\left|\int_A f(y)\,dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[\left(\left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)(t-R_1)^{k+1} + \right.\right.\right.$$

$$\left.\left.(-1)^k \left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)(R_2-t)^{k+1}\right]\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_1}{\Gamma(\nu+2)} \left[(t-R_1)^{\nu+1} + (R_2-t)^{\nu+1}\right], \tag{76}$$

$\forall\ t \in [R_1, R_2]$,

ii) at $t = \frac{R_1+R_2}{2}$, the right hand side of (76) is minimized, and we get:

$$\left|\int_A f(y)\,dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(R_2-R_1)^{k+1}}{2^{k+1}} \left[\int_{S^{N-1}} \left(f(s\omega) s^{N-1}\right)^{(k)}(R_1)\,d\omega + \right.\right.$$

$$\left.\left.(-1)^k \int_{S^{N-1}} \left(f(s\omega) s^{N-1}\right)^{(k)}(R_2)\,d\omega\right]\right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_1}{\Gamma(\nu+2)} \frac{(R_2-R_1)^{\nu+1}}{2^{\nu-1}}, \tag{77}$$

iii) if $\left(f(s\omega) s^{N-1}\right)^{(k)}(R_1) = \left(f(s\omega) s^{N-1}\right)^{(k)}(R_2) = 0$, $\forall\ \omega \in S^{N-1}$, (i.e. $\frac{\partial^k\left(f(s\omega)s^{N-1}\right)}{\partial s^k}$ vanish on $\partial B(0, R_1)$ and $\partial B(0, R_2)$) for all $k = 0, 1, ..., n-1$, we obtain

$$\left|\int_A f(y)\,dy\right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_1}{\Gamma(\nu+2)} \frac{(R_2-R_1)^{\nu+1}}{2^{\nu-1}}, \tag{78}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., \overline{N} \in \mathbb{N}$, it holds

$$\left|\int_A f(y)\,dy - \sum_{k=0}^{\nu-1} \frac{1}{(k+1)!}\left(\frac{R_2-R_1}{\overline{N}}\right)^{k+1}\right.$$

$$\left[j^{k+1} \left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1} \right)^{(k)} (R_1) \, d\omega \right) + \right.$$

$$\left. (-1)^k \left(\overline{N} - j \right)^{k+1} \left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1} \right)^{(k)} (R_2) \, d\omega \right) \right] \Bigg| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_1}{\Gamma(\nu+2)} \left(\frac{R_2 - R_1}{\overline{N}} \right)^{\nu+1} \left[j^{\nu+1} + \left(\overline{N} - j \right)^{\nu+1} \right], \tag{79}$$

v) if $\left(f(s\omega) s^{N-1} \right)^{(k)} (R_1) = \left(f(s\omega) s^{N-1} \right)^{(k)} (R_2) = 0$, $\forall\ \omega \in S^{N-1}$, *(i.e.* $\frac{\partial^k \left(f(s\omega) s^{N-1} \right)}{\partial s^k}$ *vanish on* $\partial B(0, R_1)$ *and* $\partial B(0, R_2)$*) for* $k = 1, \ldots, n-1$*, from (79) we get:*

$$\left| \int_A f(y) \, dy - \left(\frac{R_2 - R_1}{\overline{N}} \right) \left[j R_1^{N-1} \left(\int_{S^{N-1}} f(R_1 \omega) \, d\omega \right) + \right. \right.$$

$$\left. \left. \left(\overline{N} - j \right) R_2^{N-1} \left(\int_{S^{N-1}} f(R_2 \omega) \, d\omega \right) \right] \right| \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot$$

$$\frac{K_1}{\Gamma(\nu+2)} \left(\frac{R_2 - R_1}{\overline{N}} \right)^{\nu+1} \left[j^{\nu+1} + \left(\overline{N} - j \right)^{\nu+1} \right], \tag{80}$$

for $j = 0, 1, 2, \ldots, \overline{N} \in \mathbb{N}$,

vi) when $\overline{N} = 2$ *and* $j = 1$*, (80) turns to*

$$\left| \int_A f(y) \, dy - \left(\frac{R_2 - R_1}{2} \right) \left(R_1^{N-1} \int_{S^{N-1}} f(R_1 \omega) \, d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2 \omega) \, d\omega \right) \right|$$

$$\leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_1}{\Gamma(\nu+2)} \frac{(R_2 - R_1)^{\nu+1}}{2^{\nu-1}}, \tag{81}$$

vii) when $0 < \nu \leq 1$ *(without any boundary conditions), we get again*

$$\left| \int_A f(y) \, dy - \left(\frac{R_2 - R_1}{2} \right) \left(R_1^{N-1} \int_{S^{N-1}} f(R_1 \omega) \, d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2 \omega) \, d\omega \right) \right|$$

$$\leq \frac{\pi^{\frac{N}{2}} K_1}{\Gamma\left(\frac{N}{2}\right) \Gamma(\nu+2)} \frac{(R_2 - R_1)^{\nu+1}}{2^{\nu-1}}. \tag{82}$$

Proof. We apply Theorem 4 along with (74). See in the 3. Appendix the general proving method in this article. ■

Theorem 18 *Let* $\nu \geq 1$, $n = \lceil \nu \rceil$*. Consider* $f : \overline{A} \to \mathbb{R}$ *be Lebesgue integrable, which is not necessarily radial. Assume that* $f(s\omega) s^{N-1} \in AC^n([R_1, R_2])$ *(i.e.* $\left(f(s\omega) s^{N-1} \right)^{(n-1)} \in AC([R_1, R_2])$ *- absolutely continuous functions),* $\forall\ \omega \in S^{N-1}$, $N \geq 2$*. We assume that* $D_{*R_1}^{\nu} \left(f(s\omega) s^{N-1} \right)$, $D_{R_2-}^{\nu} \left(f(s\omega) s^{N-1} \right) \in L_1([R_1, R_2])$, $\forall\ \omega \in S^{N-1}$*, and that* $\max\{ \left\| D_{*R_1}^{\nu} \left(f(s\omega) s^{N-1} \right) \right\|_{L_1([R_1, R_2])}$,

$\left\| D_{R_2-}^{\nu}\left(f\left(s\omega\right)s^{N-1}\right)\right\|_{L_1([R_1,R_2])}\} \leq K_2$, *where* $K_2 > 0$, $s \in [R_1, R_2]$, $\forall\ \omega \in S^{N-1}$.

Then

i)

$$\left|\int_A f\left(y\right)dy - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\left(\int_{S^{N-1}}\left(f\left(s\omega\right)s^{N-1}\right)^{(k)}\left(R_1\right)d\omega\right)\left(t-R_1\right)^{k+1} + \right.\right.$$

$$\left.\left.(-1)^k\left(\int_{S^{N-1}}\left(f\left(s\omega\right)s^{N-1}\right)^{(k)}\left(R_2\right)d\omega\right)\left(R_2-t\right)^{k+1}\right]\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_2}{\Gamma\left(\nu+1\right)}\left[\left(t-R_1\right)^{\nu}+\left(R_2-t\right)^{\nu}\right], \tag{83}$$

$\forall\ t \in [R_1, R_2]$,

ii) when $\nu = 1$, *from (83), we have*

$$\left|\int_A f\left(y\right)dy - \left[\left(\int_{S^{N-1}} f\left(R_1\omega\right)d\omega\right)R_1^{N-1}\left(t-R_1\right) + \right.\right.$$

$$\left.\left.\left(\int_{S^{N-1}} f\left(R_2\omega\right)d\omega\right)R_2^{N-1}\left(R_2-t\right)\right]\right| \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}K_2\left(R_2-R_1\right), \tag{84}$$

$\forall\ t \in [R_1, R_2]$,

iii) from (84), we obtain ($\nu = 1$ *case,* $t = \frac{R_1+R_2}{2}$*)*

$$\left|\int_A f\left(y\right)dy - \left(\frac{R_2-R_1}{2}\right)\left[\left(\int_{S^{N-1}} f\left(R_1\omega\right)d\omega\right)R_1^{N-1} + \left(\int_{S^{N-1}} f\left(R_2\omega\right)d\omega\right)R_2^{N-1}\right]\right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}K_2\left(R_2-R_1\right), \tag{85}$$

iv) at $t = \frac{R_1+R_2}{2}$, $\nu > 1$, *the right hand side of (83) is minimized, and we get:*

$$\left|\int_A f\left(y\right)dy - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{\left(R_2-R_1\right)^{k+1}}{2^{k+1}}\left[\int_{S^{N-1}}\left(f\left(s\omega\right)s^{N-1}\right)^{(k)}\left(R_1\right)d\omega + \right.\right.$$

$$\left.\left.(-1)^k\int_{S^{N-1}}\left(f\left(s\omega\right)s^{N-1}\right)^{(k)}\left(R_2\right)d\omega\right]\right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_2}{\Gamma\left(\nu+1\right)}\frac{\left(R_2-R_1\right)^{\nu}}{2^{\nu-2}}, \tag{86}$$

v) if $\left(f\left(s\omega\right)s^{N-1}\right)^{(k)}\left(R_1\right) = \left(f\left(s\omega\right)s^{N-1}\right)^{(k)}\left(R_2\right) = 0$, $\forall\ \omega \in S^{N-1}$, *(i.e.* $\frac{\partial^k\left(f(s\omega)s^{N-1}\right)}{\partial s^k}$ *vanish on* $\partial B\left(0,R_1\right)$ *and* $\partial B\left(0,R_2\right)$*) for all* $k = 0, 1, ..., n-1$; $\nu > 1$, *we obtain from (86) that*

$$\left|\int_A f\left(y\right)dy\right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_2}{\Gamma\left(\nu+1\right)}\frac{\left(R_2-R_1\right)^{\nu}}{2^{\nu-2}}, \tag{87}$$

which is a sharp inequality,

vi) more generally, for $j = 0, 1, 2, ..., \overline{N} \in \mathbb{N}$*, it holds*

$$\left| \int_A f(y)\, dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{R_2 - R_1}{\overline{N}} \right)^{k+1} \right.$$

$$\left[j^{k+1} \left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1} \right)^{(k)} (R_1)\, d\omega \right) + \right.$$

$$\left. \left. (-1)^k \left(\overline{N} - j \right)^{k+1} \left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1} \right)^{(k)} (R_2)\, d\omega \right) \right] \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_2}{\Gamma(\nu+1)} \left(\frac{R_2 - R_1}{\overline{N}} \right)^{\nu} \left[j^{\nu} + \left(\overline{N} - j \right)^{\nu} \right], \tag{88}$$

vii) if $\left(f(s\omega) s^{N-1} \right)^{(k)} (R_1) = \left(f(s\omega) s^{N-1} \right)^{(k)} (R_2) = 0$*,* $\forall\ \omega \in S^{N-1}$*, (i.e.* $\frac{\partial^k \left(f(s\omega) s^{N-1} \right)}{\partial s^k}$ *vanish on* $\partial B(0, R_1)$ *and* $\partial B(0, R_2)$*) for* $k = 1, ..., n-1$*, from (88) we get:*

$$\left| \int_A f(y)\, dy - \left(\frac{R_2 - R_1}{\overline{N}} \right) \left[j R_1^{N-1} \left(\int_{S^{N-1}} f(R_1\omega)\, d\omega \right) + \right. \right.$$

$$\left. \left. \left(\overline{N} - j \right) R_2^{N-1} \left(\int_{S^{N-1}} f(R_2\omega)\, d\omega \right) \right] \right| \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot$$

$$\frac{K_2}{\Gamma(\nu+1)} \left(\frac{R_2 - R_1}{\overline{N}} \right)^{\nu} \left[j^{\nu} + \left(\overline{N} - j \right)^{\nu} \right], \tag{89}$$

for $j = 0, 1, 2, ..., \overline{N} \in \mathbb{N}$*,*

viii) when $\overline{N} = 2$ *and* $j = 1$*, (89) turns to*

$$\left| \int_A f(y)\, dy - \left(\frac{R_2 - R_1}{2} \right) \left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega)\, d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega)\, d\omega \right) \right|$$

$$\leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_2}{\Gamma(\nu+1)} \frac{(R_2 - R_1)^{\nu}}{2^{\nu-2}}. \tag{90}$$

Proof. We apply Theorem 5 along with (74). See in the 3. Appendix the general proving method in this article. ■

We continue with

Theorem 19 *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*,* $\nu > \frac{1}{q}$*,* $n = \lceil \nu \rceil$*. Consider* $f : \overline{A} \to \mathbb{R}$ *be Lebesgue integrable, which is not necessarily radial. Assume that* $f(s\omega) s^{N-1} \in AC^n([R_1, R_2])$ *(i.e.* $\left(f(s\omega) s^{N-1} \right)^{(n-1)} \in AC([R_1, R_2])$ *- absolutely continuous functions),* $\forall\ \omega \in S^{N-1}$*,* $N \geq 2$*. We assume that* $D_{*R_1}^{\nu} \left(f(s\omega) s^{N-1} \right)$*,* $D_{R_2-}^{\nu} \left(f(s\omega) s^{N-1} \right) \in L_q([R_1, R_2])$*,* $\forall\ \omega \in S^{N-1}$*, and that*

$$\max \left\{ \left\| D_{*R_1}^{\nu} \left(f(s\omega) s^{N-1} \right) \right\|_{L_q([R_1, R_2])}, \left\| D_{R_2-}^{\nu} \left(f(s\omega) s^{N-1} \right) \right\|_{L_q([R_1, R_2])} \right\} \leq K_3, \tag{91}$$

where $K_3 > 0$*,* $s \in [R_1, R_2]$*,* $\forall\ \omega \in S^{N-1}$*.*

Then

i)

$$\left| \int_A f(y)\, dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\left(\int_{S^{N-1}} \left(f(s\omega)\, s^{N-1} \right)^{(k)} (R_1)\, d\omega \right) (t - R_1)^{k+1} + \right.\right.$$

$$\left.\left. (-1)^k \left(\int_{S^{N-1}} \left(f(s\omega)\, s^{N-1} \right)^{(k)} (R_2)\, d\omega \right) (R_2 - t)^{k+1} \right] \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_3}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu - 1) + 1)^{\frac{1}{p}}} \left[(t - R_1)^{\nu + \frac{1}{p}} + (R_2 - t)^{\nu + \frac{1}{p}} \right], \qquad (92)$$

$\forall\ t \in [R_1, R_2]$,

ii) at $t = \frac{R_1 + R_2}{2}$*, the right hand side of (92) is minimized, and we get:*

$$\left| \int_A f(y)\, dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(R_2 - R_1)^{k+1}}{2^{k+1}} \cdot \right.$$

$$\left. \left[\int_{S^{N-1}} \left(f(s\omega)\, s^{N-1} \right)^{(k)} (R_1)\, d\omega + (-1)^k \int_{S^{N-1}} \left(f(s\omega)\, s^{N-1} \right)^{(k)} (R_2)\, d\omega \right] \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_3}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu - 1) + 1)^{\frac{1}{p}}} \frac{(R_2 - R_1)^{\nu + \frac{1}{p}}}{2^{\nu - 1 - \frac{1}{q}}}, \qquad (93)$$

iii) if $\left(f(s\omega)\, s^{N-1} \right)^{(k)} (R_1) = \left(f(s\omega)\, s^{N-1} \right)^{(k)} (R_2) = 0$*,* $\forall\ \omega \in S^{N-1}$*, (i.e.* $\frac{\partial^k \left(f(s\omega) s^{N-1} \right)}{\partial s^k}$ *vanish on* $\partial B(0, R_1)$ *and* $\partial B(0, R_2)$*) for all* $k = 0, 1, ..., n - 1$*, we obtain*

$$\left| \int_A f(y)\, dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_3}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu - 1) + 1)^{\frac{1}{p}}} \frac{(R_2 - R_1)^{\nu + \frac{1}{p}}}{2^{\nu - 1 - \frac{1}{q}}}, \qquad (94)$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., \overline{N} \in \mathbb{N}$*, it holds*

$$\left| \int_A f(y)\, dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{R_2 - R_1}{\overline{N}} \right)^{k+1} \right.$$

$$\left[j^{k+1} \left(\int_{S^{N-1}} \left(f(s\omega)\, s^{N-1} \right)^{(k)} (R_1)\, d\omega \right) + \right.$$

$$\left.\left. (-1)^k \left(\overline{N} - j \right)^{k+1} \left(\int_{S^{N-1}} \left(f(s\omega)\, s^{N-1} \right)^{(k)} (R_2)\, d\omega \right) \right] \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_3}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\cdot$$

$$\left(\frac{R_2-R_1}{\overline{N}}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+\left(\overline{N}-j\right)^{\nu+\frac{1}{p}}\right], \tag{95}$$

v) if $\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)=\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)=0$, $\forall\ \omega\in S^{N-1}$, *(i.e.* $\frac{\partial^k\left(f(s\omega)s^{N-1}\right)}{\partial s^k}$ *vanish on* $\partial B(0,R_1)$ *and* $\partial B(0,R_2)$*) for* $k=1,...,n-1$*, from (95) we get:*

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{\overline{N}}\right)\left[jR_1^{N-1}\left(\int_{S^{N-1}}f(R_1\omega)\,d\omega\right)+\right.\right.$$

$$\left.\left.\left(\overline{N}-j\right)R_2^{N-1}\left(\int_{S^{N-1}}f(R_2\omega)\,d\omega\right)\right]\right|\leq\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\cdot$$

$$\frac{K_3}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\left(\frac{R_2-R_1}{\overline{N}}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+\left(\overline{N}-j\right)^{\nu+\frac{1}{p}}\right], \tag{96}$$

for $j=0,1,2,...,\overline{N}\in\mathbb{N}$,

vi) when $\overline{N}=2$ *and* $j=1$, *(96) turns to*

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}}f(R_1\omega)\,d\omega+R_2^{N-1}\int_{S^{N-1}}f(R_2\omega)\,d\omega\right)\right|$$

$$\leq\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_3}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(R_2-R_1)^{\nu+\frac{1}{p}}}{2^{\nu-1-\frac{1}{q}}}, \tag{97}$$

vii) when $\frac{1}{q}<\nu\leq 1$ *(without any boundary conditions), we get again (97).*

Proof. By Theorem 6 and (74). See also 3. Appendix for the general proving method here. ∎

We give Caputo results on the ball:

Theorem 20 *Let* $0<\nu<1$ *and* $f:\overline{B(0,R)}\to\mathbb{R}$ *be Lebesgue integrable, which is not necessarily radial. Assume that* $f(s\omega)s^{N-1}\in AC([0,R])$ *(absolutely continuous functions),* $\forall\ \omega\in S^{N-1}$, $N\geq 2$. *We assume that* $D_{*0}^{\nu}\left(f(s\omega)s^{N-1}\right)$, $D_{R-}^{\nu}\left(f(s\omega)s^{N-1}\right)\in L_\infty([0,R])$, $\forall\ \omega\in S^{N-1}$ *and that*
$\max\left\{\left\|D_{*0}^{\nu}\left(f(s\omega)s^{N-1}\right)\right\|_{L_\infty([0,R])},\left\|D_{R-}^{\nu}\left(f(s\omega)s^{N-1}\right)\right\|_{L_\infty([0,R])}\right\}\leq M_1$, *where* $M_1>0$, $s\in[0,R]$, $\forall\ \omega\in S^{N-1}$.

Then

i)

$$\left|\int_{B(0,R)}f(y)\,dy-\left(\int_{S^{N-1}}f(R\omega)\,d\omega\right)R^{N-1}(R-t)\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}} M_1}{\Gamma(\nu+2)\,\Gamma\left(\frac{N}{2}\right)}\left[t^{\nu+1}+(R-t)^{\nu+1}\right], \tag{98}$$

$\forall\ t \in [0,R]$,

ii) at $t=\frac{R}{2}$*, the right hand side of (98) is minimized, and we get:*

$$\left|\int_{B(0,R)} f(y)\,dy-\left(\int_{S^{N-1}} f(R\omega)\,d\omega\right)\frac{R^N}{2}\right| \leq \frac{\pi^{\frac{N}{2}} M_1 R^{\nu+1}}{\Gamma(\nu+2)\,\Gamma\left(\frac{N}{2}\right)2^{\nu-1}}, \tag{99}$$

iii) if $f(R\omega)=0$*,* $\forall\ \omega \in S^{N-1}$*, (i.e.* $f(\cdot\omega)$ *vanish on* $\partial B(0,R)$*), we obtain*

$$\left|\int_{B(0,R)} f(y)\,dy\right| \leq \frac{\pi^{\frac{N}{2}} M_1 R^{\nu+1}}{\Gamma(\nu+2)\,\Gamma\left(\frac{N}{2}\right)2^{\nu-1}}, \tag{100}$$

which is a sharp inequality,

iv) more generally, for $j=0,1,2,...,\overline{N} \in \mathbb{N}$*, it holds*

$$\left|\int_{B\ \ (0,R)} f(y)\,dy-\frac{R^N}{\overline{N}}\left(\overline{N}-j\right)\int_{S^{N-1}} f(R\omega)\,d\omega\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}} M_1}{\Gamma(\nu+2)\,\Gamma\left(\frac{N}{2}\right)}\left(\frac{R}{\overline{N}}\right)^{\nu+1}\left[j^{\nu+1}+\left(\overline{N}-j\right)^{\nu+1}\right], \tag{101}$$

v) when $\overline{N}=2$ *and* $j=1$*, (101) turns to*

$$\left|\int_{B(0,R)} f(y)\,dy-\frac{R^N}{2}\int_{S^{N-1}} f(R\omega)\,d\omega\right| \leq \frac{\pi^{\frac{N}{2}} M_1 R^{\nu+1}}{\Gamma(\nu+2)\,\Gamma\left(\frac{N}{2}\right)2^{\nu-1}}. \tag{102}$$

Proof. Same as the proof of Theorem 17, just set there $R_1=0$ and $R_2=R$ and use (73). ∎

We continue with

Theorem 21 *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$*,* $\frac{1}{q}<\nu<1$*, and* $f:\overline{B(0,R)} \to \mathbb{R}$ *be Lebesgue integrable, which is not necessarily radial. Assume that* $f(s\omega)\,s^{N-1} \in AC([0,R])$ *(absolitely continuous functions),* $\forall\ \omega \in S^{N-1}$*,* $N \geq 2$*. We assume that* $D_{*0}^{\nu}\left(f(s\omega)\,s^{N-1}\right)$*,* $D_{R-}^{\nu}\left(f(s\omega)\,s^{N-1}\right) \in L_q([0,R])$*,* $\forall\ \omega \in S^{N-1}$ *and that* $\max\left\{\left\|D_{*0}^{\nu}\left(f(s\omega)\,s^{N-1}\right)\right\|_{L_q([0,R])},\left\|D_{R-}^{\nu}\left(f(s\omega)\,s^{N-1}\right)\right\|_{L_q([0,R])}\right\} \leq M_2$*, where* $M_2>0$*,* $s \in [0,R]$*,* $\forall\ \omega \in S^{N-1}$*.*

Then

i)

$$\left|\int_{B(0,R)} f(y)\,dy-\left(\int_{S^{N-1}} f(R\omega)\,d\omega\right)R^{N-1}(R-t)\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}} M_2}{\Gamma(\nu)\,\Gamma\left(\frac{N}{2}\right)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\left[t^{\nu+\frac{1}{p}}+(R-t)^{\nu+\frac{1}{p}}\right], \tag{103}$$

$\forall\ t \in [0, R]$,

ii) at $t = \frac{R}{2}$, the right hand side of (103) is minimized, and we get:

$$\left| \int_{B(0,R)} f(y)\,dy - \left(\int_{S^{N-1}} f(R\omega)\,d\omega \right) \frac{R^N}{2} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}} M_2 R^{\nu+\frac{1}{p}}}{\Gamma(\nu)\,\Gamma\left(\frac{N}{2}\right)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}\, 2^{\nu-1-\frac{1}{q}}}, \tag{104}$$

iii) if $f(R\omega) = 0$, $\forall\ \omega \in S^{N-1}$, (i.e. $f(\cdot\omega)$ vanish on $\partial B(0,R)$), we obtain

$$\left| \int_{B(0,R)} f(y)\,dy \right| \leq \frac{\pi^{\frac{N}{2}} M_2 R^{\nu+\frac{1}{p}}}{\Gamma(\nu)\,\Gamma\left(\frac{N}{2}\right)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}\, 2^{\nu-1-\frac{1}{q}}}, \tag{105}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., \overline{N} \in \mathbb{N}$, it holds

$$\left| \int_{B\ (0,R)} f(y)\,dy - \frac{R^N}{\overline{N}} \left(\overline{N} - j\right) \int_{S^{N-1}} f(R\omega)\,d\omega \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}} M_2}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}\,\Gamma\left(\frac{N}{2}\right)} \left(\frac{R}{\overline{N}}\right)^{\nu+\frac{1}{p}} \left[j^{\nu+\frac{1}{p}} + \left(\overline{N} - j\right)^{\nu+\frac{1}{p}} \right], \tag{106}$$

v) when $\overline{N} = 2$ and $j = 1$, (106) turns to

$$\left| \int_{B(0,R)} f(y)\,dy - \frac{R^N}{2} \int_{S^{N-1}} f(R\omega)\,d\omega \right| \leq$$

$$\frac{\pi^{\frac{N}{2}} M_2 R^{\nu+\frac{1}{p}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}\, 2^{\nu-1-\frac{1}{q}}}. \tag{107}$$

Proof. Same as the proof of Theorem 19, just set there $R_1 = 0$ and $R_2 = R$ and use (73). ■

Next we give Canavati type results on the shell:

Theorem 22 *Let $\nu \geq 1$, $n = [\nu]$. Consider $f : \overline{A} \to \mathbb{R}$ be Lebesgue integrable, which is not necessarily radial. Assume that $f(s\omega)\, s^{N-1} \in C^{\nu}_{R_1+}([R_1, R_2]) \cap C^{\nu}_{R_2-}([R_1, R_2])$ in $s \in [R_1, R_2]$, $\forall\ \omega \in S^{N-1}$, $N \geq 2$. Suppose there exists $\psi_1 > 0$ such that $\max\left\{ \left\| D^{\nu}_{R_1+}\left(f(s\omega)\, s^{N-1}\right)\right\|_{\infty,[R_1,R_2]}, \left\| \overline{D}^{\nu}_{R_2-}\left(f(s\omega)\, s^{N-1}\right)\right\|_{\infty,[R_1,R_2]} \right\} \leq \psi_1$, where $s \in [R_1, R_2]$, $\forall\ \omega \in S^{N-1}$. Then*

i)

$$\left| \int_A f(y)\,dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\left(\int_{S^{N-1}} \left(f(s\omega)\, s^{N-1}\right)^{(k)} (R_1)\,d\omega \right) (t - R_1)^{k+1} + \right.\right.$$

$$(-1)^k \left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1}\right)^{(k)} (R_2)\, d\omega\right) (R_2 - t)^{k+1}\Bigg]\Bigg| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\psi_1}{\Gamma(\nu+2)} \left[(t-R_1)^{\nu+1} + (R_2-t)^{\nu+1}\right], \tag{108}$$

$\forall\ t \in [R_1, R_2]$,

ii) at $t = \frac{R_1+R_2}{2}$, *the right hand side of (108) is minimized, and we get:*

$$\Bigg|\int_A f(y)\, dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(R_2-R_1)^{k+1}}{2^{k+1}} \left[\int_{S^{N-1}} \left(f(s\omega) s^{N-1}\right)^{(k)} (R_1)\, d\omega + \right.$$

$$\left. (-1)^k \int_{S^{N-1}} \left(f(s\omega) s^{N-1}\right)^{(k)} (R_2)\, d\omega\right]\Bigg| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\psi_1}{\Gamma(\nu+2)} \frac{(R_2-R_1)^{\nu+1}}{2^{\nu-1}}, \tag{109}$$

iii) if $\left(f(s\omega) s^{N-1}\right)^{(k)} (R_1) = \left(f(s\omega) s^{N-1}\right)^{(k)} (R_2) = 0$, $\forall\ \omega \in S^{N-1}$, *(i.e.* $\frac{\partial^k \left(f(s\omega)s^{N-1}\right)}{\partial s^k}$ *vanish on* $\partial B(0,R_1)$ *and* $\partial B(0,R_2)$*) for all* $k = 0,1,...,n-1$, *we obtain*

$$\left|\int_A f(y)\, dy\right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\psi_1}{\Gamma(\nu+2)} \frac{(R_2-R_1)^{\nu+1}}{2^{\nu-1}}, \tag{110}$$

which is a sharp inequality,

iv) more generally, for $j = 0,1,2,...,\overline{N} \in \mathbb{N}$, *it holds*

$$\Bigg|\int_A f(y)\, dy - \sum_{k=0}^{\nu-1} \frac{1}{(k+1)!} \left(\frac{R_2-R_1}{\overline{N}}\right)^{k+1}$$

$$\left[j^{k+1} \left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1}\right)^{(k)} (R_1)\, d\omega\right) + \right.$$

$$\left. (-1)^k \left(\overline{N} - j\right)^{k+1} \left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1}\right)^{(k)} (R_2)\, d\omega\right)\right]\Bigg| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\psi_1}{\Gamma(\nu+2)} \left(\frac{R_2-R_1}{\overline{N}}\right)^{\nu+1} \left[j^{\nu+1} + \left(\overline{N}-j\right)^{\nu+1}\right], \tag{111}$$

v) if $\left(f(s\omega) s^{N-1}\right)^{(k)} (R_1) = \left(f(s\omega) s^{N-1}\right)^{(k)} (R_2) = 0$, $\forall\ \omega \in S^{N-1}$, *(i.e.* $\frac{\partial^k \left(f(s\omega)s^{N-1}\right)}{\partial s^k}$ *vanish on* $\partial B(0,R_1)$ *and* $\partial B(0,R_2)$*) for* $k = 1,...,n-1$, *from (111) we get:*

$$\Bigg|\int_A f(y)\, dy - \left(\frac{R_2-R_1}{\overline{N}}\right) \left[j R_1^{N-1} \left(\int_{S^{N-1}} f(R_1\omega)\, d\omega\right) + \right.$$

$$\left. \left(\overline{N}-j\right) R_2^{N-1} \left(\int_{S^{N-1}} f(R_2\omega)\, d\omega\right)\right]\Bigg| \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}.$$

$$\frac{\psi_1}{\Gamma(\nu+2)}\left(\frac{R_2-R_1}{\overline{N}}\right)^{\nu+1}\left[j^{\nu+1}+\left(\overline{N}-j\right)^{\nu+1}\right], \tag{112}$$

for $j=0,1,2,...,\overline{N}\in\mathbb{N}$,

vi) when $\overline{N}=2$ *and* $j=1$, *(112) turns to*

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}}f(R_1\omega)\,d\omega+R_2^{N-1}\int_{S^{N-1}}f(R_2\omega)\,d\omega\right)\right|$$

$$\le\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\psi_1}{\Gamma(\nu+2)}\frac{(R_2-R_1)^{\nu+1}}{2^{\nu-1}}, \tag{113}$$

Proof. We apply Theorem 7 along with (74). Se also in the 3. Appendix the general proving method in this article. ■

We continue with

Theorem 23 *Let* $\nu\ge1$, $n=[\nu]$. *Consider* $f:\overline{A}\to\mathbb{R}$ *be Lebesgue integrable, which is not necessarily radial. Assume that* $f(s\omega)s^{N-1}\in C_{R_1+}^{\nu}([R_1,R_2])\cap C_{R_2-}^{\nu}([R_1,R_2])$ *in* $s\in[R_1,R_2]$, $\forall\ \omega\in S^{N-1}$, $N\ge2$. *Suppose there exists* $\psi_2>0$ *such that*

$$\max\left\{\left\|D_{R_1+}^{\nu}\left(f(s\omega)s^{N-1}\right)\right\|_{L_1([R_1,R_2])},\left\|\overline{D}_{R_2-}^{\nu}\left(f(s\omega)s^{N-1}\right)\right\|_{L_1([R_1,R_2])}\right\}\le\psi_2,$$

where $s\in[R_1,R_2]$, $\forall\ \omega\in S^{N-1}$.

Then

i)

$$\left|\int_A f(y)\,dy-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)(t-R_1)^{k+1}+\right.\right.$$

$$\left.\left.(-1)^k\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)(R_2-t)^{k+1}\right]\right|\le$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\psi_2}{\Gamma(\nu+1)}\left[(t-R_1)^{\nu}+(R_2-t)^{\nu}\right], \tag{114}$$

$\forall\ t\in[R_1,R_2]$,

ii) when $\nu=1$, *from (114), we have*

$$\left|\int_A f(y)\,dy-\left[\left(\int_{S^{N-1}}f(R_1\omega)\,d\omega\right)R_1^{N-1}(t-R_1)+\right.\right.$$

$$\left.\left.\left(\int_{S^{N-1}}f(R_2\omega)\,d\omega\right)R_2^{N-1}(R_2-t)\right]\right|\le\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\psi_2(R_2-R_1), \tag{115}$$

$\forall\ t\in[R_1,R_2]$,

iii) from (115), we obtain ($\nu=1$ *case,* $t=\frac{R_1+R_2}{2}$*)*

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{2}\right)\left[\left(\int_{S^{N-1}}f(R_1\omega)\,d\omega\right)R_1^{N-1}+\left(\int_{S^{N-1}}f(R_2\omega)\,d\omega\right)R_2^{N-1}\right]\right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \psi_2 \left(R_2 - R_1\right), \tag{116}$$

iv) at $t = \frac{R_1+R_2}{2}$, $\nu > 1$, *the right hand side of (114) is minimized, and we get:*

$$\left| \int_A f(y)\, dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{\left(R_2 - R_1\right)^{k+1}}{2^{k+1}} \left[\int_{S^{N-1}} \left(f(s\omega) s^{N-1}\right)^{(k)} (R_1)\, d\omega + \right.\right.$$

$$\left.\left. (-1)^k \int_{S^{N-1}} \left(f(s\omega) s^{N-1}\right)^{(k)} (R_2)\, d\omega \right] \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\psi_2}{\Gamma(\nu+1)} \frac{\left(R_2 - R_1\right)^{\nu}}{2^{\nu-2}}, \tag{117}$$

v) if $\left(f(s\omega) s^{N-1}\right)^{(k)} (R_1) = \left(f(s\omega) s^{N-1}\right)^{(k)} (R_2) = 0$, $\forall\ \omega \in S^{N-1}$, *(i.e.* $\frac{\partial^k \left(f(s\omega) s^{N-1}\right)}{\partial s^k}$ *vanish on* $\partial B(0, R_1)$ *and* $\partial B(0, R_2)$*) for all* $k = 0, 1, ..., n-1$; $\nu > 1$, *we obtain from (117) that*

$$\left| \int_A f(y)\, dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\psi_2}{\Gamma(\nu+1)} \frac{\left(R_2 - R_1\right)^{\nu}}{2^{\nu-2}}, \tag{118}$$

which is a sharp inequality,

vi) more generally, for $j = 0, 1, 2, ..., \overline{N} \in \mathbb{N}$, *it holds*

$$\left| \int_A f(y)\, dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{R_2 - R_1}{\overline{N}}\right)^{k+1} \right.$$

$$\left[j^{k+1} \left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1}\right)^{(k)} (R_1)\, d\omega \right) + \right.$$

$$\left.\left. (-1)^k \left(\overline{N} - j\right)^{k+1} \left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1}\right)^{(k)} (R_2)\, d\omega \right) \right] \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\psi_2}{\Gamma(\nu+1)} \left(\frac{R_2 - R_1}{\overline{N}}\right)^{\nu} \left[j^{\nu} + \left(\overline{N} - j\right)^{\nu} \right], \tag{119}$$

vii) if $\left(f(s\omega) s^{N-1}\right)^{(k)} (R_1) = \left(f(s\omega) s^{N-1}\right)^{(k)} (R_2) = 0$, $\forall\ \omega \in S^{N-1}$, *(i.e.* $\frac{\partial^k \left(f(s\omega) s^{N-1}\right)}{\partial s^k}$ *vanish on* $\partial B(0, R_1)$ *and* $\partial B(0, R_2)$*) for* $k = 1, ..., n-1$, *from (119) we get:*

$$\left| \int_A f(y)\, dy - \left(\frac{R_2 - R_1}{\overline{N}}\right) \left[j R_1^{N-1} \left(\int_{S^{N-1}} f(R_1 \omega)\, d\omega \right) + \right.\right.$$

$$\left.\left. \left(\overline{N} - j\right) R_2^{N-1} \left(\int_{S^{N-1}} f(R_2 \omega)\, d\omega \right) \right] \right| \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}.$$

$$\frac{\psi_2}{\Gamma(\nu+1)}\left(\frac{R_2-R_1}{\overline{N}}\right)^{\nu}\left[j^{\nu}+\left(\overline{N}-j\right)^{\nu}\right], \tag{120}$$

for $j=0,1,2,...,\overline{N}\in\mathbb{N}$,

viii) when $\overline{N}=2$ *and* $j=1$, *(120) turns to*

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}}f(R_1\omega)\,d\omega+R_2^{N-1}\int_{S^{N-1}}f(R_2\omega)\,d\omega\right)\right|$$

$$\leq\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\psi_2}{\Gamma(\nu+1)}\frac{(R_2-R_1)^{\nu}}{2^{\nu-2}}. \tag{121}$$

Proof. We apply Theorem 8 along with (74). See also in the 3. Appendix the general proving method in this article. ■

We also give

Theorem 24 *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\nu\geq1$, $n=[\nu]$. *Consider* $f:\overline{A}\to\mathbb{R}$ *be Lebesgue integrable, which is not necessarily radial. Assume that* $f(s\omega)s^{N-1}\in C^{\nu}_{R_1+}([R_1,R_2])\cap C^{\nu}_{R_2-}([R_1,R_2])$, *in* $s\in[R_1,R_2]$, $\forall\ \omega\in S^{N-1}$, $N\geq2$. *Suppose that there exists* $\psi_3>0$ *such that*

$$\max\left\{\left\|D^{\nu}_{R_1+}\left(f(s\omega)s^{N-1}\right)\right\|_{L_q([R_1,R_2])},\left\|\overline{D}^{\nu}_{R_2-}\left(f(s\omega)s^{N-1}\right)\right\|_{L_q([R_1,R_2])}\right\}\leq\psi_3,$$

where $s\in[R_1,R_2]$, $\forall\ \omega\in S^{N-1}$.

Then

i)

$$\left|\int_A f(y)\,dy-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)(t-R_1)^{k+1}+\right.\right.$$

$$\left.\left.(-1)^k\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)(R_2-t)^{k+1}\right]\right|\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\psi_3}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\left[(t-R_1)^{\nu+\frac{1}{p}}+(R_2-t)^{\nu+\frac{1}{p}}\right], \tag{122}$$

$\forall\ t\in[R_1,R_2]$,

ii) at $t=\frac{R_1+R_2}{2}$, *the right hand side of (122) is minimized, and we get:*

$$\left|\int_A f(y)\,dy-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(R_2-R_1)^{k+1}}{2^{k+1}}\cdot\right.$$

$$\left.\left[\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega+(-1)^k\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right]\right|\leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\psi_3}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)\left(p(\nu-1)+1\right)^{\frac{1}{p}}}\frac{\left(R_2-R_1\right)^{\nu+\frac{1}{p}}}{2^{\nu-1-\frac{1}{q}}}, \tag{123}$$

iii) if $\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)=\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)=0$, $\forall\ \omega\in S^{N-1}$, *(i.e.* $\frac{\partial^k\left(f(s\omega)s^{N-1}\right)}{\partial s^k}$ *vanish on* $\partial B(0,R_1)$ *and* $\partial B(0,R_2)$*) for all* $k=0,1,...,n-1$*, we obtain*

$$\left|\int_A f(y)\,dy\right|\le\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\psi_3}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)\left(p(\nu-1)+1\right)^{\frac{1}{p}}}\frac{\left(R_2-R_1\right)^{\nu+\frac{1}{p}}}{2^{\nu-1-\frac{1}{q}}}, \tag{124}$$

which is a sharp inequality,

iv) more generally, for $j=0,1,2,...,\overline{N}\in\mathbb{N}$*, it holds*

$$\left|\int_A f(y)\,dy-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{R_2-R_1}{\overline{N}}\right)^{k+1}\right.$$

$$\left[j^{k+1}\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)+\right.$$

$$\left.\left.(-1)^k\left(\overline{N}-j\right)^{k+1}\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)\right]\right|\le$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\psi_3}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)\left(p(\nu-1)+1\right)^{\frac{1}{p}}}\cdot$$

$$\left(\frac{R_2-R_1}{\overline{N}}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+\left(\overline{N}-j\right)^{\nu+\frac{1}{p}}\right], \tag{125}$$

v) if $\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)=\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)=0$, $\forall\ \omega\in S^{N-1}$, *(i.e.* $\frac{\partial^k\left(f(s\omega)s^{N-1}\right)}{\partial s^k}$ *vanish on* $\partial B(0,R_1)$ *and* $\partial B(0,R_2)$*) for* $k=1,...,n-1$*, from (125) we get:*

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{\overline{N}}\right)\left[jR_1^{N-1}\left(\int_{S^{N-1}}f(R_1\omega)\,d\omega\right)+\right.\right.$$

$$\left.\left.\left(\overline{N}-j\right)R_2^{N-1}\left(\int_{S^{N-1}}f(R_2\omega)\,d\omega\right)\right]\right|\le\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\cdot$$

$$\frac{\psi_3}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)\left(p(\nu-1)+1\right)^{\frac{1}{p}}}\left(\frac{R_2-R_1}{\overline{N}}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+\left(\overline{N}-j\right)^{\nu+\frac{1}{p}}\right], \tag{126}$$

for $j=0,1,2,...,\overline{N}\in\mathbb{N}$,

vi) when $\overline{N}=2$ *and* $j=1$*, (126) turns to*

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}}f(R_1\omega)\,d\omega+R_2^{N-1}\int_{S^{N-1}}f(R_2\omega)\,d\omega\right)\right|$$

$$\leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\psi_3}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(R_2-R_1)^{\nu+\frac{1}{p}}}{2^{\nu-1-\frac{1}{q}}}, \tag{127}$$

Proof. By Theorem 9 and (74). See also 3. Appendix for the general proving method here. ■

Next we give Canavati type results on the ball:

Theorem 25 *Let $1 \leq \nu < 2$. Consider $f : \overline{B(0,R)} \to \mathbb{R}$ be Lebesgue integrable, which is not necessarily radial. Assume that $f(s\omega)s^{N-1} \in C^{\nu}_{0+}([0,R]) \cap C^{\nu}_{R-}([0,R])$, in $s \in [0,R]$, $\forall\ \omega \in S^{N-1}$, $N \geq 2$. Suppose there exists $\phi_1 > 0$ such that $\max\left\{\left\|D^{\nu}_{0+}\left(f(s\omega)s^{N-1}\right)\right\|_{\infty,[0,R]}, \left\|\overline{D}^{\nu}_{R-}\left(f(s\omega)s^{N-1}\right)\right\|_{\infty,[0,R]}\right\} \leq \phi_1$, where $s \in [0,R]$, $\forall\ \omega \in S^{N-1}$.*

Then

i)

$$\left|\int_{B(0,R)} f(y)\,dy - \left(\int_{S^{N-1}} f(R\omega)\,d\omega\right) R^{N-1}(R-t)\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}\phi_1}{\Gamma(\nu+2)\Gamma\left(\frac{N}{2}\right)}\left[t^{\nu+1}+(R-t)^{\nu+1}\right], \tag{128}$$

$\forall\ t \in [0,R]$,

ii) at $t = \frac{R}{2}$, the right hand side of (128) is minimized, and we get:

$$\left|\int_{B(0,R)} f(y)\,dy - \left(\int_{S^{N-1}} f(R\omega)\,d\omega\right)\frac{R^N}{2}\right| \leq \frac{\pi^{\frac{N}{2}}\phi_1 R^{\nu+1}}{\Gamma(\nu+2)\Gamma\left(\frac{N}{2}\right)2^{\nu-1}}, \tag{129}$$

iii) if $f(R\omega) = 0$, $\forall\ \omega \in S^{N-1}$, (i.e. $f(\cdot\omega)$ vanish on $\partial B(0,R)$), we obtain

$$\left|\int_{B(0,R)} f(y)\,dy\right| \leq \frac{\pi^{\frac{N}{2}}\phi_1 R^{\nu+1}}{\Gamma(\nu+2)\Gamma\left(\frac{N}{2}\right)2^{\nu-1}}, \tag{130}$$

which is a sharp inequality,

iv) more generally, for $j = 0,1,2,...,\overline{N} \in \mathbb{N}$, it holds

$$\left|\int_{B\ (0,R)} f(y)\,dy - \frac{R^N}{\overline{N}}\left(\overline{N}-j\right)\int_{S^{N-1}} f(R\omega)\,d\omega\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}\phi_1}{\Gamma(\nu+2)\Gamma\left(\frac{N}{2}\right)}\left(\frac{R}{\overline{N}}\right)^{\nu+1}\left[j^{\nu+1}+\left(\overline{N}-j\right)^{\nu+1}\right], \tag{131}$$

v) when $\overline{N} = 2$ and $j = 1$, (131) turns to

$$\left|\int_{B(0,R)} f(y)\,dy - \frac{R^N}{2}\int_{S^{N-1}} f(R\omega)\,d\omega\right| \leq \frac{\pi^{\frac{N}{2}}\phi_1 R^{\nu+1}}{\Gamma(\nu+2)\Gamma\left(\frac{N}{2}\right)2^{\nu-1}}. \tag{132}$$

Proof. Same as the proof of Theorem 22, just set there $R_1 = 0$ and $R_2 = R$ and use (73). $\blacksquare$

We continue with

Theorem 26 *Let $1 \leq \nu < 2$. Consider $f : \overline{B(0,R)} \rightarrow \mathbb{R}$ be Lebesgue integrable, which is not necessarily radial. Assume that $f(s\omega) s^{N-1} \in C_{0+}^{\nu}([0,R]) \cap C_{R-}^{\nu}([0,R])$, in $s \in [0,R]$, $\forall\, \omega \in S^{N-1}$, $N \geq 2$. Suppose there exists $\phi_2 > 0$ such that $\max\left\{ \left\| D_{0+}^{\nu} \left(f(s\omega) s^{N-1} \right) \right\|_{L_1([0,R])}, \left\| \overline{D}_{R-}^{\nu} \left(f(s\omega) s^{N-1} \right) \right\|_{L_1([0,R])} \right\} \leq \phi_2$, where $s \in [0,R]$, $\forall\, \omega \in S^{N-1}$.*

Then

i)

$$\left| \int_{B(0,R)} f(y)\, dy - \left(\int_{S^{N-1}} f(R\omega)\, d\omega \right) R^{N-1} (R-t) \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}} \phi_2}{\Gamma\left(\frac{N}{2}\right) \Gamma(\nu+1)} \left[t^{\nu} + (R-t)^{\nu} \right], \tag{133}$$

$\forall\, t \in [0,R]$,

ii) when $\nu = 1$, from (133), we have

$$\left| \int_{B(0,R)} f(y)\, dy - \left(\int_{S^{N-1}} f(R\omega)\, d\omega \right) R^{N-1} (R-t) \right| \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \phi_2 R, \tag{134}$$

$\forall\, t \in [0,R]$,

iii) from (134), we obtain ($\nu = 1$ case, $t = \frac{R}{2}$)

$$\left| \int_{B(0,R)} f(y)\, dy - \frac{R^N}{2} \left(\int_{S^{N-1}} f(R\omega)\, d\omega \right) \right| \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \phi_2 R, \tag{135}$$

iv) at $t = \frac{R}{2}$, $\nu > 1$, the right hand side of (133) is minimized, and we get:

$$\left| \int_{B(0,R)} f(y)\, dy - \frac{R^N}{2} \left(\int_{S^{N-1}} f(R\omega)\, d\omega \right) \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right) \Gamma(\nu+1)} \phi_2 \frac{R^{\nu}}{2^{\nu-2}}, \tag{136}$$

v) if $f(R\omega) = 0$, $\forall\, \omega \in S^{N-1}$, (from (136)), we get

$$\left| \int_{B(0,R)} f(y)\, dy \right| \leq \frac{\pi^{\frac{N}{2}} \phi_2 R^{\nu}}{\Gamma\left(\frac{N}{2}\right) \Gamma(\nu+1)\, 2^{\nu-2}}, \tag{137}$$

which is a sharp inequality,

vi) more generally, for $j = 0, 1, 2, ..., \overline{N} \in \mathbb{N}$, it holds

$$\left| \int_{B\ (0,R)} f(y)\, dy - \frac{R^N}{\overline{N}} \left(\overline{N} - j \right) \int_{S^{N-1}} f(R\omega)\, d\omega \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\phi_2}{\Gamma(\nu+1)} \left(\frac{R}{\overline{N}}\right)^{\nu} \left[j^{\nu} + \left(\overline{N}-j\right)^{\nu} \right], \tag{138}$$

vii) when $\overline{N}=2$ and $j=1$, (138) turns to

$$\left| \int_{B(0,R)} f(y)\,dy - \frac{R^N}{2} \int_{S^{N-1}} f(R\omega)\,d\omega \right| \leq \frac{\pi^{\frac{N}{2}}\phi_2}{\Gamma\left(\frac{N}{2}\right)\Gamma(\nu+1)} \frac{R^{\nu}}{2^{\nu-2}}. \tag{139}$$

Proof. Same as the proof of Theorem 23, just set there $R_1=0$ and $R_2=R$ and use (73). ∎

We continue with

Theorem 27 *Let $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$, $1\leq\nu<2$. Consider $f:\overline{B(0,R)}\to\mathbb{R}$ be Lebesgue integrable, which is not necessarily radial. Assume that $f(s\omega)s^{N-1}\in C_{0+}^{\nu}([0,R])\cap C_{R-}^{\nu}([0,R])$, in $s\in[0,R]$, $\forall\,\omega\in S^{N-1}$, $N\geq 2$. Suppose there exists $\phi_3>0$ such that $\max\left\{ \left\| D_{0+}^{\nu}\left(f(s\omega)s^{N-1}\right)\right\|_{L_q([0,R])}, \left\| \overline{D}_{R-}^{\nu}\left(f(s\omega)s^{N-1}\right)\right\|_{L_q([0,R])} \right\}$ $\leq\phi_3$, where $s\in[0,R]$, $\forall\,\omega\in S^{N-1}$.*

Then

i)

$$\left| \int_{B(0,R)} f(y)\,dy - \left(\int_{S^{N-1}} f(R\omega)\,d\omega\right) R^{N-1}(R-t) \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}\phi_3}{\Gamma(\nu)\Gamma\left(\frac{N}{2}\right)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \left[t^{\nu+\frac{1}{p}} + (R-t)^{\nu+\frac{1}{p}} \right], \tag{140}$$

$\forall\ t\in[0,R]$,

ii) at $t=\frac{R}{2}$, the right hand side of (140) is minimized, and we get:

$$\left| \int_{B(0,R)} f(y)\,dy - \left(\int_{S^{N-1}} f(R\omega)\,d\omega\right) \frac{R^N}{2} \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}\phi_3 R^{\nu+\frac{1}{p}}}{\Gamma(\nu)\Gamma\left(\frac{N}{2}\right)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}\, 2^{\nu-1-\frac{1}{q}}}, \tag{141}$$

iii) if $f(R\omega)=0$, $\forall\,\omega\in S^{N-1}$, (i.e. $f(\cdot\omega)$ vanish on $\partial B(0,R)$), we obtain

$$\left| \int_{B(0,R)} f(y)\,dy \right| \leq \frac{\pi^{\frac{N}{2}}\phi_3 R^{\nu+\frac{1}{p}}}{\Gamma(\nu)\Gamma\left(\frac{N}{2}\right)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}\, 2^{\nu-1-\frac{1}{q}}}, \tag{142}$$

which is a sharp inequality,

iv) more generally, for $j=0,1,2,...,\overline{N}\in\mathbb{N}$, it holds

$$\left| \int_{B\ (0,R)} f(y)\,dy - \frac{R^N}{\overline{N}}\left(\overline{N}-j\right) \int_{S^{N-1}} f(R\omega)\,d\omega \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}\phi_3}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}\Gamma\left(\frac{N}{2}\right)}\left(\frac{R}{\overline{N}}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+\left(\overline{N}-j\right)^{\nu+\frac{1}{p}}\right], \quad (143)$$

v) when $\overline{N}=2$ and $j=1$, (143) turns to

$$\left|\int_{B(0,R)} f(y)\,dy-\frac{R^N}{2}\int_{S^{N-1}} f(R\omega)\,d\omega\right| \leq$$

$$\frac{\pi^{\frac{N}{2}}\phi_3 R^{\nu+\frac{1}{p}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}2^{\nu-1-\frac{1}{q}}}. \quad (144)$$

Proof. Same as the proof of Theorem 24, just set there $R_1=0$ and $R_2=R$ and use (73). ■

Next we give Conformable type results on the shell:

Theorem 28 *Let $\alpha\in(n,n+1]$, $n\in\mathbb{N}$; $\beta=\alpha-n$. Consider $f:\overline{A}\to\mathbb{R}$ be Lebesgue integrable, which is not necessarily radial. Assume that $f(s\omega)\in C^{n+1}([R_1,R_2])$, in $s\in[R_1,R_2]$, $\forall\ \omega\in S^{N-1}$, $N\geq 2$. Suppose there exists $W_1>0$ such that* $\max\left\{\left\|\mathbf{T}_{\alpha}^{R_1}\left(f(s\omega)s^{N-1}\right)\right\|_{\infty,[R_1,R_2]},\left\|{}_{\alpha}^{R_2}\mathbf{T}\left(f(s\omega)s^{N-1}\right)\right\|_{\infty,[R_1,R_2]}\right\}$ $\leq W_1$, *$\forall\ \omega\in S^{N-1}$, where $s\in[R_1,R_2]$.*

Then

i)

$$\left|\int_A f(y)\,dy-\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)(t-R_1)^{k+1}+\right.\right.$$

$$\left.\left.(-1)^k\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)(R_2-t)^{k+1}\right]\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\Gamma(\beta)W_1}{\Gamma(\alpha+2)}\left[(z-R_1)^{\alpha+1}+(R_2-z)^{\alpha+1}\right], \quad (145)$$

$\forall\ z\in[R_1,R_2]$,

ii) at $z=\frac{R_1+R_2}{2}$, the right hand side of (145) is minimized, and we get:

$$\left|\int_A f(y)\,dy-\sum_{k=0}^{n}\frac{1}{(k+1)!}\frac{(R_2-R_1)^{k+1}}{2^{k+1}}\cdot\right.$$

$$\left.\left[\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega+(-1)^k\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right]\right| \leq$$

$$\frac{\pi^{\frac{N}{2}}\Gamma(\beta)W_1}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha+2)}\frac{(R_2-R_1)^{\alpha+1}}{2^{\alpha-1}}, \quad (146)$$

iii) assuming $\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1) = \left(f(s\omega)s^{N-1}\right)^{(k)}(R_2) = 0$*, for* $k = 0,1,...,n$*,* $\forall\ \omega \in S^{N-1}$*, we obtain*

$$\left|\int_A f(y)\,dy\right| \leq \frac{\pi^{\frac{N}{2}}\Gamma(\beta)W_1}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha+2)}\frac{(R_2-R_1)^{\alpha+1}}{2^{\alpha-1}}, \tag{147}$$

which is a sharp inequality,

iv) more generally, for $j = 0,1,2,...,\overline{N} \in \mathbb{N}$*, it holds*

$$\left|\int_A f(y)\,dy - \sum_{k=0}^{n}\frac{1}{(k+1)!}\left(\frac{R_2-R_1}{\overline{N}}\right)^{k+1}\right.$$

$$\left[j^{k+1}\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)+\right.$$

$$\left.\left.(-1)^k\left(\overline{N}-j\right)^{k+1}\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)\right]\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\Gamma(\beta)W_1}{\Gamma(\alpha+2)}\left(\frac{R_2-R_1}{\overline{N}}\right)^{\alpha+1}\left[j^{\alpha+1}+\left(\overline{N}-j\right)^{\alpha+1}\right], \tag{148}$$

v) if $\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1) = \left(f(s\omega)s^{N-1}\right)^{(k)}(R_2) = 0$*, for* $k = 1,...,n$*,* $\forall\ \omega \in S^{N-1}$*, from (148) we get:*

$$\left|\int_A f(y)\,dy - \left(\frac{R_2-R_1}{\overline{N}}\right)\left[jR_1^{N-1}\left(\int_{S^{N-1}} f(R_1\omega)\,d\omega\right)+\right.\right.$$

$$\left.\left.\left(\overline{N}-j\right)R_2^{N-1}\left(\int_{S^{N-1}} f(R_2\omega)\,d\omega\right)\right]\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\Gamma(\beta)W_1}{\Gamma(\alpha+2)}\left(\frac{R_2-R_1}{\overline{N}}\right)^{\alpha+1}\left[j^{\alpha+1}+\left(\overline{N}-j\right)^{\alpha+1}\right], \tag{149}$$

$j = 0,1,2,...,\overline{N}$,

vi) when $\overline{N} = 2$ *and* $j = 1$*, (149) turns to*

$$\left|\int_A f(y)\,dy - \left(\frac{R_2-R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}} f(R_1\omega)\,d\omega + R_2^{N-1}\int_{S^{N-1}} f(R_2\omega)\,d\omega\right)\right|$$

$$\leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\Gamma(\beta)W_1}{\Gamma(\alpha+2)}\frac{(R_2-R_1)^{\alpha+1}}{2^{\alpha-1}}. \tag{150}$$

Proof. It is based on Theorem 13 and (74). See also 3. Appendix. ■

We give

Theorem 29 *Let $\alpha \in (n, n+1]$, $n \in \mathbb{N}$; $\beta = \alpha - n$. Let also $p_1, p_2, p_3 > 1$: $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, with $\beta > \frac{1}{p_1} + \frac{1}{p_3}$. Consider $f : \overline{A} \to \mathbb{R}$ be Lebesgue integrable, which is not necessarily radial. Assume that $f(s\omega) \in C^{n+1}([R_1, R_2])$, in $s \in [R_1, R_2]$, $\forall\ \omega \in S^{N-1}$, $N \geq 2$. Suppose there exists $W_2 > 0$ such that $\max\left\{\left\|\mathbf{T}_{\alpha}^{R_1}\left(f(s\omega)s^{N-1}\right)\right\|_{p_3,[R_1,R_2]}, \left\|{}_{\alpha}^{R_2}\mathbf{T}\left(f(s\omega)s^{N-1}\right)\right\|_{p_3,[R_1,R_2]}\right\} \leq W_2$, $\forall$ $\omega \in S^{N-1}$, where $s \in [R_1, R_2]$.*

Then

i)

$$\left|\int_A f(y)\,dy - \sum_{k=0}^{n} \frac{1}{(k+1)!}\left[\left(\int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)(t-R_1)^{k+1} + \right.\right.$$

$$\left.\left.(-1)^k\left(\int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)(R_2-t)^{k+1}\right]\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{W_2}{n!\,(p_1 n+1)^{\frac{1}{p_1}}\left(p_2(\beta-1)+1\right)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}\cdot$$

$$\left[(z-R_1)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} + (R_2-z)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\right], \tag{151}$$

$\forall\ z \in [R_1, R_2]$,

ii) at $z = \frac{R_1+R_2}{2}$, the right hand side of (151) is minimized, and we get:

$$\left|\int_A f(y)\,dy - \sum_{k=0}^{n} \frac{1}{(k+1)!}\frac{(R_2-R_1)^{k+1}}{2^{k+1}}\cdot\right.$$

$$\left.\left[\int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega + (-1)^k\int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right]\right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{W_2}{n!\,(p_1 n+1)^{\frac{1}{p_1}}\left(p_2(\beta-1)+1\right)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)} \frac{(R_2-R_1)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-1-\frac{1}{p_3}}}, \tag{152}$$

iii) assuming $\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1) = \left(f(s\omega)s^{N-1}\right)^{(k)}(R_2) = 0$, for $k = 0, 1, ..., n$, $\forall\ \omega \in S^{N-1}$, we obtain

$$\left|\int_A f(y)\,dy\right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\cdot$$

$$\frac{W_2}{n!\,(p_1 n+1)^{\frac{1}{p_1}}\left(p_2(\beta-1)+1\right)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)} \frac{(R_2-R_1)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-1-\frac{1}{p_3}}}, \tag{153}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., \overline{N} \in \mathbb{N}$, it holds

$$\left| \int_A f(y)\, dy - \sum_{k=0}^{n} \frac{1}{(k+1)!} \left(\frac{R_2 - R_1}{\overline{N}} \right)^{k+1} \right.$$

$$\left[j^{k+1} \left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1} \right)^{(k)} (R_1)\, d\omega \right) + \right.$$

$$\left. \left. (-1)^k \left(\overline{N} - j \right)^{k+1} \left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1} \right)^{(k)} (R_2)\, d\omega \right) \right] \right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{W_2}{n!\, (p_1 n + 1)^{\frac{1}{p_1}} (p_2 (\beta - 1) + 1)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_1} + \frac{1}{p_2} \right)} \cdot$$

$$\left(\frac{R_2 - R_1}{\overline{N}} \right)^{\alpha + \frac{1}{p_1} + \frac{1}{p_2}} \left[j^{\alpha + \frac{1}{p_1} + \frac{1}{p_2}} + \left(\overline{N} - j \right)^{\alpha + \frac{1}{p_1} + \frac{1}{p_2}} \right], \tag{154}$$

v) if $\left(f(s\omega) s^{N-1} \right)^{(k)} (R_1) = \left(f(s\omega) s^{N-1} \right)^{(k)} (R_2) = 0$, for $k = 1, ..., n$, $\forall$ $\omega \in S^{N-1}$, from (154) we get:

$$\left| \int_A f(y)\, dy - \left(\frac{R_2 - R_1}{\overline{N}} \right) \left[j R_1^{N-1} \left(\int_{S^{N-1}} f(R_1 \omega)\, d\omega \right) + \right. \right.$$

$$\left. \left. \left(\overline{N} - j \right) R_2^{N-1} \left(\int_{S^{N-1}} f(R_2 \omega)\, d\omega \right) \right] \right| \leq$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{W_2}{n!\, (p_1 n + 1)^{\frac{1}{p_1}} (p_2 (\beta - 1) + 1)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_1} + \frac{1}{p_2} \right)}$$

$$\left(\frac{R_2 - R_1}{\overline{N}} \right)^{\alpha + \frac{1}{p_1} + \frac{1}{p_2}} \left[j^{\alpha + \frac{1}{p_1} + \frac{1}{p_2}} + \left(\overline{N} - j \right)^{\alpha + \frac{1}{p_1} + \frac{1}{p_2}} \right], \tag{155}$$

$j = 0, 1, 2, ..., \overline{N}$,

vi) when $\overline{N} = 2$ and $j = 1$, (155) turns to

$$\left| \int_A f(y)\, dy - \left(\frac{R_2 - R_1}{2} \right) \left[R_1^{N-1} \left(\int_{S^{N-1}} f(R_1 \omega)\, d\omega \right) + R_2^{N-1} \left(\int_{S^{N-1}} f(R_2 \omega) \right) d\omega \right] \right|$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{W_2}{n!\, (p_1 n + 1)^{\frac{1}{p_1}} (p_2 (\beta - 1) + 1)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_1} + \frac{1}{p_2} \right)}$$

$$\frac{(R_2 - R_1)^{\alpha + \frac{1}{p_1} + \frac{1}{p_2}}}{2^{\alpha - 1 - \frac{1}{p_3}}}, \tag{156}$$

Proof. It is based on Theorem 14 and (74). See also 3. Appendix. ■

Next we give conformable results on the ball.

Theorem 30 *Let $\alpha \in (0,1]$ and consider $f : \overline{B(0,R)} \to \mathbb{R}$ to be Lebesgue integrable, which is not necessarily radial. Assume that $f(s\omega) \in C^1([0,R])$, in $s \in [0,R]$, $\forall\ \omega \in S^{N-1}$, $N \geq 2$. Suppose there exists $\theta_1 > 0$ such that* $\max\left\{\left\|\mathbf{T}_\alpha^R\left(f(s\omega)s^{N-1}\right)\right\|_{\infty,[0,R]}, \left\|{}_\alpha^R\mathbf{T}\left(f(s\omega)s^{N-1}\right)\right\|_{\infty,[0,R]}\right\} \leq \theta_1, \forall\, \omega \in S^{N-1}$, *where $s \in [0,R]$.*

Then

i)

$$\left|\int_{B(0,R)} f(y)\,dy - \left(\int_{S^{N-1}} f(R\omega)\,d\omega\right) R^{N-1}(R-t)\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\theta_1}{\alpha(\alpha+1)} \left[z^{\alpha+1} + (R-z)^{\alpha+1}\right], \tag{157}$$

$,\forall\ z \in [0,R]$,

ii) at $t = \frac{R}{2}$, the right hand side of (157) is minimized, and we get:

$$\left|\int_{B(0,R)} f(y)\,dy - \left(\int_{S^{N-1}} f(R\omega)\,d\omega\right) \frac{R^N}{2}\right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\theta_1}{\alpha(\alpha+1)} \frac{R^{\alpha+1}}{2^{\alpha-1}}, \tag{158}$$

iii) if $f(R\omega) = 0$, $\forall\ \omega \in S^{N-1}$, we obtain

$$\left|\int_{B(0,R)} f(y)\,dy\right| \leq \frac{\pi^{\frac{N}{2}}\theta_1 R^{\alpha+1}}{\Gamma\left(\frac{N}{2}\right)\alpha(\alpha+1)2^{\alpha-1}}, \tag{159}$$

which is a sharp inequality,

iv) more generally, for $j = 0,1,2,...,\overline{N} \in \mathbb{N}$, it holds

$$\left|\int_{B\ (0,R)} f(y)\,dy - \frac{R^N}{\overline{N}}\left(\overline{N} - j\right)\int_{S^{N-1}} f(R\omega)\,d\omega\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\theta_1}{\alpha(\alpha+1)} \left(\frac{R}{\overline{\overline{N}}}\right)^{\alpha+1} \left[j^{\alpha+1} + \left(\overline{N} - j\right)^{\alpha+1}\right], \tag{160}$$

v) when $\overline{N} = 2$ and $j = 1$, (160) turns to

$$\left|\int_{B(0,R)} f(y)\,dy - \frac{R^N}{2}\int_{S^{N-1}} f(R\omega)\,d\omega\right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\theta_1}{\alpha(\alpha+1)} \frac{R^{\alpha+1}}{2^{\alpha-1}}. \tag{161}$$

Proof. It is based on Theorem 28, just there $R_1 = 0$ and $R_2 = R$ and use (73). Notice here $n = 0$, and $\alpha = \beta$. ■

Next we give conformable results on the ball.

Theorem 31 *Let $\alpha \in (0,1]$ and let also $p_1, p_2, p_3 > 1 : \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, with $\alpha > \frac{1}{p_1} + \frac{1}{p_3}$ and consider $f : \overline{B(0,R)} \to \mathbb{R}$ be Lebesgue integrable, which is not necessarily radial. Assume that $f(s\omega) \in C^1([0,R])$, in $s \in [0,R]$, $\forall\ \omega \in S^{N-1}$, $N \geq 2$. Suppose there exists $\theta_2 > 0$ such that $\max\{\left\|\mathbf{T}_{\alpha}^{R}\left(f(s\omega)s^{N-1}\right)\right\|_{p_3,[0,R]},$ $\left\|{}_{\alpha}^{R}\mathbf{T}\left(f(s\omega)s^{N-1}\right)\right\|_{p_3,[0,R]}\} \leq \theta_2$, $\forall\ \omega \in S^{N-1}$, where $s \in [0,R]$.*

Then

i)

$$\left|\int_{B(0,R)} f(y)\,dy - \left(\int_{S^{N-1}} f(R\omega)\,d\omega\right) R^{N-1}(R-t)\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\theta_2}{\left(p_2(\alpha-1)+1\right)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)} \cdot$$

$$\left[z^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} + (R-z)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\right], \tag{162}$$

$\forall\ z \in [0,R]$,

ii) at $t = \frac{R}{2}$, the right hand side of (162) is minimized, and we get:

$$\left|\int_{B(0,R)} f(y)\,dy - \left(\int_{S^{N-1}} f(R\omega)\,d\omega\right) \frac{R^N}{2}\right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\theta_2}{\left(p_2(\alpha-1)+1\right)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)} \frac{R^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-1-\frac{1}{p_3}}}, \tag{163}$$

iii) if $f(R\omega) = 0$, $\forall\ \omega \in S^{N-1}$, we obtain

$$\left|\int_{B(0,R)} f(y)\,dy\right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot$$

$$\frac{\theta_2}{\left(p_2(\alpha-1)+1\right)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)} \frac{R^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-1-\frac{1}{p_3}}}, \tag{164}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., \overline{N} \in \mathbb{N}$, it holds

$$\left|\int_{B\ (0,R)} f(y)\,dy - \frac{R^N}{\overline{N}}\left(\overline{N}-j\right)\int_{S^{N-1}} f(R\omega)\,d\omega\right| \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\theta_2}{\left(p_2(\alpha-1)+1\right)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)} \cdot$$

$$\left(\frac{R}{\overline{N}}\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} \left[j^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} + \left(\overline{N}-j\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\right], \tag{165}$$

v) when $\overline{N} = 2$ and $j = 1$, (165) turns to

$$\left| \int_{B(0,R)} f(y)\, dy - \frac{R^N}{2} \int_{S^{N-1}} f(R\omega)\, d\omega \right| \leq$$

$$\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\theta_2}{\left(p_2(\alpha-1)+1\right)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_1} + \frac{1}{p_2}\right)} \frac{R^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-1-\frac{1}{p_3}}}. \tag{166}$$

Proof. It is based on Theorem 29, just there $R_1 = 0$ and $R_2 = R$ and use (73). Notice here $n = 0$, and $\alpha = \beta$. ■

3 Appendix

Proof. (**Detailed proof of Theorem 17 -** serving as a model proof for the rest of this article.)

We apply Theorem 4 (i) for $f(s\omega)\, s^{N-1}$:

$$\left| \int_{R_1}^{R_2} f(s\omega)\, s^{N-1} ds - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\left(f(s\omega)\, s^{N-1} \right)^{(k)} (R_1)(t-R_1)^{k+1} + \right.\right.$$

$$\left.\left. (-1)^k \left(f(s\omega)\, s^{N-1} \right)^{(k)} (R_2)(R_2-t)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{*R_1}^{\nu} \left(f(s\omega)\, s^{N-1} \right) \right\|_{L_\infty([R_1,R_2])}, \left\| D_{R_2-}^{\nu} \left(f(s\omega)\, s^{N-1} \right) \right\|_{L_\infty([R_1,R_2])} \right\}}{\Gamma(\nu+2)} \cdot$$

$$\left[(t-R_1)^{\nu+1} + (R_2-t)^{\nu+1} \right] \leq \tag{167}$$

$$\frac{K_1}{\Gamma(\nu+2)} \left[(t-R_1)^{\nu+1} + (R_2-t)^{\nu+1} \right] =: \rho(t),$$

$\forall\, t \in [R_1, R_2]$, and $\forall\, \omega \in S^{N-1}$.

Equivalently, we have that

$$-\rho(t) \leq \int_{R_1}^{R_2} f(s\omega)\, s^{N-1} ds - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\left(f(s\omega)\, s^{N-1} \right)^{(k)} (R_1)(t-R_1)^{k+1} \right.$$

$$\left. + (-1)^k \left(f(s\omega)\, s^{N-1} \right)^{(k)} (R_2)(R_2-t)^{k+1} \right] \leq \rho(t), \tag{168}$$

$\forall\, t \in [R_1, R_2]$, and $\forall\, \omega \in S^{N-1}$.

Therefore it holds

$$-\rho(t) \int_{S^{N-1}} d\omega \leq \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(s\omega)\, s^{N-1} ds \right) d\omega -$$

$$\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)(t-R_1)^{k+1}+\right.$$

$$\left.(-1)^k\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)(R_2-t)^{k+1}\right]\Bigg|\leq$$

$$\leq\rho(t)\int_{S^{N-1}}d\omega,\quad\forall\ t\in[R_1,R_2]. \tag{169}$$

By (74) we obtain

$$-\rho(t)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\leq\int_A f(y)\,dy-$$

$$\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)(t-R_1)^{k+1}+\right.$$

$$\left.(-1)^k\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)(R_2-t)^{k+1}\right]\Bigg|\leq$$

$$\rho(t)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)},\quad\forall\ t\in[R_1,R_2]. \tag{170}$$

Consequently it holds

$$\left|\int_A f(y)\,dy-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)(t-R_1)^{k+1}\right.\right.$$

$$\left.\left.+(-1)^k\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)(R_2-t)^{k+1}\right]\right|\leq\rho(t)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}=$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_1}{\Gamma(\nu+2)}\left[(t-R_1)^{\nu+1}+(R_2-t)^{\nu+1}\right], \tag{171}$$

$\forall\ t\in[R_1,R_2]$,

proving Theorem 17 (i).

Next consider

$$\varphi(t):=(t-R_1)^{\nu+1}+(R_2-t)^{\nu+1},\quad\forall\ t\in[R_1,R_2].$$

Then

$$\varphi'(t)=(\nu+1)\left[(t-R_1)^{\nu}-(R_2-t)^{\nu}\right]=0,$$

and φ has the only critical number $t=\frac{R_1+R_2}{2}$. Hence $\varphi(t)$ has a minimum over $[R_1,R_2]$ which is $\varphi\left(\frac{R_1+R_2}{2}\right)=\frac{(R_2-R_1)^{\nu+1}}{2^{\nu}}$.

Consequently, it holds (by (171))

$$\left|\int_A f(y)\,dy-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(R_2-R_1)^{k+1}}{2^{k+1}}\right.$$

$$\left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1} \right)^{(k)} (R_1) \, d\omega \right) + (-1)^k \left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1} \right)^{(k)} (R_2) \, d\omega \right) \Bigg|$$

$$\leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_1}{\Gamma(\nu+2)} \frac{(R_2 - R_1)^{\nu+1}}{2^{\nu-1}}, \tag{172}$$

proving Theorem 17 (ii).

The rest of Theorem 17 is obvious or follows the same way as above. ■

The rest of the proofs of this article as similar are omitted.

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