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SOME INEQUALITIES OF JENSEN TYPE FOR TRIGONOMETRICALLY ρ-CONVEX FUNCTIONS

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Abstract. In this paper we establish several Jensen type integral inequalities for trigonometrically ρ-convex functions. Some examples for power function and applications for continuous functions of selfadjoint operators on Hilbert spaces are provided as well.

1. Introduction

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $\Phi : I \to \mathbb{R}$ is a convex function on I. Then Φ is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x,\ y \in \mathring{I}$ and $x < y$, then $\Phi'_{-}(x) \leq \Phi'_{+}(x) \leq \Phi'_{-}(y) \leq \Phi'_{+}(y)$ which shows that both Φ'_{-} and Φ'_{+} are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $\Phi : I \to \mathbb{R}$, the subdifferential of Φ denoted by $\partial\Phi$ is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$\Phi(x) \geq \Phi(a) + (x-a)\varphi(a) \text{ for any } x,\ a \in I. \tag{1.1}$$

It is also well known that if Φ is convex on I, then $\partial\Phi$ is nonempty, Φ'_{-}, $\Phi'_{+} \in \partial\Phi$ and if $\varphi \in \partial\Phi$, then

$$\Phi'_{-}(x) \leq \varphi(x) \leq \Phi'_{+}(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If Φ is differentiable and convex on $\mathring{I}$, then $\partial\Phi = \{\Phi'\}$.

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ -a.e. (almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)|\, w(x)\, d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\, d\mu(x)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, we obtained in 2002 [7] the following result:

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Theorem 1. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_w (\Omega, \mu)$, where $w \geq 0$ μ-a.e. (almost everywhere) on Ω with $\int_\Omega w d\mu = 1$. Then we have the inequality:*

$$\begin{aligned} 0 &\leq \int_\Omega (\Phi \circ f)\, w d\mu - \Phi \left(\int_\Omega f w d\mu \right) \\ &\leq \int_\Omega (\Phi' \circ f) f w d\mu - \int_\Omega (\Phi' \circ f)\, w d\mu \int_\Omega f w d\mu \\ &\leq \frac{1}{2} \left[\Phi'_- (M) - \Phi'_- (m) \right] \int_\Omega \left| f - \int_\Omega f w d\mu \right| w d\mu. \end{aligned} \tag{1.2}$$

We also have the following result which provides a general Fejér's type inequality [11] for the general Lebesgue integral [8]:

Theorem 2. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[m, M]$ and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f \in L_w (\Omega, \mu)$, where $w \geq 0$ μ-a.e. on Ω with $\int_\Omega w d\mu = 1$. Then we have the inequalities:*

$$\begin{aligned} &\Phi \left(\frac{m + M}{2} \right) + \varphi \left(\frac{m + M}{2} \right) \int_\Omega \left(f - \frac{m + M}{2} \right) w d\mu \\ &\leq \int_\Omega (\Phi \circ f)\, w d\mu \\ &\leq \frac{\Phi(m) + \Phi(M)}{2} + \frac{\Phi(M) - \Phi(m)}{M - m} \int_\Omega \left(f - \frac{m + M}{2} \right) w d\mu, \end{aligned} \tag{1.3}$$

where $\varphi \left(\frac{m+M}{2} \right) \in \left[\Phi'_- \left(\frac{m+M}{2} \right), \Phi'_+ \left(\frac{m+M}{2} \right) \right]$.

In order to extend these results for trigonometrically ρ-convex functions, we need the following preparations.

Let I be a finite or infinite open interval of real numbers and $\rho > 0$.

In the following we present the basic definitions and results concerning the class of trigonometrically ρ-convex function, see for example [14], [15] and [3], [5], [6], [12], [16], [17] and [18].

Following [1], we say that a function $\Phi : I \to \mathbb{R}$ is *trigonometrically ρ-convex* on I if for any closed subinterval $[a, b]$ of I with $0 < b - a < \frac{\pi}{\rho}$ we have

$$\Phi(x) \leq \frac{\sin[\rho(b - x)]}{\sin[\rho(b - a)]} \Phi(a) + \frac{\sin[\rho(x - a)]}{\sin[\rho(b - a)]} \Phi(b) \tag{1.4}$$

for all $x \in [a, b]$.

If the inequality (1.4) holds with " $\geq$ ", then the function will be called *trigonometrically ρ-concave* on I.

Geometrically speaking, this means that the graph of Φ on $[a, b]$ lies nowhere above the ρ-trigonometric function determined by the equation

$$H(x) = H(x; a, b, \Phi) := A \cos(\rho x) + B \sin(\rho x)$$

where A and B are chosen such that $H(a) = \Phi(a)$ and $H(b) = \Phi(b)$.

If we take $x = (1 - t) a + t b \in [a, b]$, $t \in [0, 1]$, then the condition (1.4) becomes

$$\Phi((1 - t) a + t b) \leq \frac{\sin[\rho(1 - t)(b - a)]}{\sin[\rho(b - a)]} \Phi(a) + \frac{\sin[\rho t (b - a)]}{\sin[\rho(b - a)]} \Phi(b) \tag{1.5}$$

for any $t \in [0, 1]$.

We have the following properties of trigonometrically ρ-convex functions on I, [1]:

(i) A trigonometrically ρ-convex function $\Phi : I \to \mathbb{R}$ has finite right and left derivatives $\Phi'_+(x)$ and $\Phi'_-(x)$ at every point $x \in I$ and $\Phi'_-(x) \leq \Phi'_+(x)$. The function Φ is differentiable on I with the exception of an at most countable set.

(ii) A necessary and sufficient condition for the function $\Phi : I \to \mathbb{R}$ to be trigonometrically ρ-convex function on I is that it satisfies the gradient inequality

$$\Phi(y) \geq \Phi(x) \cos[\rho(y-x)] + K_{x,\Phi} \sin[\rho(y-x)] \tag{1.6}$$

for any $x,\ y \in I$ where $K_{x,\Phi} \in \left[\Phi'_-(x), \Phi'_+(x)\right]$. If Φ is differentiable at the point x then $K_{x,\Phi} = \Phi'(x)$.

(iii) A necessary and sufficient condition for the function Φ to be a trigonometrically ρ-convex in I, is that the function

$$\varphi(x) = \Phi'(x) + \rho^2 \int_a^x \Phi(t)\, dt$$

is nondecreasing on I, where $a \in I$.

(iv) Let $\Phi : I \to \mathbb{R}$ be a two times continuously differentiable function on I. Then Φ is trigonometrically ρ-convex on I if and only if for all $x \in I$ we have

$$\Phi''(x) + \rho^2 \Phi(x) \geq 0. \tag{1.7}$$

For other properties of trigonometrically ρ-convex functions, see [1].

As general examples of trigonometrically ρ-convex functions we can give the indicator function

$$h_F(\theta) := \limsup_{r \to \infty} \frac{\log\left|F\left(re^{i\theta}\right)\right|}{r^\rho},\ \theta \in (\alpha, \beta),$$

where F is an entire function of order $\rho \in (0, \infty)$.

If $0 < \beta - \alpha < \frac{\pi}{\rho}$, then, it was shown in 1908 by Phragmén and Lindelöf, see [14], that h_F is trigonometrically ρ-convex on (α, β).

Using the condition (1.7) one can also observe that any nonnegative twice differentiable and convex function on I is also trigonometrically ρ-convex on I for any $\rho > 0$.

There exists also concave functions on an interval that are trigonometrically ρ-convex on that interval for some $\rho > 0$.

Consider for example $\Phi(x) = \cos x$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then

$$\Phi''(x) + \rho^2 \Phi(x) = -\cos x + \rho^2 \cos x = \left(\rho^2 - 1\right) \cos x,$$

which shows that it is trigonometrically ρ-convex on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for all $\rho > 1$ and trigonometrically ρ-concave for $\rho \in (0, 1)$.

In this paper we establish several Jensen type integral inequalities for trigonometrically ρ-convex functions. Some examples for power function and applications for continuous functions of selfadjoint operators on Hilbert spaces are provided as well.

2. Main Results

In the following we assume that $\rho > 0$ and m, M are real numbers such that $0 < M - m < \frac{\pi}{\rho}$.

We have the following result:

Theorem 3. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a trigonometrically ρ-convex function on $[m, M]$ and $f : \Omega \to [m, M]$ so that $\Phi \circ f$, $\Phi \circ (m + M - f)$, $f \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. on Ω. Then*

$$\begin{aligned} &\Phi\left(\frac{m+M}{2}\right) \int_\Omega w \cos\left[\rho\left(f - \frac{m+M}{2}\right)\right] d\mu \\ &\leq \frac{1}{2}\left[\int_\Omega (\Phi \circ f)\, w d\mu + \int_\Omega \Phi \circ (m + M - f)\, w d\mu\right] \\ &\leq \frac{\Phi(m) + \Phi(M)}{2} \frac{\int_\Omega w \cos\left[\rho\left(f - \frac{m+M}{2}\right)\right] d\mu}{\cos\left[\frac{\rho(M-m)}{2}\right]}. \end{aligned} \tag{2.1}$$

Proof. From (1.4) we have by replacing x with $m + M - x$ that

$$\Phi(m + M - x) \leq \frac{\sin[\rho(x - m)]}{\sin[\rho(M - m)]}\Phi(m) + \frac{\sin[\rho(M - x)]}{\sin[\rho(M - m)]}\Phi(M) \tag{2.2}$$

for any $x \in [m, M]$.

If we add (1.4) with (2.11) we get

$$\begin{aligned} &\Phi(x) + \Phi(m + M - x) \\ &\leq \frac{\sin[\rho(M - x)]}{\sin[\rho(M - m)]}\Phi(m) + \frac{\sin[\rho(x - m)]}{\sin[\rho(M - m)]}\Phi(M) \\ &+ \frac{\sin[\rho(x - m)]}{\sin[\rho(M - m)]}\Phi(m) + \frac{\sin[\rho(M - x)]}{\sin[\rho(M - m)]}\Phi(M) \\ &= \frac{\sin[\rho(M - x)] + \sin[\rho(x - m)]}{\sin[\rho(M - m)]}\Phi(m) \\ &+ \frac{\sin[\rho(M - x)] + \sin[\rho(x - m)]}{\sin[\rho(M - m)]}\Phi(M) \\ &= \frac{\sin[\rho(M - x)] + \sin[\rho(x - m)]}{\sin[\rho(M - m)]}[\Phi(m) + \Phi(M)] \end{aligned} \tag{2.3}$$

for any $x \in [m, M]$.

Observe that

$$\begin{aligned} &\frac{\sin[\rho(M - x)] + \sin[\rho(x - m)]}{\sin[\rho(M - m)]} \\ &= \frac{2\sin\left[\frac{\rho(M-m)}{2}\right]\cos\left[\rho\left(x - \frac{m+M}{2}\right)\right]}{2\sin\left[\frac{\rho(M-m)}{2}\right]\cos\left[\frac{\rho(M-m)}{2}\right]} = \frac{\cos\left[\rho\left(x - \frac{m+M}{2}\right)\right]}{\cos\left[\frac{\rho(M-m)}{2}\right]} \end{aligned} \tag{2.4}$$

for any $x \in [m, M]$.

Using the equality (2.4) and dividing by 2 in (2.3) we get

$$\frac{1}{2}[\Phi(x) + \Phi(m + M - x)] \leq \frac{\cos\left[\rho\left(x - \frac{m+M}{2}\right)\right]}{\cos\left[\frac{\rho(M-m)}{2}\right]}\left[\frac{\Phi(m) + \Phi(M)}{2}\right] \tag{2.5}$$

for any $x \in [m, M]$.

From (1.5) for $t = \frac{1}{2}$ and $m = u$, $M = v$ we get

$$\begin{aligned}\Phi\left(\frac{u+v}{2}\right) &\leq \frac{\sin\left[\rho\left(\frac{v-u}{2}\right)\right]}{\sin\left[\rho\left(v-u\right)\right]}\Phi\left(u\right)+\frac{\sin\left[\rho\left(\frac{v-u}{2}\right)\right]}{\sin\left[\rho\left(v-u\right)\right]}\Phi\left(v\right)\\ &= \frac{\sin\left[\rho\left(\frac{v-u}{2}\right)\right]}{\sin\left[\rho\left(v-u\right)\right]}\left[\Phi\left(u\right)+\Phi\left(v\right)\right]\\ &= \frac{\sin\left[\rho\left(\frac{v-u}{2}\right)\right]}{2\sin\left[\rho\left(\frac{v-u}{2}\right)\right]\cos\left[\rho\left(\frac{v-u}{2}\right)\right]}\left[\Phi\left(u\right)+\Phi\left(v\right)\right]\\ &= \frac{1}{\cos\left[\rho\left(\frac{v-u}{2}\right)\right]}\frac{\Phi\left(u\right)+\Phi\left(v\right)}{2},\end{aligned}$$

which implies that

$$\Phi\left(\frac{u+v}{2}\right)\cos\left[\rho\left(\frac{v-u}{2}\right)\right] \leq \frac{\Phi\left(u\right)+\Phi\left(v\right)}{2} \tag{2.6}$$

for any u, $v \in I$.

Now, if in (2.6) we take $v = x$ and $u = m + M - x$, then we get

$$\Phi\left(\frac{m+M}{2}\right)\cos\left[\rho\left(x-\frac{m+M}{2}\right)\right] \leq \frac{1}{2}\left[\Phi\left(x\right)+\Phi\left(m+M-x\right)\right] \tag{2.7}$$

for any $x \in [m, M]$.

By taking $x = f(s)$, $s \in \Omega$ in (2.5) and (2.7), we get

$$\begin{aligned}&\Phi\left(\frac{m+M}{2}\right)\cos\left[\rho\left(f\left(s\right)-\frac{m+M}{2}\right)\right]\\ &\leq \frac{1}{2}\left[\Phi\left(f\left(s\right)\right)+\Phi\left(m+M-f\left(s\right)\right)\right]\\ &\leq \frac{\Phi\left(m\right)+\Phi\left(M\right)}{2}\frac{\cos\left[\rho\left(f\left(s\right)-\frac{m+M}{2}\right)\right]}{\cos\left[\frac{\rho\left(M-m\right)}{2}\right]}\end{aligned} \tag{2.8}$$

for any $s \in \Omega$.

By multiplying (2.8) with $w(s) \geq 0$ and integrate on Ω, we get the desired result (2.1). □

Corollary 1. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a trigonometrically ρ-convex function on $[m, M]$ and $f : [a, b] \to [m, M]$ so that $\Phi \circ f$, $\Phi \circ (m + M - f)$, $f \in L_w[a, b]$, where $w \geq 0$ μ-a.e. on $[a, b]$. Then*

$$\begin{aligned}&\Phi\left(\frac{m+M}{2}\right)\int_a^b w\left(t\right)\cos\left[\rho\left(f\left(t\right)-\frac{m+M}{2}\right)\right]dt\\ &\leq \frac{1}{2}\left[\int_a^b \Phi\left(f\left(t\right)\right)w\left(t\right)dt+\int_a^b \Phi\left(m+M-f\left(t\right)\right)w\left(t\right)dt\right]\\ &\leq \frac{\Phi\left(m\right)+\Phi\left(M\right)}{2}\frac{\int_a^b w\left(t\right)\cos\left[\rho\left(f\left(t\right)-\frac{m+M}{2}\right)\right]dt}{\cos\left[\frac{\rho\left(M-m\right)}{2}\right]}.\end{aligned} \tag{2.9}$$

We also have the Jensen's type inequality:

Theorem 4. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a trigonometrically ρ-convex function on $[m, M]$ and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. on Ω. Assume that $\int_\Omega \cos(\rho f)\, w d\mu \neq 0$ and*

$$\overline{f}_{\rho,w} := \frac{1}{\rho} \arctan\left(\frac{\int_\Omega \sin(\rho f)\, w d\mu}{\int_\Omega \cos(\rho f)\, w d\mu}\right) \in [m, M], \tag{2.10}$$

then we have:

$$\int_\Omega (\Phi \circ f)\, w d\mu \geq \Phi\left(\overline{f}_{\rho,w}\right) \int_\Omega \cos\left[\rho\left(f - \overline{f}_{\rho,w}\right)\right] w d\mu. \tag{2.11}$$

Proof. By the gradient inequality (1.6) we have

$$\Phi(y) \geq \Phi\left(\overline{f}_{\rho,w}\right) \cos\left[\rho\left(y - \overline{f}_{\rho,w}\right)\right] + K_{\overline{f}_{\rho,w},\Phi} \sin\left[\rho\left(y - \overline{f}_{\rho,w}\right)\right] \tag{2.12}$$

for any $y \in [m, M]$.

If we replace y with $f(s) \in [m, M]$, multiply by $w(s) \geq 0$, with $s \in \Omega$ and integrate on Ω, we get

$$\begin{aligned} \int_\Omega (\Phi \circ f)\, w d\mu &\geq \Phi\left(\overline{f}_{\rho,w}\right) \int_\Omega \cos\left[\rho\left(f - \overline{f}_{\rho,w}\right)\right] w d\mu \\ &\quad + K_{\overline{f}_{\rho,w},\Phi} \int_\Omega \sin\left[\rho\left(f - \overline{f}_{\rho,w}\right)\right] w d\mu. \end{aligned} \tag{2.13}$$

We have, by using the definition of $\overline{f}_{\rho,w}$, that

$$\begin{aligned} &\int_\Omega \sin\left[\rho\left(f - \overline{f}_{\rho,w}\right)\right] w d\mu \\ &= \int_\Omega \left[\sin(\rho f) \cos\left(\rho \overline{f}_{\rho,w}\right) - \sin\left(\rho \overline{f}_{\rho,w}\right) \cos(\rho f)\right] w d\mu \\ &= \cos\left(\rho \overline{f}_{\rho,w}\right) \int_\Omega \sin(\rho f)\, w d\mu - \sin\left(\rho \overline{f}_{\rho,w}\right) \int_\Omega \cos(\rho f)\, w d\mu \\ &= \cos\left(\rho \overline{f}_{\rho,w}\right) \int_\Omega \cos(\rho f)\, w d\mu \left[\frac{\int_\Omega \sin(\rho f)\, w d\mu}{\int_\Omega \cos(\rho f)\, w d\mu} - \tan\left(\rho \overline{f}_{\rho,w}\right)\right] \\ &= \cos\left(\rho \overline{f}_{\rho,w}\right) \int_\Omega \cos(\rho f)\, w d\mu \left[\frac{\int_\Omega \sin(\rho f)\, w d\mu}{\int_\Omega \cos(\rho f)\, w d\mu} - \frac{\int_\Omega \sin(\rho f)\, w d\mu}{\int_\Omega \cos(\rho f)\, w d\mu}\right] \\ &= 0 \end{aligned}$$

and by using the inequality (2.13) we deduce the desired result (2.11). □

The case of functions of a real variable is as follows:

Corollary 2. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a trigonometrically ρ-convex function on $[m, M]$ and $f : [a, b] \to [m, M]$ so that $\Phi \circ f,\ f \in L_w[a, b]$, where $w \geq 0$ μ-a.e. on $[a, b]$. Assume that $\int_a^b \cos(\rho f(t))\, w(t)\, dt \neq 0$ and*

$$\overline{f}_{\rho,w} := \frac{1}{\rho} \arctan\left(\frac{\int_a^b \sin(\rho f(t))\, w(t)\, dt}{\int_a^b \cos(\rho f(t))\, w(t)\, dt}\right) \in [m, M], \tag{2.14}$$

then we have:

$$\int_a^b \Phi(f(t))\, w(t)\, dt \geq \Phi\left(\overline{f}_{\rho,w}\right) \int_a^b \cos\left[\rho\left(f(t) - \overline{f}_{\rho,w}\right)\right] w(t)\, dt. \tag{2.15}$$

We have the reverse of Jensen's inequality:

Theorem 5. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a trigonometrically ρ-convex function on $[m, M]$ and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. on Ω. Then*

$$\int_\Omega (\Phi \circ f)\, w d\mu \leq \frac{\Phi(m) + \Phi(M)}{2} \frac{\int_\Omega w \cos\left[\rho\left(f - \frac{m+M}{2}\right)\right] d\mu}{\cos\left[\frac{\rho(M-m)}{2}\right]} + \frac{\Phi(M) - \Phi(m)}{2} \frac{\int_\Omega w \sin\left[\rho\left(f - \frac{m+M}{2}\right)\right] d\mu}{\sin\left[\frac{\rho(M-m)}{2}\right]}. \tag{2.16}$$

Proof. We have

$$\begin{aligned}
&\frac{\sin[\rho(M-x)]\Phi(m) + \sin[\rho(x-m)]\Phi(M)}{\sin[\rho(M-m)]} \\
&- \frac{\sin[\rho(M-x)] + \sin[\rho(x-m)]}{\sin[\rho(M-m)]} \frac{\Phi(m) + \Phi(M)}{2} \\
&= \frac{\sin[\rho(M-x)]}{\sin[\rho(M-m)]}\left(\Phi(m) - \frac{\Phi(m)+\Phi(M)}{2}\right) \\
&+ \frac{\sin[\rho(x-m)]}{\sin[\rho(M-m)]}\left(\Phi(M) - \frac{\Phi(m)+\Phi(M)}{2}\right) \\
&= \frac{\Phi(M) - \Phi(m)}{2}\left[\frac{\sin[\rho(x-m)] - \sin[\rho(M-x)]}{\sin[\rho(M-m)]}\right] \\
&= \frac{\Phi(M) - \Phi(m)}{2} \frac{2\sin\left[\rho\left(x - \frac{m+M}{2}\right)\right]\cos\left[\frac{\rho(M-m)}{2}\right]}{2\sin\left[\frac{\rho(M-m)}{2}\right]\cos\left[\frac{\rho(M-m)}{2}\right]} \\
&= \frac{\Phi(M) - \Phi(m)}{2} \frac{\sin\left[\rho\left(x - \frac{m+M}{2}\right)\right]}{\sin\left[\frac{\rho(M-m)}{2}\right]}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\sin[\rho(M-x)] + \sin[\rho(x-m)]}{\sin[\rho(M-m)]} \\
&= \frac{2\sin\left[\frac{\rho(M-m)}{2}\right]\cos\left[\rho\left(x - \frac{m+M}{2}\right)\right]}{2\sin\left[\frac{\rho(M-m)}{2}\right]\cos\left[\frac{\rho(M-m)}{2}\right]} = \frac{\cos\left[\rho\left(x - \frac{m+M}{2}\right)\right]}{\cos\left[\frac{\rho(M-m)}{2}\right]}
\end{aligned}$$

for any $x \in [m, M]$.

Therefore

$$\begin{aligned}
&\frac{\sin[\rho(M-x)]\Phi(m) + \sin[\rho(x-m)]\Phi(M)}{\sin[\rho(M-m)]} \\
&= \frac{\cos\left[\rho\left(x - \frac{m+M}{2}\right)\right]}{\cos\left[\frac{\rho(M-m)}{2}\right]} \frac{\Phi(m) + \Phi(M)}{2} \\
&+ \frac{\Phi(M) - \Phi(m)}{2} \frac{\sin\left[\rho\left(x - \frac{m+M}{2}\right)\right]}{\sin\left[\frac{\rho(M-m)}{2}\right]},
\end{aligned} \tag{2.17}$$

for any $x \in [m, M]$.

Now, let $s \in \Omega$ and by using the identity (2.17) for $x = f(s)$ we have, by multiplying with $w(s) \geq 0$ and integrating, that

$$\begin{aligned} &\frac{\Phi(m)\int_{\Omega} w\sin[\rho(M-f)]\,d\mu + \Phi(M)\int_{\Omega} w\sin[\rho(f-m)]\,d\mu}{\sin[\rho(M-m)]} \\ &= \frac{\Phi(m)+\Phi(M)}{2}\frac{\int_{\Omega} w\cos\left[\rho\left(f-\frac{m+M}{2}\right)\right]d\mu}{\cos\left[\frac{\rho(M-m)}{2}\right]} \\ &+ \frac{\Phi(M)-\Phi(m)}{2}\frac{\int_{\Omega} w\sin\left[\rho\left(f-\frac{m+M}{2}\right)\right]d\mu}{\sin\left[\frac{\rho(M-m)}{2}\right]}. \end{aligned} \tag{2.18}$$

From the definition (1.4) we have

$$\Phi(f(s)) \leq \frac{\sin[\rho(M-f(s))]}{\sin[\rho(M-m)]}\Phi(m) + \frac{\sin[\rho(f(s)-m)]}{\sin[\rho(M-m)]}\Phi(M) \tag{2.19}$$

for any $s \in \Omega$.

If we multiply this inequality by $w(s) \geq 0$ and integrate, we get

$$\begin{aligned} &\int_{\Omega} (\Phi \circ f)\, w d\mu \\ &\leq \frac{\Phi(m)\int_{\Omega} w\sin[\rho(M-f)]\,d\mu + \Phi(M)\int_{\Omega} w\sin[\rho(f-m)]\,d\mu}{\sin[\rho(M-m)]} \end{aligned}$$

and by (2.18) we deduce the desired result (2.16) □

The case of functions of a real variable is as follows:

Corollary 3. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a trigonometrically ρ-convex function on $[m, M]$ and $f : [a, b] \to [m, M]$ so that $\Phi \circ f,\ f \in L_w[a, b]$, where $w \geq 0$ μ-a.e. on $[a, b]$. Then*

$$\begin{aligned} \int_a^b \Phi(f(t))\,w(t)\,dt &\leq \frac{\Phi(m)+\Phi(M)}{2}\frac{\int_a^b w(t)\cos\left[\rho\left(f(t)-\frac{m+M}{2}\right)\right]dt}{\cos\left[\frac{\rho(M-m)}{2}\right]} \\ &+ \frac{\Phi(M)-\Phi(m)}{2}\frac{\int_a^b w(t)\sin\left[\rho\left(f(t)-\frac{m+M}{2}\right)\right]dt}{\sin\left[\frac{\rho(M-m)}{2}\right]}. \end{aligned} \tag{2.20}$$

3. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda(A) \tag{3.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [13, p. 256]:

Theorem 6 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, called the spectral family of A, with the following properties*

a) *$E_\lambda \leq E_{\lambda'}$ for $\lambda \leq \lambda'$;*
b) *$E_{a-0} = 0, E_b = I$ and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$;*
c) *We have the representation*

$$A = \int_{a-0}^{b} \lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ there exists a unique operator $\varphi(A) \in \mathcal{B}(H)$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the inequality

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi\left(\lambda_k'\right)\left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \leq \varepsilon$$

whenever

$$\begin{cases} \lambda_0 < a = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = b, \\ \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases}$$

this means that

$$\varphi(A) = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda, \tag{3.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 4. *With the assumptions of Theorem 6 for A, E_λ and φ we have the representations*

$$\varphi(A)\, x = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(A)\, x, y\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_\lambda x, y\rangle \quad \text{for all } x,\ y \in H. \tag{3.3}$$

Theorem 7. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. Let $\Phi : J \subset \mathbb{R} \to \mathbb{R}$ be a trigonometrically ρ-convex function on J, $f : I \to J$, $w : I \to [0, \infty)$ continuous functions and such that $[a, b] \subset I$ and*

$f([a,b]) \subset [m,M] \subset J$ where $0 < M - m < \frac{\pi}{\rho}$. Then

$$\Phi\left(\frac{m+M}{2}\right) w(A) \cos\left[\rho\left(f(A) - \frac{m+M}{2}\right)\right] \tag{3.4}$$
$$\leq \frac{1}{2}\left[\Phi(f(A)) + \Phi(m+M-f(A))\right] w(A)$$
$$\leq \frac{\Phi(m)+\Phi(M)}{2} \sec\left[\frac{\rho(M-m)}{2}\right] w(A) \cos\left[\rho\left(f(A) - \frac{m+M}{2}\right)\right]$$

and

$$\Phi(f(A)) w(A) \tag{3.5}$$
$$\leq \frac{\Phi(m)+\Phi(M)}{2} \sec\left[\frac{\rho(M-m)}{2}\right] w(A) \cos\left[\rho\left(f(A) - \frac{m+M}{2}\right)\right]$$
$$+ \frac{\Phi(M)-\Phi(m)}{2} \csc\left[\frac{\rho(M-m)}{2}\right] w(A) \sin\left[\rho\left(f(A) - \frac{m+M}{2}\right)\right].$$

in the operator order of $\mathcal{B}(H)$.

Proof. For small $\varepsilon > 0$, since Φ is continuous an $\langle E_t x, x\rangle$ (with $x \in H$) is of bounded variation on any closed interval, the Riemann-Stieltjes integrals exists in the following inequalities obtained from (2.1)

$$\Phi\left(\frac{m+M}{2}\right) \int_{a-\varepsilon}^{b} w(t) \cos\left[\rho\left(f(t) - \frac{m+M}{2}\right)\right] d\langle E_t x, x\rangle \tag{3.6}$$
$$\leq \frac{1}{2}\left[\int_{a-\varepsilon}^{b} \Phi(f(t)) w(t)\, d\langle E_t x, x\rangle + \int_{a-\varepsilon}^{b} \Phi(m+M-f(t)) w(t)\, d\langle E_t x, x\rangle\right]$$
$$\leq \frac{\Phi(m)+\Phi(M)}{2} \frac{\int_{a-\varepsilon}^{b} w(t) \cos\left[\rho\left(f(t) - \frac{m+M}{2}\right)\right] d\langle E_t x, x\rangle}{\cos\left[\frac{\rho(M-m)}{2}\right]},$$

for any $x \in H$.

By taking the limit over $\varepsilon \to 0+$ in (3.6) and utilising Corollary 4, we deduce

$$\Phi\left(\frac{m+M}{2}\right) \left\langle w(A) \cos\left[\rho\left(f(A) - \frac{m+M}{2}\right)\right] x, x\right\rangle$$
$$\leq \frac{1}{2}\left[\langle \Phi(f(A)) w(A) x, x\rangle + \langle \Phi(m+M-f(A)) w(A) x, x\rangle\right]$$
$$\leq \frac{\Phi(m)+\Phi(M)}{2} \frac{\left\langle w(A) \cos\left[\rho\left(f(A) - \frac{m+M}{2}\right)\right] x, x\right\rangle}{\cos\left[\frac{\rho(M-m)}{2}\right]}$$

for any $x \in H$, which is equivalent to the desired operator inequality (3.4).

The inequality (3.5) follows in a similar way from the inequality (2.16). □

The following result also holds:

Theorem 8. *With the assumptions of Theorem 7 and if*

$$\overline{f}_{\rho,w,A,x} := \frac{1}{\rho} \arctan\left(\frac{\langle \sin(\rho f(A)) w(A) x, x\rangle}{\langle \cos(\rho f(A)) w(A) x, x\rangle}\right) \in [m,M], \tag{3.7}$$

and $\langle \cos(\rho f(A)) w(A) x, x\rangle \neq 0$ *for* $x \in H$, *then*

$$\langle \Phi(f(A)) w(A) x, x\rangle \geq \Phi\left(\overline{f}_{\rho,w,A,x}\right)\left\langle w(A)\cos\left[\rho\left(f(A) - \overline{f}_{\rho,w,A,x}1_H\right)\right] x, x\right\rangle. \tag{3.8}$$

The proof follows by the integral inequality (2.11) in a similar manner to the one from Theorem 7 and we omit the details.

4. Examples for Power Function

Consider the function $\Phi_r : (0,\infty) \to (0,\infty)$, $\Phi_r(x) = x^r$ with $r \in \mathbb{R}\setminus\{0\}$. If $r \in (-\infty, 0)\cup[1,\infty)$ the function is convex and therefore trigonometrically ρ-convex for any $\rho > 0$. If $r \in (0,1)$ then the function is concave and

$$\Phi_r''(x) + \rho^2\Phi_r(x) = \rho^2 x^r - r(1-r)x^{r-2} = \rho^2 x^{r-2}\left(x^2 - \frac{r(1-r)}{\rho^2}\right), \ x > 0.$$

This shows that for $r \in (0,1)$ and $\rho > 0$ the function $\Phi_r(x) = x^r$ is trigonometrically ρ-convex on $\left(\frac{1}{\rho}\sqrt{r(1-r)}, \infty\right)$ and trigonometrically ρ-concave on $\left(0, \frac{1}{\rho}\sqrt{r(1-r)}\right)$.

Assume that $\rho > 0$ and m, M are real numbers such that $0 < M - m < \frac{\pi}{\rho}$. We observe that if $r \in (-\infty, 0) \cup [1,\infty)$ and $[m, M] \subset (0,\infty)$ or $r \in (0,1)$ and $[m, M] \subset \left(\frac{1}{\rho}\sqrt{r(1-r)}, \infty\right)$, then $\Phi_r(x) = x^r$ is trigonometrically ρ-convex on $[m, M]$ and by (2.1) we get

$$\begin{aligned}\left(\frac{m+M}{2}\right)^r \int_\Omega w\cos\left[\rho\left(f - \frac{m+M}{2}\right)\right] d\mu &\leq \frac{1}{2}\left[\int_\Omega f^r w d\mu + \int_\Omega (m+M-f)^r w d\mu\right] \\ &\leq \frac{m^r + M^r}{2}\frac{\int_\Omega w\cos\left[\rho\left(f - \frac{m+M}{2}\right)\right] d\mu}{\cos\left[\frac{\rho(M-m)}{2}\right]},\end{aligned} \tag{4.1}$$

where $f : \Omega \to [m, M]$ so that f^r, $(m+M-f)^r$, $f \in L_w(\Omega,\mu)$, and $w \geq 0$ μ-a.e. on Ω.

Under these assumptions, by making use of (2.15) we have

$$\int_\Omega f^r w d\mu \geq \overline{f}^r_{\rho,w}\int_\Omega \cos\left[\rho\left(f - \overline{f}_{\rho,w}\right)\right] w d\mu, \tag{4.2}$$

provided

$$\overline{f}_{\rho,w} := \frac{1}{\rho}\arctan\left(\frac{\int_\Omega \sin(\rho f) w d\mu}{\int_\Omega \cos(\rho f) w d\mu}\right) \in [m, M]. \tag{4.3}$$

Finally, by utilising (2.16), we get

$$\begin{aligned}\int_\Omega f^r w d\mu \leq{}& \frac{m^r + M^r}{2}\frac{\int_\Omega w\cos\left[\rho\left(f - \frac{m+M}{2}\right)\right] d\mu}{\cos\left[\frac{\rho(M-m)}{2}\right]} \\ &+ \frac{M^r - m^r}{2}\frac{\int_\Omega w\sin\left[\rho\left(f - \frac{m+M}{2}\right)\right] d\mu}{\sin\left[\frac{\rho(M-m)}{2}\right]}.\end{aligned} \tag{4.4}$$

If $r \in (0,1)$ and $[m,M] \subset \left(0, \frac{1}{\rho}\sqrt{r(1-r)}\right)$, then the sign of inequality reverses in (4.1), (4.2) and (4.4).

References

[1] M. S. S. Ali, On certain properties of trigonometrically ρ-convex functions, *Advances in Pure Mathematics*, 2012, 2, 337-340 http://dx.doi.org/10.4236/apm.2012.25047.

[2] M. S. S. Ali, On Hadamard's inequality for trigonometrically ρ-convex functions," accepted to appear in Theoretical Mathematics & Applications., March, 2013.

[3] E. F. Beckenbach, Convex functions, *Bulletin of the American Mathematical Society*, Vol. **54**, No. 5, 1948, pp. 439-460. doi:10.1090/S0002-9904-1948-08994-7

[4] M. Bessenyei, Hermite–Hadamard-type inequalities for generalized convex functions, *PhD Thesis*, Univ. Debrecen, Hungary, 2004. RGMIA Monograps, [http://rgmia.org/papers/monographs/dissertatio.pdf]

[5] F. F. Bonsall, The characterization of generalized convex functions,*The Quarterly Journal of Mathematics Oxford Series*, Vol. **1**, 1950, pp. 100-111. doi:10.1093/qmath/1.1.100

[6] A. M. Bruckner and E. Ostrow, Some functions classes related to the class of convex functions," *Pacific Journal of Mathematics*, Vol. **12**, 1962, pp. 1203-1215.

[7] S. S. Dragomir, A Grüss type inequality for isotonic linear functionals and applications. *Demonstratio Math.* **36** (2003), no. 3, 551–562. Preprint RGMIA *Res. Rep. Coll.* **5** (2002), Suplement, Art. 12. [http://rgmia.org/papers/v5e/GTIILFApp.pdf].

[8] S. S. Dragomir, Integral inequalities for convex functions and applications for divergence measures. *Miskolc Math. Notes* **17** (2016), no. 1, 151–169. DOI: 10.18514/MMN.2016.1794. Preprint RGMIA *Res. Rep. Coll.* **18** (2015), Art. 18. [http://rgmia.org/papers/v18/v18a10.pdf].

[9] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results, *Australian J. Math. Anal. Appl.*, Volume **14**, Issue 1, Article 1, pp. 1-287, 2017. [Online http://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex].

[10] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000. [Online http://rgmia.org/monographs/hermite_hadamard.html].

[11] L. Fejér, Über die Fourierreihen, II, *Math. Naturwiss. Anz Ungar. Akad. Wiss.* **24** (1906),369-390. (In Hungarian).

[12] J. W. Green, Support, Convergence, and Differentiability Properties of Generalized Convex Functions, *Proceedings of the American Mathematical Society*, Vol. **4**, No. 3, 1953, pp. 391-396. doi:10.1090/S0002-9939-1953-0056039-2

[13] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc.-New York, 1969.

[14] B. Ya. Levin, *Lectures on Entire Functions*, American Mathematical Society, 1996.

[15] L. S. Maergoiz, *Asymptotic Characteristics of Entire Functions and Their Applications in Mathematics and Biophysics*, Kluwer Academic Publishers, New York, 2003.

[16] M. J. Miles, An extremum property of convex functions, *American Mathematical Monthly*, Vol. **76**, 1969, pp. 921-922. doi:10.2307/2317948

[17] M. M. Peixoto, On the existence of derivatives of generalized convex functions, *Summa Brasilian Mathematics,* Vol. 2, No. 3, 1948, pp. 35-42.

[18] M. M. Peixoto, Generalized convex functions and second order differential inequlities, *Bulletin of the American Mathematical Society,* Vol. **55**, No. 6, 1949, pp. 563- 572. doi:10.1090/S0002-9904-1949-09246-7

[19] A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, New York, 1973.

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