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NEW INTEGRAL INEQUALITIES FOR $r-$CONVEX FUNCTION

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ABSTRACT. In this paper, we established new integral inequalities for $r-$convex functions via two integral identities.

1. INTRODUCTION

In [3],The power mean $M_r(x,y;\lambda)$ of order r of positive numbers x,y is defined as the following:

$$M_r(x,y;\lambda)=\begin{cases}(\lambda x^r+(1-\lambda)y^r), & if\ r\neq 0\\ \left(x^r y^{1-\lambda}\right), & if\ r=0\end{cases}. \tag{1.1}$$

In [4], Gill et. all used the definition of $M_r(x,y;\lambda)$ to introduce the concept of $r-$convex functions.

Definition 1. *A positive function f is $r-$convex on $[a,b]$ if for all $x,y\in[a,b]$ and $\lambda\in[0,1]$*
(1.2)

$$f(\lambda x+(1-\lambda)y)\leq M_r(f(x),f(y);\lambda)=\begin{cases}(\lambda f^r(x)+(1-\lambda)f^r(y))^{\frac{1}{r}}, & r\neq 0\\ f^{\lambda}(x)f^{1-\lambda}(y), & r=0\end{cases}.$$

In the definition of $r-$convex functions if we choose $r=1$ and $r=0$, we have ordinary convex functions and $\log-$convex functions respectively.
It is obvious that if f is $r-$convex in $[a,b]$ where $r>0$, then f^r is convex on $[a,b]$.

The logarithmic mean of two positive number x,y which is given by

$$L(x,y)=\begin{cases}\frac{x-y}{\ln x-\ln y}, & x\neq y\\ x, & x=y\end{cases}$$

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and the generalized logarithmic means of order r of positive numbers x, y defined by

$$L_r(x,y) = \begin{cases} \frac{r}{r+1}\frac{x-y}{\ln x - \ln y}, & r \neq 0, -1; x \neq y \\ \frac{x-y}{\ln x - \ln y}, & r = 0, x \neq y \\ xy\frac{x-y}{\ln x - \ln y}, & r = -1, x \neq y \\ x & x = y \end{cases}$$

,see [4].

Gill et. all proved the following theorem for $r-$convex functions:

Theorem 1. *Suppose f is a positive $r-$convex function on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq L_r(f(a), f(b)). \tag{1.3}$$

If f is a positive $r-$concave function, then the inequality is reversed, see [4].

For several results concerning of $r-$convexity, see [1]-[2], [5]-[7] and [11]-[14] where further references are listed.

The main aim of this paper is to obtain some new integral inequalities for $r-$convex functions by using Lemma 1 and Lemma 2.

We will give two integral identities which are emboided in the following lemmas to obtain our main theorems.

Lemma 1. *If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° and $f^{''} \in L_1[a,b]$, then*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x - \frac{a+b}{2}\right) f^{'}(x) \\ = &\frac{(x-a)^3}{2(b-a)}\int_0^1 t^2 f^{''}(tx+(1-t)a)dt + \frac{(b-x)^3}{2(b-a)}\int_0^1 t^2 f^{''}(tx+(1-t)b)dt \end{aligned} \tag{1.4}$$

this inequality, see [8].

Lemma 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a three times differentable function on I° with $a, b \in I$ and $a < b$. If $f^{'''} \in L[a,b]$ then*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}\left[f^{'}(b) - f^{'}(a)\right] \\ = &\frac{(b-a)^3}{12}\int_0^1 t(1-t)(2t-1) f^{'''}(ta+(1-t)b)\, dt \end{aligned} \tag{1.5}$$

, see [10].

2. New Results via Lemma 1

We will start with the following theorem:

Theorem 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^{+}$ be a positive twice differentable function on I° such that $f^{''} \in L[a,b]$ where $a, b \in I$ with $a < b$. If $\left|f^{''}\right|^{q}$ is r-convex on $[a,b]$ for $q > 1$, $\frac{1}{q} + \frac{1}{p} = 1$, then the following inequalitiy holds:*

$$
\begin{aligned}
(2.1)\qquad & \left|\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x - \frac{a+b}{2}\right) f^{'}(x)\right| \\
\leq\ & \frac{(x-a)^3}{2(b-a)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left[L_r\left(\left|f^{''}(x)\right|^{q}, \left|f^{''}(a)\right|^{q}\right)\right]^{\frac{1}{q}} \\
& \frac{(b-x)^3}{2(b-a)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left[L_r\left(\left|f^{''}(x)\right|^{q}, \left|f^{''}(b)\right|^{q}\right)\right]^{\frac{1}{q}}
\end{aligned}
$$

where $L_r(.,.)$ is the generalized logarithmic mean.

Proof. From Lemma 1, using properties of modulus and Hölder inequality we have

$$
\begin{aligned}
I &= \left|\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x - \frac{a+b}{2}\right) f^{'}(x)\right| \\
&\leq \frac{(x-a)^3}{2(b-a)}\int_0^1 t^2 \left|f^{''}(tx + (1-t)a)\right| dt \\
&\quad + \frac{(b-x)^3}{2(b-a)}\int_0^1 t^2 \left|f^{''}(tx + (1-t)b)\right| dt \\
&\leq \frac{(x-a)^3}{2(b-a)}\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f^{''}(tx + (1-t)a)\right|^{q} dt\right)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^3}{2(b-a)}\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f^{''}(tx + (1-t)b)\right|^{q} dt\right)^{\frac{1}{q}}. \qquad (2.2)
\end{aligned}
$$

If we use $r-$convextity of $\left|f^{''}\right|^{q}$, we can write the following inequalities via 1.3:

(2.3)
$$
\int_0^1 \left|f^{''}(tx + (1-t)a)\right|^{q} dt = \frac{1}{x-a}\int_a^x \left|f^{''}(u)\right|^{q} du \leq L_r\left(\left|f^{''}(x)\right|^{q}, \left|f^{''}(a)\right|^{q}\right)
$$

and

$$
(2.4)\qquad \int_0^1 \left|f^{''}(tx + (1-t)b)\right|^{q} dt \leq L_r\left(\left|f^{''}(x)\right|^{q}, \left|f^{''}(b)\right|^{q}\right).
$$

If we use 2.3 and 2.4 in 2.2 we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x-\frac{a+b}{2}\right)f^{'}(x)\right| \\
\leq\ & \frac{(x-a)^3}{2(b-a)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left[L_r\left(\left|f^{''}(x)\right|^q, \left|f^{''}(a)\right|^q\right)\right]^{\frac{1}{q}} \\
&+\frac{(b-x)^3}{2(b-a)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left[L_r\left(\left|f^{''}(x)\right|^q, \left|f^{''}(b)\right|^q\right)\right]^{\frac{1}{q}}.
\end{aligned}$$

□

Corollary 1. *If we choose $x=\frac{a+b}{2}$ in Theorem 2, we have following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\begin{array}{c}\left[L_r\left(\left|f^{''}\left(\frac{a+b}{2}\right)\right|^q, \left|f^{''}(a)\right|^q\right)\right]^{\frac{1}{q}} \\ +\left[L_r\left(\left|f^{''}\left(\frac{a+b}{2}\right)\right|^q, \left|f^{''}(b)\right|^q\right)\right]^{\frac{1}{q}}\end{array}\right).$$

Theorem 3. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentable function on I° such that $f^{''}\in L[a,b]$ with $a,b\in I$ and $a<b$. If $\left|f^{''}\right|$ is r-convex on $[a,b]$ for $r>1$, then the following inequalitiy holds:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x-\frac{a+b}{2}\right)f^{'}(x)\right| \\
\leq\ & \frac{1}{(b-a)}\left[\frac{r\left|f^{''}(x)\right|\left[(x-a)^3+(b-x)^3\right]}{3r+1} + \frac{2r^3\left[(x-a)^3\left|f^{''}(a)\right|+(b-x)^3\left|f^{''}(b)\right|\right]}{(1+r)(1+2r)(3r+1)}\right].
\end{aligned}$$

Proof. From Lemma 1, using properties of modulus and $r-$convextity of $\left|f^{''}\right|$, we can write

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x-\frac{a+b}{2}\right)f^{'}(x)\right| \\
\leq\ & \frac{(x-a)^3}{2(b-a)}\int_0^1 t^2\left[t\left|f^{''}(x)\right|^r+(1-t)\left|f^{''}(a)\right|^r\right]^{\frac{1}{r}}dt \\
&+\frac{(b-x)^3}{2(b-a)}\int_0^1 t^2\left[t\left|f^{''}(x)\right|^r+(1-t)\left|f^{''}(b)\right|^r\right]^{\frac{1}{r}}dt.
\end{aligned}$$

If we use the fact that

$$\sum_{i=1}^n (a_i+b_i)^k \leq \sum_{i=1}^n a_i^k + \sum_{i=1}^n b_i^k \tag{2.5}$$

for $0 < k < 1$; $a_1, a_2, ..., a_n \geq 0$ and $b_1, b_2, ..., b_n \geq 0$ we obtain

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(u)du - f(x) + \left(x - \frac{a+b}{2} \right) f^{'}(x) \right| \\
\leq\ & \frac{(x-a)^3}{2(b-a)} \int_0^1 t^{2+\frac{1}{r}} \left| f^{''}(x) \right| + t^2 (1-t)^{\frac{1}{r}} \left| f^{''}(a) \right| dt \\
&+ \frac{(b-x)^3}{2(b-a)} \int_0^1 t^{2+\frac{1}{r}} \left| f^{''}(x) \right| + t^2 (1-t)^{\frac{1}{r}} \left| f^{''}(b) \right| dt. \qquad (2.6)
\end{aligned}$$

Where

$$\int_0^1 t^{2+\frac{1}{r}} dt = \frac{r}{3r+1} \qquad (2.7)$$

and

$$\int_0^1 t^2 (1-t)^{\frac{1}{r}} dt = \frac{2r^3}{(1+r)(1+2r)(1+3r)}. \qquad (2.8)$$

If we use 2.7 and 2.8 in 2.6 we have

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(u)du - f(x) + \left(x - \frac{a+b}{2} \right) f^{'}(x) \right| \\
\leq\ & \frac{(x-a)^3}{2(b-a)} \left[\frac{r}{3r+1} \left| f^{''}(x) \right| + \frac{2r^3}{(1+r)(1+2r)(1+3r)} \left| f^{''}(a) \right| \right] \\
&+ \frac{(b-x)^3}{2(b-a)} \left[\frac{r}{3r+1} \left| f^{''}(x) \right| + \frac{2r^3}{(1+r)(1+2r)(1+3r)} \left| f^{''}(b) \right| \right] \\
=\ & \frac{1}{b-a} \left[\frac{r \left| f^{''}(x) \right| \left[(x-a)^3 + (b-x)^3 \right]}{3r+1} + 2r^3 \frac{(x-a)^3 \left| f^{''}(a) \right| + (b-x)^3 \left| f^{''}(b) \right|}{(1+r)(1+2r)(1+3r)} \right]
\end{aligned}$$

By a simple computation, we get

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(u)du - f(x) + \left(x - \frac{a+b}{2} \right) f^{'}(x) \right| \\
\leq\ & \frac{1}{b-a} \left[\frac{r \left| f^{''}(x) \right| \left[(x-a)^3 + (b-x)^3 \right]}{3r+1} + 2r^3 \frac{(x-a)^3 \left| f^{''}(a) \right| + (b-x)^3 \left| f^{''}(b) \right|}{(1+r)(1+2r)(1+3r)} \right].
\end{aligned}$$

□

Corollary 2. *If we choose $x = \frac{a+b}{2}$ in Theorem 3, we have following inequality:*

$$\left| \frac{1}{b-a} \int_a^b f(u)du - f\left(\frac{a+b}{2} \right) \right| \leq \frac{(b-a)^2}{8} \left[\frac{r \left| f^{''}(\frac{a+b}{2}) \right|}{3r+1} + \frac{r^3 \left[\left| f^{''}(a) \right| + \left| f^{''}(b) \right| \right]}{(1+r)(1+2r)(1+3r)} \right].$$

Theorem 4. *Under the assumptions of Theorem 3, we obtain the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)du-f(x)+\left(x-\frac{a+b}{2}\right)f^{'}(x)\right| \\
\leq\ & \frac{1}{2\left(b-a\right)\left(p+1\right)^{\frac{1}{p}}}\left(\frac{r}{1+r}\right)^{\frac{1}{q}}\left[\begin{array}{c}(b-x)^3\left[\left|f^{''}(x)\right|^q+\left|f^{''}(b)\right|^q\right]^{\frac{1}{q}} \\ +\left(x-a\right)^3\left[\left|f^{''}(a)\right|^q+\left|f^{''}(x)\right|^q\right]^{\frac{1}{q}}\end{array}\right]
\end{aligned}$$

where $r>1$.

Proof. From Lemma 1, using properties of modulus and Hölder inequalality we can write

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)du-f(x)+\left(x-\frac{a+b}{2}\right)f^{'}(x)\right| \\
\leq\ & \frac{(x-a)^3}{2(b-a)}\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{''}(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^3}{2(b-a)}\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{''}(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

If we use the $r-$convexity of $\left|f^{''}\right|^q$ and the inequalitiy in 2.5 we can write

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)du-f(x)+\left(x-\frac{a+b}{2}\right)f^{'}(x)\right| \\
\leq\ & \frac{(x-a)^3}{2(b-a)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[t\left|f^{''}(x)\right|^{qr}+(1-t)\left|f^{''}(a)\right|^{qr}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^3}{2(b-a)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[t\left|f^{''}(x)\right|^{qr}+(1-t)\left|f^{''}(b)\right|^{qr}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{(x-a)^3}{2(b-a)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\int_0^1 t^{\frac{1}{r}}\left|f^{''}(x)\right|^q+(1-t)^{\frac{1}{r}}\left|f^{''}(a)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^3}{2(b-a)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\int_0^1 t^{\frac{1}{r}}\left|f^{''}(x)\right|^q+(1-t)^{\frac{1}{r}}\left|f^{''}(b)\right|^q dt\right)^{\frac{1}{q}} \\
=\ & \frac{(x-a)^3}{2(b-a)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{r}{1+r}\right)^{\frac{1}{q}}\left(\left|f^{''}(x)\right|^q+\left|f^{''}(a)\right|^q\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^3}{2(b-a)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{r}{1+r}\right)^{\frac{1}{q}}\left(\left|f^{''}(x)\right|^q+\left|f^{''}(b)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

which completes the proof. □

Corollary 3. *If we choose $x=\frac{a+b}{2}$ in Theorem 4, we obtain the following inequality:*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(u)du-f\left(\frac{a+b}{2}\right)\right|\\ \leq\ & \frac{(b-a)^2}{16\,(2p+1)^{\frac{1}{p}}}\left(\frac{r}{1+r}\right)^{\frac{1}{q}}\left(\left[\left|f^{''}(\frac{a+b}{2})\right|^q+\left|f^{''}(b)\right|^q\right]^{\frac{1}{q}}+\left[\left|f^{''}(a)\right|^q+\left|f^{''}(\frac{a+b}{2})\right|^q\right]^{\frac{1}{q}}\right).\end{aligned}$$

3. New Results Via Lemma 2

Theorem 5. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}^+$ be a three times differentable function on I° such that $f^{'''}\in L[a,b]$ with $a,b\in I$ and $a<b$. If $\left|f^{'''}\right|$ is r-convex on $[a,b]$, then the following inequalitiy holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\frac{b-a}{12}\left[f^{'}(b)-f^{'}(a)\right]\right|\\ \leq\ & \frac{(b-a)^3}{12}\left[\left|f^{'''}(a)\right|+\left|f^{'''}(b)\right|\right]\frac{r^2\left[\left(\frac{1}{2}\right)^{2+\frac{1}{r}}(1+6r)+1\right]}{(2r+1)\,(3r+1)\,(4r+1)}.\end{aligned}$$

Proof. From Lemma 2, using properties of modulus and $r-$convexity of $\left|f^{'''}\right|$ we can write

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\frac{b-a}{12}\left[f^{'}(b)-f^{'}(a)\right]\right|\\ \leq\ & \frac{(b-a)^3}{12}\int_0^{\frac{1}{2}} t(1-t)\,(1-2t)\left[t\left|f^{'''}(a)\right|^r+(1-t)\left|f^{'''}(b)\right|^r\right]^{\frac{1}{r}}dt\\ &+\frac{(b-a)^3}{12}\int_{\frac{1}{2}}^1 t(1-t)\,(2t-1)\left[t\left|f^{'''}(a)\right|^r+(1-t)\left|f^{'''}(b)\right|^r\right]^{\frac{1}{r}}dt.\end{aligned}$$

If we use inequality of 2.5, we can write

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\frac{b-a}{12}\left[f^{'}(b)-f^{'}(a)\right]\right|\\ \leq\ & \frac{(b-a)^3}{12}\int_0^{\frac{1}{2}} t(1-t)\,(1-2t)\left[t^{1/r}\left|f^{'''}(a)\right|+(1-t)^{1/r}\left|f^{'''}(b)\right|\right]dt\\ &+\frac{(b-a)^3}{12}\int_{\frac{1}{2}}^1 t(1-t)\,(2t-1)\left[t^{1/r}\left|f^{'''}(a)\right|+(1-t)^{1/r}\left|f^{'''}(b)\right|\right]dt.\end{aligned}$$

If we calculate the integrals above, we get the desired result. □

Theorem 6. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a three times differentable function on I° such that $f''' \in L[a,b]$ with $a,b \in I$ and $a < b$. If $\left|f'''\right|^q$ is r-convex on $[a,b]$, for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $r > 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}\left[f'(b) - f'(a)\right]\right| \\ \leq\ & \frac{(b-a)^3}{12}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\beta\left(\frac{r+rq+1}{r}, q+1\right)\left(\left|f'''(b)\right|^q + \left|f'''(a)\right|^q\right)\right]^{\frac{1}{q}} \end{aligned}$$

where $\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ $(x,y>0)$ is Beta function.

Proof. From lemma 2 and using properties of modulus, $r-$convexity of $\left|f'''\right|^q$ and the inequality in 2.5 we can write

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}\left[f'(b) - f'(a)\right]\right| \\ \leq\ & \frac{(b-a)^3}{12}\left(\int_0^1 |2t-1|^p\, dt\right)^{\frac{1}{p}}\left(\int_0^1 t^q(1-t)^q\left|f'''(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\ \leq\ & \frac{(b-a)^3}{12}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 t^q(1-t)^q\left[t\left|f'''(a)\right|^{qr} + (1-t)\left|f'''(b)\right|^{qr}\right]^{\frac{1}{r}} dt\right)^{\frac{1}{q}} \\ \leq\ & \frac{(b-a)^3}{12}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 t^{q+\frac{1}{r}}(1-t)^q\left|f'''(a)\right|^q + t^q(1-t)^{q+\frac{1}{r}}\left|f'''(b)\right|^q dt\right)^{\frac{1}{q}}. \end{aligned}$$

If we calculate the integrals above, we get the desired result. □

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