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SIMPSON TYPE INTEGRAL INEQUALITIES FOR r-CONVEX FUNCTION

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ABSTRACT. In this paper, we established new integral inequalities of Simpson type for r-convex functions via two integral identities.

1. Introduction

Firstly, we start one of the most famous inequality for convex functions is so called Simpson's inequality as follows: Let mapping $f:[a,b]\to\mathbb{R}$ is supposed four times differntiable on interval (a,b)

$$\left|\int_a^b f(x)dx-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\leq\frac{1}{2880}\left|\left|f^{(4)}\right|\right|_\infty(b-a)^5$$

and having the fourth derivative bound on (a,b), that is $\left|\left|f^{(4)}\right|\right|_\infty:=\sup_{x\in(a,b)}\left|f^{(4)}\right|<\infty$,see [5]. Follwing found inequalities are had Simpson Type.

In [5],The power mean $M_r(x,y;\lambda)$ of order r of positive numbers x,y is defined as the following:

$$M_r(x,y;\lambda)=\begin{cases}(\lambda x^r+(1-\lambda)y^r), & if\ r\neq 0\\ \left(x^r y^{1-\lambda}\right), & if\ r=0\end{cases}. \tag{1.1}$$

In [6], Gill et. all used the definition of $M_r(x,y;\lambda)$ to introduce the concept of r-convex functions.

Definition 1. *A positive function f is $r-$convex on $[a,b]$ if for all $x,y\in[a,b]$ and $\lambda\in[0,1]$*

(1.2)

$$f(\lambda x+(1-\lambda)y)\leq M_r(f(x),f(y);\lambda)=\begin{cases}(\lambda f^r(x)+(1-\lambda)f^r(y))^{\frac{1}{r}}, & r\neq 0\\ f^\lambda(x)f^{1-\lambda}(y), & r=0\end{cases}.$$

In the definition of r-convex functions if we choose $r=1$ and $r=0$, we have ordinary convex functions and log-convex functions respectively.

It is obvious that if f is r-convex in $[a,b]$ where $r>0$, then f^r is convex on $[a,b]$.

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The generalized logarithmic means of order r of positive numbers x, y defined by

$$L_r(x,y)=\begin{cases} \frac{r}{r+1}\frac{x-y}{\ln x-\ln y}, & r\neq 0,-1; x\neq y \\ \frac{x-y}{\ln x-\ln y}, & r=0, x\neq y \\ xy\frac{x-y}{\ln x-\ln y}, & r=-1, x\neq y \\ x & x=y \end{cases}$$

,see [6].

Gill et. all proved the following theorem for r-convex functions:

Theorem 1. *Suppose f is a positive r-convex function on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq L_r(f(a), f(b)). \tag{1.3}$$

If f is a positive r-concave function, then the inequality is reversed, see [6].

For several results concerning of r-convexity, see [3],[4] and [6] -[14] where further references are listed.

The main aim of this paper is to obtain some new integral inequalities for r-convex functions by using Lemma 1 and Lemma 2.

We will give two integral identities which are emboided in the following lemmas to obtain our main theorems.

Lemma 1. *Let $f^{''}: I\subseteq R\to R$ be an absolutely continuous function on I° such that $f^{'''}\in L[a;b]$, where $a,b\in I$ with $a<b$. If $\left|f^{'''}\right|$ is quasi-convex on $[a,b]$, then the following inequality holds:*

$$\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)f(b)\right]=(b-a)^4\int_0^1 m(t)f^{'''}(ta+(1-t)b)dt,$$

where

$$m(t)=\begin{cases} \frac{1}{6}t^2\left(t-\frac{1}{2}\right), & t\in\left[0,\frac{1}{2}\right] \\ \frac{1}{6}(t-1)^2\left(t-\frac{1}{2}\right), & t\in\left(\frac{1}{2},1\right] \end{cases}$$

,see [1].

Lemma 1 was proved by Alomari and Hussain in [1]. But they didn't use quasi-convexty for Lemma 1 of proved. Naturally, we change quasi-convexty of rule to r-covexty of rule.

Lemma 2. *Let $f^{'''}: I\subseteq R\to R$ be an absolutely continuous mapping on I°, where $a,b\in I$ with $a<b$. If $f^{(4)}\in L[a,b]$, then the following equality holds:*

$$\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)f(b)\right]=\frac{(b-a)^5}{24}\int_0^1 h(t)f^{(4)}(ta+(1-t)b)dt,$$

where

$$h(t)=\begin{cases} \frac{1}{24}t^3\left(t-\frac{2}{3}\right), & t\in\left[0,\frac{1}{2}\right] \\ \frac{1}{24}(t-1)^3\left(t-\frac{1}{3}\right), & t\in\left(\frac{1}{2},1\right] \end{cases}$$

,see [2].

2. New Inequqlities Simpson's Type via Lemma 1

We will start with the following theorem:

Theorem 2. *Let $f^{''}: I \subseteq R \to R$ be an absolutely continuous function on I° such that $f^{'''} \in L[a;b]$, where $a, b \in I$ with $a < b$. If $\left|f^{'''}\right|$ is r-convex on $[a,b]$ for $q > 1$, $\frac{1}{q} + \frac{1}{p} = 1$, then the following inequality holds:*

$$\begin{aligned}\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right)f(b)\right]\right| &\leq \frac{(b-a)^4}{3}\left(\frac{1}{2}\right)^{1+1/q}\left(\frac{8^{-p}\Gamma(2p)\Gamma(p+1)}{\Gamma(3p+2)}\right)^{1/p} \\ &\times\left\{\left[L_r\left(\left|f^{'''}(a)\right|^q, \left|f^{'''}\left(\frac{a+b}{2}\right)\right|^q\right)\right]^{1/q}\right. \\ &\left.+\left[L_r\left(\left|f^{'''}(b)\right|^q, \left|f^{'''}\left(\frac{a+b}{2}\right)\right|^q\right)\right]^{1/q}\right\},\end{aligned}$$

where Γ is gama function. $\Gamma(z) = \int_0^\infty x^{z-1}\exp(-x)dx$.

Proof. From Lemma 1 and using modulus of properties

$$\begin{aligned}\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right)f(b)\right]\right| &\leq \frac{(b-a)^4}{6}\left\{\int_0^{1/2} t^2\left(\frac{1}{2}-t\right)\left|f^{'''}(ta+(1-t)b)\right|dt\right. \\ &\left.+\int_{1/2}^1 (1-t)^2\left(t-\frac{1}{2}\right)\left|f^{'''}(ta+(1-t)b)\right|dt\right\}.\end{aligned}$$

Using well-known Hölder inequality, we have

$$\begin{aligned}A &= \left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right)f(b)\right]\right| \\ &\leq \frac{(b-a)^4}{6}\left\{\left[\int_0^{1/2}\left(t^2\left(\frac{1}{2}-t\right)\right)^p dt\right]^{1/p}\left[\int_0^{1/2}\left|f^{'''}(ta+(1-t)b)\right|^q dt\right]^{1/q}\right. \\ &\left.+\left[\int_{1/2}^1\left((1-t)^2\left(t-\frac{1}{2}\right)\right)^p\right]^{1/p}\left[\int_{1/2}^1\left|f^{'''}(ta+(1-t)b)\right|^q dt\right]^{1/q}\right\}.\end{aligned}$$

If we calculate above integrals and use 1.3, then we following inequality

$$\begin{aligned}A &\leq \frac{(b-a)^4}{3}\left(\frac{1}{2}\right)^{1+1/q}\left(\frac{8^{-p}\Gamma(2p)\Gamma(p+1)}{\Gamma(3p+2)}\right)^{1/p} \\ &\times\left\{\left[L_r\left(\left|f^{'''}(a)\right|^q, \left|f^{'''}\left(\frac{a+b}{2}\right)\right|^q\right)\right]^{1/q} + \left[L_r\left(\left|f^{'''}(b)\right|^q, \left|f^{'''}\left(\frac{a+b}{2}\right)\right|^q\right)\right]^{1/q}\right\}.\end{aligned}$$

The proof is completed. □

Corollary 1. *If choose $q = 2$ in Teorem 2, then we have*

$$\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) f(b)\right]\right| \leq \frac{(b-a)^4}{48}\left(\frac{1}{420}\right)^{1/2} \times \left\{\left[L_r\left(\left|f^{'''}(a)\right|^2, \left|f^{'''}\left(\frac{a+b}{2}\right)\right|^2\right)\right]^{1/2} + \left[L_r\left(\left|f^{'''}(b)\right|^2, \left|f^{'''}\left(\frac{a+b}{2}\right)\right|^2\right)\right]^{1/2}\right\}.$$

Corollary 2. *If choose $r = 1$ in Corollary 1, then we have*

$$\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) f(b)\right]\right| \leq \frac{(b-a)^4}{48\sqrt{2}}\left(\frac{1}{420}\right)^{1/2} \times \left\{\left[L_1\left(\left|f^{'''}(a)\right|^2, \left|f^{'''}\left(\frac{a+b}{2}\right)\right|^2\right)\right]^{1/2} + \left[L_1\left(\left|f^{'''}(b)\right|^2, \left|f^{'''}\left(\frac{a+b}{2}\right)\right|^2\right)\right]^{1/2}\right\}.$$

Theorem 3. *Let $f^{''} : I \subseteq R \to R$ be an absolutely continuous function on I° such that $f^{'''} \in L[a;b]$, where $a, b \in I$ with $a < b$. If $\left|f^{'''}\right|$ is r-convex on $[a, b]$ for $q > 1$, $\frac{1}{q} + \frac{1}{p} = 1$, then the following inequality holds:*

$$\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) f(b)\right]\right| \leq \frac{(b-a)^4}{3}\left(\frac{1}{2}\right)^{4+(p^2+p+q)/p^2q} \times \left(\frac{1}{2p^2+1}\right)^{1/p^2}\left(\frac{1}{pq+1}\right)^{1/pq} \times \left\{\left[L_r\left(\left|f^{'''}(a)\right|^q, \left|f^{'''}\left(\frac{a+b}{2}\right)\right|^q\right)\right]^{1/q} + \left[L_r\left(\left|f^{'''}(b)\right|^q, \left|f^{'''}\left(\frac{a+b}{2}\right)\right|^q\right)\right]^{1/q}\right\}.$$

Proof. From Lemma 1 and using modulus of properties

$$\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) f(b)\right]\right| \leq \frac{(b-a)^4}{6}\left\{\int_0^{1/2} t^2\left(\frac{1}{2} - t\right)\left|f^{'''}(ta + (1-t)b)\right| dt + \int_{1/2}^1 (1-t)^2\left(t - \frac{1}{2}\right)\left|f^{'''}(ta + (1-t)b)\right| dt\right\}.$$

Using well-known Hölder inequality, we have

$$\begin{aligned} A &= \left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)f(b)\right]\right| \\ &\leq \frac{(b-a)^4}{6}\left\{\left[\int_0^{1/2}\left(t^2\left(\frac{1}{2}-t\right)\right)^p dt\right]^{1/p}\left[\int_0^{1/2}\left|f^{'''}(ta+(1-t)b)\right|^q dt\right]^{1/q}\right. \\ &\left. + \left[\int_{1/2}^1\left((1-t)^2\left(t-\frac{1}{2}\right)\right)^p dt\right]^{1/p}\left[\int_{1/2}^1\left|f^{'''}(ta+(1-t)b)\right|^q dt\right]^{1/q}\right\}. \end{aligned}$$

If we use hölder inequality again and choose $u = ta + (1-t)b$, we have

$$\begin{aligned} A &\leq \frac{(b-a)^4}{6}\left\{\left(\left[\int_0^{1/2} t^{2p^2} dt\right]^{1/p}\left[\int_0^{1/2}\left(\frac{1}{2}-t\right)^{pq} dt\right]^{1/q}\right)^{1/p}\left[\frac{1}{b-a}\int_{(a+b)/2}^b\left|f^{'''}(u)\right|^q du\right]^{1/q}\right. \\ &\left. + \left(\left[\int_{1/2}^1 (1-t)^{2p^2}\right]^{1/p}\left[\int_{1/2}^1\left(t-\frac{1}{2}\right)^{pq} dt\right]^{1/q}\right)^{1/p}\left[\frac{1}{b-a}\int_a^{(a+b)/2}\left|f^{'''}(u)\right|^q du\right]^{1/q}\right\}. \end{aligned}$$

If we calculate integrals above and use 1.3, then we obtain result in Theorem 3. □

Corollary 3. *If choose $q = 2$ in Teorem 3, then we have*

$$\begin{aligned} \left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)f(b)\right]\right| &\leq \frac{(b-a)^4}{96}\left(\frac{1}{45}\right)^{1/4} \\ &\times\left\{\left[L_r\left(\left|f^{'''}(a)\right|^2, \left|f^{'''}\left(\frac{a+b}{2}\right)\right|^2\right)\right]^{1/2}\right. \\ &\left. + \left[L_r\left(\left|f^{'''}(b)\right|^2, \left|f^{'''}\left(\frac{a+b}{2}\right)\right|^2\right)\right]^{1/2}\right\}. \end{aligned}$$

Corollary 4. *If choose $r = 1$ in Corollary 3, then we have*

$$\begin{aligned} \left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)f(b)\right]\right| &\leq \frac{(b-a)^4}{96\sqrt[4]{180}}\times\left\{\left[L_1\left(\left|f^{'''}(a)\right|^2, \left|f^{'''}\left(\frac{a+b}{2}\right)\right|^2\right)\right]^{1/2}\right. \\ &\left. + \left[L_1\left(\left|f^{'''}(b)\right|^2, \left|f^{'''}\left(\frac{a+b}{2}\right)\right|^2\right)\right]^{1/2}\right\}. \end{aligned}$$

Theorem 4. *Let $f^{''} : I \subseteq R \to R$ be an absolutely continuous function on I° such that $f^{'''} \in L[a;b]$, where $a, b \in I$ with $a < b$. If $\left|f^{'''}\right|$ is r-convex on $[a,b]$ for*

$r>q>1$, $\frac{1}{q}+\frac{1}{p}=1$, then the following inequality holds:

$$\begin{aligned}\left|\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)f(b)\right]\right| &\leq \frac{(b-a)^4}{3}\left(\frac{1}{2}\right)^{(2r+1)/qr}\left(\frac{8^{-p}\Gamma(2p)\Gamma(p+1)}{\Gamma(3p+2)}\right)^{1/p}\\ &\times\left[\left(\left|f^{'''}(a)\right|^q+\left|f^{'''}(b)\right|^q\left(2^{1+1/r}\right)\right)^{1/q}\right.\\ &\left.+\left(\left|f^{'''}(a)\right|^q\left(2^{1+1/r}\right)+\left|f^{'''}(b)\right|^q\right)^{1/q}\right].\end{aligned}$$

Proof. From Lemma 1 and using modulus of properties

$$\begin{aligned}\left|\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)f(b)\right]\right| &\leq \frac{(b-a)^4}{6}\left\{\int_0^{1/2}t^2\left(\frac{1}{2}-t\right)\left|f^{'''}(ta+(1-t)b)\right|dt\right.\\ &\left.+\int_{1/2}^1(1-t)^2\left(t-\frac{1}{2}\right)\left|f^{'''}(ta+(1-t)b)\right|dt\right\}.\end{aligned}$$

Using well-known Hölder inequality, we have

$$\begin{aligned}A &= \left|\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)f(b)\right]\right|\\ &\leq \frac{(b-a)^4}{6}\left\{\left[\int_0^{1/2}\left(t^2\left(\frac{1}{2}-t\right)\right)^p dt\right]^{1/p}\left[\int_0^{1/2}\left|f^{'''}(ta+(1-t)b)\right|^q dt\right]^{1/q}\right.\\ &\left.+\left[\int_{1/2}^1\left((1-t)^2\left(t-\frac{1}{2}\right)\right)^p dt\right]^{1/p}\left[\int_{1/2}^1\left|f^{'''}(ta+(1-t)b)\right|^q dt\right]^{1/q}\right\}.\end{aligned}$$

If we use definition of r-convexty and $\sum_{i=1}^n(a_i+b_i)^k\leq\sum_{i=1}^n a_i^k+\sum_{i=1}^n b_i^k$ for $0<k<1$; $a_1,a_2,...,a_n\geq 0$ and $b_1,b_2,...,b_n\geq 0$, we obtain

$$\begin{aligned}A &\leq \frac{(b-a)^4}{6}\left(\frac{8^{-p}\Gamma(2p)\Gamma(p+1)}{\Gamma(3p+2)}\right)^{1/p}\\ &\left\{\left[\int_0^{1/2}t^{1/r}\left|f^{'''}(a)\right|^q+(1-t)^{1/r}\left|f^{'''}(b)\right|^q dt\right]^{1/q}+\left[\int_{1/2}^1 t^{1/r}\left|f^{'''}(a)\right|^q+(1-t)^{1/r}\left|f^{'''}(b)\right|^q dt\right]^{1/q}\right\}.\end{aligned}$$

If we calculate integrals above, then we obtain result in Theorem 4. □

Corollary 5. *If choose $q=2$ in Teorem 4, then we have*

$$\begin{aligned}\left|\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)f(b)\right]\right| &\leq \frac{(b-a)^4}{3}\left(\frac{1}{2}\right)^{(2r+1)/2r}\left(\frac{8^{-2}}{420}\right)^{1/2}\\ &\times\left[\left(\left|f^{'''}(a)\right|^2+\left|f^{'''}(b)\right|^2\left(2^{1+1/r}\right)\right)^{1/2}\right.\\ &\left.+\left(\left|f^{'''}(a)\right|^2\left(2^{1+1/r}\right)+\left|f^{'''}(b)\right|^2\right)^{1/2}\right].\end{aligned}$$

Theorem 5. *Let $f^{''}: I \subseteq R \to R$ be an absolutely continuous function on I° such that $f^{'''} \in L[a;b]$, where $a,b \in I$ with $a<b$. If $\left|f^{'''}\right|$ is r-convex on $[a,b]$ for $r>q>1$, $\frac{1}{q}+\frac{1}{p}=1$, then the following inequality holds:*

$$\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)f(b)\right]\right| \leq \frac{(b-a)^4}{6}\left(\left|f^{'''}(a)\right|^q+\left|f^{'''}(b)\right|^q\right)\frac{2^{-(4+1/r)}r^2}{12r^2+7r+1} \times\left[1+\frac{r\left(2^{4+1/r}(1-2r)+34r+11\right)+1}{(r+1)(2r+1)(5r+1)}\right].$$

Proof. From Lemma 1, using modulus of properties and definition of r-convexty

$$\begin{aligned} A &= \left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)f(b)\right]\right| \\ &\leq \frac{(b-a)^4}{6}\left\{\int_0^{1/2} t^2\left(\frac{1}{2}-t\right)\left[t\left|f^{'''}(a)\right|^r+(1-t)\left|f^{'''}(b)\right|^r\right]^{1/r}dt\right. \\ &\left.+\int_{1/2}^1 (1-t)^2\left(t-\frac{1}{2}\right)\left[t\left|f^{'''}(a)\right|^r+(1-t)\left|f^{'''}(b)\right|^r\right]^{1/r}dt\right\}. \end{aligned}$$

Using fact that $\sum_{i=1}^n (a_i+b_i)^k \leq \sum_{i=1}^n a_i^k + \sum_{i=1}^n b_i^k$ for $0<k<1$; $a_1,a_2,...,a_n \geq 0$ and $b_1,b_2,...,b_n \geq 0$, we obtain

$$\begin{aligned} A &\leq \frac{(b-a)^4}{6}\left\{\int_0^{1/2} t^{2+1/r}\left(\frac{1}{2}-t\right)\left|f^{'''}(a)\right| + t^2(1-t)^{1+1/r}\left(\frac{1}{2}-t\right)\left|f^{'''}(b)\right|dt\right. \\ &\left.+\int_{1/2}^1 (1-t)^2\left(t-\frac{1}{2}\right)t^{+1/r}\left|f^{'''}(a)\right|^r+(1-t)^{2+1/r}\left(t-\frac{1}{2}\right)\left|f^{'''}(b)\right|dt\right\}. \end{aligned}$$

If we calculate integrals above, then we obtain result in Theorem 5. □

Corollary 6. *If choose $q=2$ in Teorem 5, then we have*

$$\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)f(b)\right]\right| \leq \frac{(b-a)^4}{6}\left(\left|f^{'''}(a)\right|^2+\left|f^{'''}(b)\right|^2\right)\frac{2^{-(4+1/r)}r^2}{12r^2+7r+1} \times\left[1+\frac{r\left(2^{4+1/r}(1-2r)+34r+11\right)+1}{(r+1)(2r+1)(5r+1)}\right].$$

3. New Inequqlities Simpson's Type via Lemma 2

Theorem 6. *Let $f^{'''}: I \subseteq R \to R$ be an absolutely continuous mapping on I°, where $a,b \in I$ with $a<b$. If $f^{(4)} \in L[a,b]$. If $\left|f^{'''}\right|$ is r-convex on $[a,b]$ for $q>1$,*

$\frac{1}{q}+\frac{1}{p}=1$, *then the following inequality holds:*

$$\begin{aligned}\left|\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)f(b)\right]\right| &\le \frac{(b-a)^5}{3}2^{-\left(6+(1/p^2)+(1/q)\right)}\left(\frac{1}{3p^2+1}\right)^{1/p^2}\left(\frac{1}{pq+1}\right)^{1/pq}\\ &\left(\frac{1}{6}\right)^{(pq+1)/pq}\left(4^{pq}-1\right)^{1/pq}\\ &\times\left\{\left[L_r\left(\left|f^{(4)}(a)\right|^q,\left|f^{(4)}\left(\frac{a+b}{2}\right)\right|^q\right)\right]^{1/q}\right.\\ &\left.+\left[L_r\left(\left|f^{(4)}(b)\right|^q,\left|f^{(4)}\left(\frac{a+b}{2}\right)\right|^q\right)\right]^{1/q}\right\}.\end{aligned}$$

Proof. From Lemma 2 and using properties of moduls, we have

$$\begin{aligned}\left|\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)f(b)\right]\right| &\le \frac{(b-a)^5}{6}\left\{\int_0^{1/2}t^3\left(\frac{2}{3}-t\right)\left|f^{(4)}(tb+(1-t)a)\right|dt\right.\\ &\left.\int_{1/2}^1(1-t)^3\left(t-\frac{1}{3}\right)\left|f^{(4)}(tb+(1-t)a)\right|dt\right\}.\end{aligned}$$

If we use well-known Hölder inequlity, we obtain following inequality

$$\begin{aligned}A &= \left|\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)f(b)\right]\right|\\ &\le \frac{(b-a)^5}{6}\left\{\left[\int_0^{1/2}\left(t^3\left(\frac{2}{3}-t\right)\right)^p dt\right]^{1/p}\left[\int_0^{1/2}\left|f^{(4)}(tb+(1-t)a)\right|^q dt\right]^{1/q}\right.\\ &\left.+\left[\int_{1/2}^1\left((1-t)^3\left(t-\frac{1}{3}\right)\right)^p dt\right]^{1/p}\left[\int_{1/2}^1\left|f^{(4)}(tb+(1-t)a)\right|^q dt\right]^{1/q}\right\}.\end{aligned}$$

If we use Hölder inequality again and choose $u=tb+(1-t)a$, we have

$$\begin{aligned}A &\le \frac{(b-a)^5}{6}\left\{\left(\left[\int_0^{1/2}t^{3p^2}dt\right]^{1/p}\left[\int_0^{1/2}\left(\frac{2}{3}-t\right)^{pq}dt\right]^{1/q}\right)^{1/p}\left[\frac{1}{b-a}\int_a^{(a+b)/2}\left|f^{(4)}(u)\right|^q du\right]^{1/q}\right.\\ &\left.+\left(\left[\int_{1/2}^1(1-t)^{3p^2}dt\right]^{1/p}\left[\int_{1/2}^1\left(t-\frac{1}{3}\right)^{pq}dt\right]^{1/q}\right)^{1/p}\left[\frac{1}{b-a}\int_{(a+b)/2}^b\left|f^{(4)}(u)\right|^q du\right]^{1/q}\right\}.\end{aligned}$$

If we calculate integrals above, then we obtain result in Theorem 6. □

Corollary 7. *If choose $q=2$ in Teorem 6, then we have*

$$\left|\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)f(b)\right]\right| \leq \frac{(b-a)^5}{2304}\left(\frac{19}{39}\right)^{1/4} \times\left\{\left[L_r\left(\left|f^{(4)}(a)\right|^2,\left|f^{(4)}\left(\frac{a+b}{2}\right)\right|^2\right)\right]^{1/2}+\left[L_r\left(\left|f^{(4)}(b)\right|^2,\left|f^{(4)}\left(\frac{a+b}{2}\right)\right|^2\right)\right]^{1/2}\right\}.$$

Corollary 8. *If choose $r=1$ in Corollary 7, then we have*

$$\left|\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)f(b)\right]\right| \leq \frac{(b-a)^5}{2304}\left(\frac{19}{154}\right)^{1/4} \times\left\{\left[L_1\left(\left|f^{(4)}(a)\right|^2,\left|f^{(4)}\left(\frac{a+b}{2}\right)\right|^2\right)\right]^{1/2}+\left[L_1\left(\left|f^{(4)}(b)\right|^2,\left|f^{(4)}\left(\frac{a+b}{2}\right)\right|^2\right)\right]^{1/2}\right\}.$$

Theorem 7. *Let $f''':I\subseteq R\to R$ be an absolutely continuous mapping on I°, where $a,b\in I$ with $a<b$. If $f^{(4)}\in L[a,b]$. If $\left|f'''\right|$ is r-convex on $[a,b]$ for $r>q>1$, $\frac{1}{q}+\frac{1}{p}=1$, then the following inequality holds:*

$$\left|\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)f(b)\right]\right| \leq \frac{(b-a)^5}{24}\left[\frac{1}{(p+1)6^{p+1}}\left(4^{p+1}-1\right)\right]^{1/p} \times\left\{\left[\left|f^{(4)}(b)\right|^q M+\left|f^{(4)}(a)\right|^q N\right]^{1/q}+\left[\left|f^{(4)}(b)\right|^q N+\left|f^{(4)}(a)\right|^q M\right]^{1/q}\right\},$$

where

$$M=\frac{r}{3qr+1}\left(\frac{1}{2}\right)^{(3qr+1)/r},$$

and

$$N=\left(\left(\frac{1}{2}\right)^{3pq+1}\frac{1}{3pq+1}\right)^{1/p}\left(\frac{r}{q+r}\left(1-\left(\frac{1}{2}\right)^{(q+r)/r}\right)\right)^{1/q}.$$

Proof. From Lemma 2 and using properties of moduls and Hölder inequality, we have

$$\begin{aligned} A &= \left| \int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) f(b) \right] \right| \\ &\leq \frac{(b-a)^5}{24} \left\{ \left[\int_0^{1/2} \left(\frac{2}{3} - t\right)^p dt \right]^{1/p} \left[\int_0^{1/2} t^{3q} \left| f^{(4)}(tb + (1-t)a) \right|^q dt \right]^{1/q} \right. \\ &\quad \left. + \left[\int_{1/2}^1 \left(t - \frac{1}{3}\right)^p dt \right]^{1/p} \left[\int_{1/2}^1 (1-t)^{3q} \left| f^{(4)}(tb + (1-t)a) \right|^q dt \right]^{1/q} \right\}. \end{aligned}$$

If we use definition of r-convexty and $\sum_{i=1}^{n} (a_i + b_i)^k \leq \sum_{i=1}^{n} a_i^k + \sum_{i=1}^{n} b_i^k$ for $0 < k < 1$; $a_1, a_2, ..., a_n \geq 0$ and $b_1, b_2, ..., b_n \geq 0$, we obtain

$$\begin{aligned} A &\leq \frac{(b-a)^5}{24} \left\{ \left[\int_0^{1/2} \left(\frac{2}{3} - t\right)^p dt \right]^{1/p} \left[\int_0^{1/2} t^{3q+1/r} \left| f^{(4)}(b) \right|^q + t^{3q}(1-t)^{1/r} \left| f^{(4)}(a) \right|^q dt \right]^{1/q} \right. \\ &\quad \left. + \left[\int_{1/2}^1 \left(t - \frac{1}{3}\right)^p dt \right]^{1/p} \left[\int_{1/2}^1 (1-t)^{3q} t^{1/r} \left| f^{(4)}(b) \right|^q + (1-t)^{3q+1/r} \left| f^{(4)}(a) \right|^q dt \right]^{1/q} \right\}. \end{aligned}$$

If we calculate integrals above, then we obtain result in Theorem 7. □

Corollary 9. *If choose $q = 2$ in Teorem 7, then we have*

$$\begin{aligned} \left| \int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) f(b) \right] \right| &\leq \frac{(b-a)^5}{432} \left[\frac{65}{2} \right]^{1/2} \\ &\quad \times \left\{ \left[\left| f^{(4)}(b) \right|^2 M + \left| f^{(4)}(a) \right|^2 N \right]^{1/2} \right. \\ &\quad \left. + \left[\left| f^{(4)}(b) \right|^2 N + \left| f^{(4)}(a) \right|^2 M \right]^{1/2} \right\}, \end{aligned}$$

where

$$M = \frac{r}{6r+1} \left(\frac{1}{2}\right)^{(6r+1)/r},$$

and

$$N = \left(\left(\frac{1}{2}\right)^{13} \frac{1}{13} \right)^{1/2} \left(\frac{r}{2+r} \left(1 - \left(\frac{1}{2}\right)^{(2+r)/r} \right) \right)^{1/2}.$$

References

[1] Alomari M., W., N., and Hussain, S., (2011). Two Inequalities Of Simpson Type For Quasi-Convex Functions and Applications, Applied Mathematics E-Notes, 11(2011), 110-117.

[2] Alomari M., W., N., (2016). An Improvement Inequality of Simpson's Type for Quasi-Convex Mappings with Applications, Arxiv:1603.08055v1 [math.CA].

[3] Avriel, M., (1972). r-Convex Functions, Mathematical Programming 2: 309-323. North-Holland Publishing Company

[4] Avriel, M., (1973). Solution of Certain Nonlinear Programs Involving r-Convex Functions, Journal of Optimization Theory and Applications Vol. 11, No. 2.

[5] Dragomir, S.S., (1998). On Simpson's Quadrature Formula for Differentiable Mappings Whose Derivatives Belong to L_p-Spaces and Applications, RGMIA Reserch Report Collection Vol. 1, No. 2.

[6] Gill, P.M., Pearce, C.E.M., and Pečarić, J., (1997). Hadamard's inequality for r-convex functions, Journal of Mathematical Analysis and Applications 215: 461–470.

[7] Ngoc, N.P.N., Vinh, N.V., and Hien, P.T.T., (2009). Integral inequalities of Hadamard type for r-convex functions. International Mathematical Forum, no:4, 1723-1728.

[8] Pearce, C.E.M., Pečarić, J., and Šimić, V., (1998). Stolarsky means and Hadamard's inequality. Journal of Mathematical Analysis and Applications 220: 99–109.

[9] Pečarić, J., and Varošanec, S., (2001). Simpson's formula for functions whose derivatives belong to Lp spaces. Applied Mathematics Letters 14: 131–1352.

[10] Sulaiman, W.T., (2010). Integral Inequalities Regarding r-Convex and r-Concave Functions, J. Korean Math. Soc. 47 (2010), No. 2, pp. 373–383.

[11] Yang, G.S., and Hwang, D.Y., (2001). Refinementes of Hadamard inequality for r-convex functions. Indian J. Pure Appl. Math. 32(10), 1571-15791.

[12] Zabandan, G., Bodaghi, A., and Kılıçman, A., (2012). The Hermite-Hadamard inequality for r-convexfunctions, Journal of Inequalities and Applications, 215.

[13] Zhao, Y.X., Wang, S.Y., and Uria, L.C., (2010). Characterizations of r-Convex Functions, J Optim Theory Appl 145: 186–195

[14] Wang, J., Deng, J., and Fečkan, M., (2013). Hermite-Hadamard Type Inequalities for r-Convex Functions Based on The Use of Riemann-Lioville Fractional Integrals, Ukrainian Mathematical Journal, Vol. 65, No. 2.

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