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NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR r-CONVEX FUNCTIONS

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ABSTRACT. In this paper, we established new integral inequalities of Hermite-Hadamard type for r-convex functions via two integral identities.

1. Introduction

Firstly, we start one of the most famous inequality for convex functions is so called Hermite-Hadamard's inequality as follows: Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$, with $a < b$. Then:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_0^1 f(x)dx \le \frac{f(a)+f(b)}{2}$$

,see [4]. Follwing found inequalities are had Hermite-Hadamard Type.

In [4],The power mean $M_r(x, y; \lambda)$ of order r of positive numbers x, y is defined as the following:

$$M_r(x,y;\lambda) = \begin{cases} (\lambda x^r + (1-\lambda)y^r), & if\ r \neq 0 \\ \left(x^r y^{1-\lambda}\right), & if\ r = 0 \end{cases}. \tag{1.1}$$

In [5], Gill et. all used the definition of $M_r(x, y; \lambda)$ to introduce the concept of r-convex functions.

Definition 1. *A positive function f is $r-$convex on $[a, b]$ if for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$*

(1.2)

$$f(\lambda x + (1-\lambda)y) \le M_r(f(x), f(y); \lambda) = \begin{cases} (\lambda f^r(x) + (1-\lambda)f^r(y))^{\frac{1}{r}}, & r \neq 0 \\ f^{\lambda}(x) f^{1-\lambda}(y), & r = 0 \end{cases}.$$

In the definition of r-convex functions if we choose $r = 1$ and $r = 0$, we have ordinary convex functions and log-convex functions respectively.

It is obvious that if f is r-convex in $[a, b]$ where $r > 0$, then f^r is convex on $[a, b]$.

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The generalized logarithmic means of order r of positive numbers x, y is defined by

$$L_r(x,y) = \begin{cases} \frac{r}{r+1}\frac{x-y}{\ln x - \ln y}, & r \neq 0, -1; x \neq y \\ \frac{x-y}{\ln x - \ln y}, & r = 0, x \neq y \\ xy\frac{x-y}{\ln x - \ln y}, & r = -1, x \neq y \\ x & x = y \end{cases}$$

,see [5].

Gill et. all proved the following theorem for r-convex functions:

Theorem 1. *Suppose f is a positive r-convex function on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq L_r(f(a), f(b)). \tag{1.3}$$

If f is a positive r-concave function, then the inequality is reversed, see [5].

For several results concerning of r-convexity, see [2]-[5], [7]-[9] and [12]-[15] where further references are listed.

The main aim of this paper is to obtain some new integral inequalities for r-convex functions by using Lemma 1 and Lemma 2.

We will give two integral identities which are emboided in the following lemmas to obtain our main theorems.

Lemma 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°(the interior of I), where where $a, b \in I$ with $a < b$. If $f^{'} \in L[a,b]$, then the following inequality holds:*

$$\frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du = \frac{(x-a)^2}{b-a}\int_0^1 (t-1)f^{'}(tx + (1-t)a)dt + \frac{(b-x)^2}{b-a}\int_0^1 (1-t)f^{'}(tx + (1-t)b)dt.$$

this inequality, see [6].

Lemma 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$, be twice differentiable function on I°, $a, b \in I^\circ$ (I° is the interior of I) with $a < b$.If $f^{''} \in L_1([a,b])$ then*

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) = \frac{(b-a)^2}{2}\left[\int_0^1 m(t)\left[f^{''}(ta + (1-t)b) + f^{''}(tb + (1-t)a)\right]dt\right],$$

where $m(t) = \begin{cases} t^2, & t \in [0, \frac{1}{2}) \\ (1-t)^2, & t \in [\frac{1}{2}, 1] \end{cases}$ *, see* [10].

2. New Inequalities Hermite-Hadamard Type via Lemma 1

We will start with the following theorem:

Theorem 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ be a positive twice differentable function on I° such that $f^{'} \in L[a,b]$ where $a, b \in I$ with $a < b$. If $\left|f^{'}\right|$ is r-convex on $[a,b]$ for $q > 1$, $\frac{1}{q} + \frac{1}{p} = 1$, then the following inequalitiy holds:*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \left(\frac{1}{p+1}\right)^{1/p}\left(\frac{(x-a)^2}{b-a}\left[L_r\left(\left|f^{'}(a)\right|^q, \left|f^{'}(x)\right|^q\right)\right]^{1/q} + \frac{(b-x)^2}{b-a}\left[L_r\left(\left|f^{'}(b)\right|^q, \left|f^{'}(x)\right|^q\right)\right]^{1/q}\right).$$

Proof. From Lemma 1, using properties of modulus and applying Hölder inequality, we have

$$\begin{aligned}
A &= \left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du\right| \\
&\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 (1-t)^p dt\right)^{1/p}\left(\int_0^1 \left|f^{'}(tx+(1-t)a)\right|^q dt\right)^{1/q} \\
&\quad + \frac{(b-x)^2}{b-a}\left(\int_0^1 (1-t)^p dt\right)^{1/p}\left(\int_0^1 \left|f^{'}(tx+(1-t)b)\right|^q dt\right)^{1/q}.
\end{aligned}$$

If we choose $v = 1-t$, $k = tx+(1-t)a$ and $u = tx+(1-t)b$, then we have following inequality:

$$\begin{aligned}
A &\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 (v)^p dv\right)^{1/p}\left(\frac{1}{x-a}\int_a^x \left|f^{'}(k)\right|^q dk\right)^{1/q} \\
&\quad + \frac{(b-x)^2}{b-a}\left(\int_0^1 (v)^p dv\right)^{1/p}\left(\frac{1}{b-x}\int_x^b \left|f^{'}(u)\right|^q du\right)^{1/q}.
\end{aligned}$$

If we use 1.3 for $\left|f^{'}\right|^q$ and calculate integrals, then we can write

$$\begin{aligned}
A &\leq \frac{(x-a)^2}{b-a}\left(\frac{1}{p+1}\right)^{1/p}\left(L_r\left(\left|f^{'}(a)\right|^q, \left|f^{'}(x)\right|^q\right)\right)^{1/p} \\
&\quad + \frac{(b-x)^2}{b-a}\left(\frac{1}{p+1}\right)^{1/p}\left(L_r\left(\left|f^{'}(b)\right|^q, \left|f^{'}(x)\right|^q\right)\right)^{1/p}.
\end{aligned}$$

The proof is completed. □

Corollary 1. *If we choose $x = \frac{a+b}{2}$ in Theorem 2, the inequality reduces to inequality in* [1] *which is obtained by M. Alomari.*

Corollary 2. *If choose $q=2$ in Corollary 1, then we have*

$$\begin{aligned}\left|\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| &\leq \left(\frac{b-a}{12}\right)^{1/2}\left(\left[L_r\left(\left|f'(a)\right|^2,\left|f'(x)\right|^2\right)\right]^{1/2}\right.\\ &\left.+\left[L_r\left(\left|f'(b)\right|^2,\left|f'(x)\right|^2\right)\right]^{1/2}\right).\end{aligned}$$

Corollary 3. *If choose $r=1$ in Corollary 2, then we have*

$$\begin{aligned}\left|\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| &\leq \left(\frac{b-a}{24}\right)^{1/2}\left(\left[L_1\left(\left|f'(a)\right|^2,\left|f'(x)\right|^2\right)\right]^{1/2}\right.\\ &\left.+\left[L_1\left(\left|f'(b)\right|^2,\left|f'(x)\right|^2\right)\right]^{1/2}\right).\end{aligned}$$

Theorem 3. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}^+$ be a positive twice differentable function on I° such that $f'\in L[a,b]$ where $a,b\in I$ with $a<b$. If $\left|f'\right|$ is r-convex on $[a,b]$ for $r>q>1$, $\frac{1}{q}+\frac{1}{p}=1$, then the following inequalitiy holds:*

$$\begin{aligned}\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| &\leq \frac{(x-a)^2}{b-a}\left(\frac{1}{p+1}\right)^{1/p}\left(\left[\left|f'(a)\right|^q+\left|f'(x)\right|^q\right]\frac{r}{q+r}\right)^{1/q}\\ &+\frac{(b-x)^2}{b-a}\left(\frac{1}{p+1}\right)^{1/p}\left(\left[\left|f'(b)\right|^q+\left|f'(x)\right|^q\right]\frac{r}{q+r}\right)^{1/q}.\end{aligned}$$

Proof. From Lemma 1, using properties of modulus and applying Hölder inequality, we have

$$\begin{aligned}A &= \left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|\\ &\leq \frac{(x-a)^2}{b-a}\left(\int_0^1(1-t)^p dt\right)^{1/p}\left(\int_0^1\left|f'(tx+(1-t)a)\right|^q dt\right)^{1/q}\\ &+\frac{(b-x)^2}{b-a}\left(\int_0^1(1-t)^p dt\right)^{1/p}\left(\int_0^1\left|f'(tx+(1-t)b)\right|^q dt\right)^{1/q}\end{aligned}$$

If we choose $v=1-t$ and use r-convexity of $\left|f'\right|$, then we have following inequality:

$$\begin{aligned}A &\leq \frac{(x-a)^2}{b-a}\left(\int_0^1(v)^p dv\right)^{1/p}\left(\frac{1}{x-a}\int_0^1\left[t\left|f'(x)\right|^r+(1-t)\left|f'(a)\right|^r\right]^{q/r}dx\right)^{1/q}\\ &+\frac{(b-x)^2}{b-a}\left(\int_0^1(v)^p dv\right)^{1/p}\left(\frac{1}{b-x}\int_0^1\left[t\left|f'(x)\right|^r+(1-t)\left|f'(b)\right|^r\right]^{q/r}dx\right)^{1/q}.\end{aligned}$$

Using fact that $\sum_{i=1}^{n}(a_i+b_i)^k \leq \sum_{i=1}^{n} a_i^k + \sum_{i=1}^{n} b_i^k$ for $0<k<1$; $a_1,a_2,...,a_n \geq 0$ and $b_1,b_2,...,b_n \geq 0$, we obtain

$$\begin{aligned}
A &\leq \frac{(x-a)^2}{b-a}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{x-a}\int_0^1\left[t^{1/r}\left|f^{'}(x)\right|^q+(1-t)^{1/r}\left|f^{'}(a)\right|^q\right]dx\right)^{1/q} \\
&\quad+\frac{(b-x)^2}{b-a}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{b-x}\int_0^1\left[t^{1/r}\left|f^{'}(x)\right|^q+(1-t)^{1/r}\left|f^{'}(b)\right|^q\right]dx\right)^{1/q}.
\end{aligned}$$

If we use basic integral calculation above, then the proof is completed. □

Corollary 4. *If we choose $x=\frac{a+b}{2}$ in Theorem 3, then we have*

$$\begin{aligned}
\left|\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| &\leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{r}{q+r}\right)^{1/q} \\
&\quad\left[\left(\left[\left|f^{'}(a)\right|^q+\left|f^{'}(x)\right|^q\right]\right)^{1/q}\right. \\
&\quad\left.+\left(\left[\left|f^{'}(b)\right|^q+\left|f^{'}(x)\right|^q\right]\right)^{1/q}\right].
\end{aligned}$$

Corollary 5. *If choose $q=2$ in Corollary 4, then we have*

$$\begin{aligned}
\left|\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| &\leq \frac{b-a}{4}\left(\frac{r}{6+3r}\right)^{1/2} \\
&\quad\left[\left(\left[\left|f^{'}(a)\right|^2+\left|f^{'}(x)\right|^2\right]\right)^{1/2}\right. \\
&\quad\left.+\left(\left[\left|f^{'}(b)\right|^2+\left|f^{'}(x)\right|^2\right]\right)^{1/2}\right].
\end{aligned}$$

Corollary 6. *Under the assumptions of Theorem 3, if $\left|f^{'}\right|^q$ is r-convex, then the following inequality holds:*

$$\begin{aligned}
\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| &\leq \frac{(x-a)^2}{b-a}\left(\frac{1}{p+1}\right)^{1/p} \\
&\quad\times\left(\left[\left|f^{'}(a)\right|^q+\left|f^{'}(x)\right|^q\right]\frac{r}{1+r}\right)^{1/q} \\
&\quad+\frac{(b-x)^2}{b-a}\left(\frac{1}{p+1}\right)^{1/p} \\
&\quad\times\left(\left[\left|f^{'}(b)\right|^q+\left|f^{'}(x)\right|^q\right]\frac{r}{1+r}\right)^{1/q}.
\end{aligned}$$

Theorem 4. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}^{+}$ be a positive twice differentable function on I° such that $f^{'}\in L[a,b]$ where $a,b\in I$ with $a<b$. If $\left|f^{'}\right|$ is r-convex on $[a,b]$ for*

$r > q > 1$, $\frac{1}{q} + \frac{1}{p} = 1$, then the following inequalitiy holds:

$$\begin{aligned}\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| &\leq \frac{1}{(1+r)(2r+1)(b-a)}\left[r^2\left[(x-a)^2+(b-x)^2\right]\left|f^{'}(x)\right|\right.\\ &\left.+(r+2r^2)\left[(x-a)^2\left|f^{'}(a)\right|+(b-x)^2\left|f^{'}(b)\right|\right]\right].\end{aligned}$$

Proof. From Lemma 1, using properties of modulus, we have

$$\begin{aligned} A &= \left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \\ &\leq \frac{(x-a)^2}{b-a}\int_0^1(1-t)\left|f^{'}(tx+(1-t)a)\right|dt \\ &+\frac{(b-x)^2}{b-a}\int_0^1(1-t)\left|f^{'}(tx+(1-t)b)\right|dt \end{aligned}$$

If we use r-convexity of definition for $\left|f^{'}\right|$, then we have following inequality:

$$\begin{aligned} A &\leq \frac{(x-a)^2}{b-a}\int_0^1(1-t)\left[t\left|f^{'}(x)\right|^r+(1-t)\left|f^{'}(a)\right|^r\right]^{1/r}dt \\ &+\frac{(b-x)^2}{b-a}\int_0^1(1-t)\left[t\left|f^{'}(x)\right|^r+(1-t)\left|f^{'}(b)\right|^r\right]^{1/r}dt. \end{aligned}$$

Using fact that $\sum_{i=1}^{n}(a_i+b_i)^k \leq \sum_{i=1}^{n} a_i^k + \sum_{i=1}^{n} b_i^k$ for $0 < k < 1$; $a_1, a_2, ..., a_n \geq 0$ and $b_1, b_2, ..., b_n \geq 0$, we obtain

$$\begin{aligned} A &\leq \frac{(x-a)^2}{b-a}\int_0^1(1-t)\left[t^{1/r}\left|f^{'}(x)\right|+(1-t)^{1/r}\left|f^{'}(a)\right|\right]^{1/r}dt \\ &+\frac{(b-x)^2}{b-a}\int_0^1(1-t)\left[t^{1/r}\left|f^{'}(x)\right|+(1-t)^{1/r}\left|f^{'}(b)\right|\right]dt. \end{aligned}$$

If we use basic integral calculation above, then the proof is completed. □

Corollary 7. *If we choose $x = \frac{a+b}{2}$ in Theorem 4, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{2(b-a)\left(r^2\left|f^{'}\left(\frac{a+b}{2}\right)\right|+(r+2r^2)\left[\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right]\right)}{4(1+r)(2r+1)}.$$

3. New Inequalities Hermite-Hadamard Type via Lemma 2

Theorem 5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ be a positive twice differentable function on I° such that $f^{''} \in L[a,b]$ where $a, b \in I$ with $a < b$. If $\left|f^{''}\right|^q$ is r-convex on $[a,b]$ for*

$q > 1$, $\frac{1}{q} + \frac{1}{p} = 1$, *then the following inequalitiy holds:*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{2^5} \left(\frac{1}{2p+1} \right)^{1/p} \left[\left(L_r \left(\left| f^{''}(b) \right|^q \left| f^{''}(\frac{a+b}{2}) \right|^q \right) \right)^{1/q} + \left(L_r \left(\left| f^{''}(a) \right|^q \left| f^{''}(\frac{a+b}{2}) \right|^q \right) \right)^{1/q} \right].$$

Proof. From Lemma 2 and using properties of modulus, we have

$$\begin{aligned} A &= \left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \\ &\leq \frac{(b-a)^2}{2} \left[\int_0^1 |m(t)| \left| f^{''}(ta+(1-t)b) \right| dt + \int_0^1 |m(t)| \left| f^{''}(tb+(1-t)a) \right| dt \right] \\ &= \frac{(b-a)^2}{2} \left[\int_0^{1/2} t^2 \left| f^{''}(ta+(1-t)b) \right| dt + \int_{1/2}^1 (1-t)^2 \left| f^{''}(ta+(1-t)b) \right| dt \right. \\ &\quad \left. + \int_0^{1/2} t^2 \left| f^{''}(tb+(1-t)a) \right| dt + \int_{1/2}^1 (1-t)^2 \left| f^{''}(tb+(1-t)a) \right| dt \right]. \end{aligned}$$

Using well-known Hölder inequality, we have

$$\begin{aligned} A &\leq \frac{(b-a)^2}{2} \left[\left(\int_0^{1/2} t^{2p} dt \right)^{1/p} \left(\int_0^{1/2} \left| f^{''}(ta+(1-t)b) \right|^q dt \right)^{1/q} \right. \\ &\quad + \left(\int_{1/2}^1 (1-t)^{2p} dt \right)^{1/p} \left(\int_{1/2}^1 \left| f^{''}(ta+(1-t)b) \right|^q dt \right)^{1/q} \\ &\quad + \left(\int_0^{1/2} t^{2p} dt \right)^{1/p} \left(\int_0^{1/2} \left| f^{''}(tb+(1-t)a) \right|^q dt \right)^{1/q} \\ &\quad \left. + \left(\int_{1/2}^1 (1-t)^{2p} dt \right)^{1/p} \left(\int_{1/2}^1 \left| f^{''}(tb+(1-t)a) \right|^q dt \right)^{1/q} \right]. \end{aligned}$$

If we choose $v = 1-t$, $k = tb+(1-t)a$ and $u = ta+(1-t)b$, then we have following inequality:

$$\begin{aligned} A &\leq \frac{(b-a)^2}{2} \left[\left(\int_0^{1/2} t^{2p} dt \right)^{1/p} \left(\frac{1}{b-a} \int_{a+b/2}^b \left| f^{''}(u) \right|^q dt \right)^{1/q} \right. \\ &\quad + \left(\int_0^{1/2} v^{2p} dv \right)^{1/p} \left(\int_a^{a+b/2} \left| f^{''}(u) \right|^q dt \right)^{1/q} \\ &\quad + \left(\int_0^{1/2} t^{2p} dt \right)^{1/p} \left(\int_a^{a+b/2} \left| f^{''}(k) \right|^q dk \right)^{1/q} \\ &\quad \left. + \left(\int_0^{1/2} v^{2p} dv \right)^{1/p} \left(\int_{a+b/2}^b \left| f^{''}(k) \right|^q dk \right)^{1/q} \right]. \end{aligned}$$

If we use 1.3 for $\left|f'\right|^q$ and calculate integrations, then we can write

$$\begin{aligned} A &\leq \frac{(b-a)^2}{2}\left(\frac{1}{2^{2p+1}2p+1}\right)^{1/p}\left[\left(\frac{1}{2}L_r\left(\left|f''(b)\right|^q,\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)\right)^{1/q}\right. \\ &+\left(\frac{1}{2}L_r\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(a)\right|^q\right)\right)^{1/q} \\ &+\left(\frac{1}{2}L_r\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(a)\right|^q\right)\right)^{1/q} \\ &\left.+\left(\frac{1}{2}L_r\left(\left|f''(b)\right|^q,\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)\right)^{1/q}\right]. \end{aligned}$$

The proof is completed. □

Corollary 8. *If choose $q=2$ in Theorem 5, then we have*

$$\begin{aligned} \left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| &\leq \frac{(b-a)^2}{2^5}\left(\frac{1}{3}\right)^{1/2}\left[\left(L_r\left(\left|f''(b)\right|^2\left|f''(\frac{a+b}{2})\right|^2\right)\right)^{1/2}\right. \\ &\left.+\left(L_r\left(\left|f''(a)\right|^2\left|f''(\frac{a+b}{2})\right|^2\right)\right)^{1/2}\right]. \end{aligned}$$

Corollary 9. *If choose $r=1$ in Corollary 8, then we have*

$$\begin{aligned} \left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| &\leq \frac{(b-a)^2}{2^5}\left(\frac{1}{6}\right)^{1/2}\left[\left(L_1\left(\left|f''(b)\right|^2\left|f''(\frac{a+b}{2})\right|^2\right)\right)^{1/2}\right. \\ &\left.+\left(L_1\left(\left|f''(a)\right|^2\left|f''(\frac{a+b}{2})\right|^2\right)\right)^{1/2}\right]. \end{aligned}$$

Theorem 6. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ be a positive twice differentable function on I° such that $f'' \in L[a,b]$ where $a,b \in I$ with $a<b$. If $\left|f''\right|$ is r-convex on $[a,b]$ for $r>q>1$, $\frac{1}{q}+\frac{1}{p}=1$, then the following inequalitiy holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq (b-a)^2\left(\left|f''(b)\right|+\left|f''(a)\right|\right)\left[\frac{2r^3-\left(\frac{1}{2}\right)^{3+1/r}(14r^3+7r^2+r)}{(r+1)(2r+1)(3r+1)}\right].$$

Proof. From Lemma 2, using properties of modulus and r-convexty of definition, we have

$$\begin{aligned} A &= \left| \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{(b-a)^2}{2}\left[\int_0^{1/2} t^2\left[t\left|f''(a)\right|^r + (1-t)\left|f''(b)\right|^r\right]^{1/r} dt \right. \\ &\quad + \int_{1/2}^1 (1-t)^2\left[t\left|f''(a)\right|^r + (1-t)\left|f''(b)\right|^r\right]^{1/r} dt \\ &\quad + \int_0^{1/2} t^2\left[t\left|f''(b)\right|^r + (1-t)\left|f''(a)\right|^r\right]^{1/r} dt \\ &\quad \left. + \int_{1/2}^1 (1-t)^2\left[t\left|f''(b)\right|^r + (1-t)\left|f''(a)\right|^r\right]^{1/r} dt\right]. \end{aligned}$$

Using fact that $\sum_{i=1}^n (a_i+b_i)^k \leq \sum_{i=1}^n a_i^k + \sum_{i=1}^n b_i^k$ for $0<k<1$; $a_1, a_2, ..., a_n \geq 0$ and $b_1, b_2, ..., b_n \geq 0$, we obtain

$$\begin{aligned} A &\leq \frac{(b-a)^2}{2}\left[\int_0^{1/2} t^2\left[t^{1/r}\left|f''(a)\right| + (1-t)^{1/r}\left|f''(b)\right|\right] dt + \right. \\ &\quad \int_{1/2}^1 (1-t)^2\left[t^{1/r}\left|f''(a)\right| + (1-t)^{1/r}\left|f''(b)\right|\right] dt \\ &\quad + \int_0^{1/2} t^2\left[t^{1/r}\left|f''(b)\right| + (1-t)^{1/r}\left|f''(a)\right|\right] dt \\ &\quad \left. + \int_{1/2}^1 (1-t)^2\left[t^{1/r}\left|f''(b)\right| + (1-t)^{1/r}\left|f''(a)\right|\right] dt\right]. \end{aligned}$$

If we use basic integral calculation above, then the proof is completed. □

Theorem 7. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ be a positive twice differentable function on I° such that $f'' \in L[a,b]$ where $a, b \in I$ with $a<b$. If $\left|f''\right|$ is r-convex on $[a,b]$ for $r>q>1$, $\frac{1}{q}+\frac{1}{p}=1$, then the following inequalitiy holds:*

$$\begin{aligned} \left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| &\leq \frac{(b-a)^2}{4}\left(\frac{1}{4p+2}\right)^{1/p}\left(\frac{r}{r+1}\right)^{1/q} \\ &\quad \times\left\{\left[\left(\frac{1}{2}\right)^{1+1/r}\left|f''(a)\right|^q + \left(1-\left(\frac{1}{2}\right)^{1+1/r}\right)\left|f''(b)\right|^q\right]^{1/q}\right. \\ &\quad \left. + \left[\left(\frac{1}{2}\right)^{1+1/r}\left|f''(b)\right|^q + \left(1-\left(\frac{1}{2}\right)^{1+1/r}\right)\left|f''(a)\right|^q\right]^{1/q}\right\}. \end{aligned}$$

Proof. From Lemma 2, using properties of modulus and applying Hölder inequality, we have

$$\begin{aligned} A &= \left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \\ &\leq \frac{(b-a)^2}{2} \left[\left(\int_0^{1/2} t^{2p} dt \right)^{1/p} \left(\int_0^{1/2} \left| f^{''}(ta+(1-t)b) \right|^q dt \right)^{1/q} \right. \\ &+ \left(\int_{1/2}^1 (1-t)^{2p} dt \right)^{1/p} \left(\int_{1/2}^1 \left| f^{''}(ta+(1-t)b) \right|^q dt \right)^{1/q} \\ &+ \left(\int_0^{1/2} t^{2p} dt \right)^{1/p} \left(\int_0^{1/2} \left| f^{''}(tb+(1-t)a) \right|^q dt \right)^{1/q} \\ &+ \left. \left(\int_{1/2}^1 (1-t)^{2p} dt \right)^{1/p} \left(\int_{1/2}^1 \left| f^{''}(tb+(1-t)a) \right|^q dt \right)^{1/q} \right]. \end{aligned}$$

If we choose $v = 1-t$, use r-convexity of $\left| f' \right|$ and $\sum_{i=1}^n (a_i + b_i)^k \leq \sum_{i=1}^n a_i^k + \sum_{i=1}^n b_i^k$ for $0 < k < 1$; $a_1, a_2, ..., a_n \geq 0$ and $b_1, b_2, ..., b_n \geq 0$, then we have following inequality:

$$\begin{aligned} A \leq \frac{(b-a)^2}{2} \left(\frac{1}{2^{2p+1} 2p+1} \right)^{1/p} & \left[\left(\int_0^{1/2} \left[t^{1/r} \left| f^{''}(a) \right|^q + (1-t)^{1/r} \left| f^{''}(b) \right|^q \right] dt \right)^{1/q} \right. \\ &+ \left(\int_{1/2}^1 \left[t^{1/r} \left| f^{''}(a) \right|^q + (1-t)^{1/r} \left| f^{''}(b) \right|^q \right] dt \right)^{1/q} \\ &+ \left(\int_0^{1/2} \left[t^{1/r} \left| f^{''}(b) \right|^q + (1-t)^{1/r} \left| f^{''}(a) \right|^q \right] dt \right)^{1/q} \\ &+ \left. \left(\int_{1/2}^1 \left[t^{1/r} \left| f^{''}(b) \right|^q + (1-t)^{1/r} \left| f^{''}(a) \right|^q \right] dt \right)^{1/q} \right]. \end{aligned}$$

If we calculate above four integrals, then the proof is completed. □

Corollary 10. *If choose $q = 2$ in Theorem 7, then we have*

$$\begin{aligned} \left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| &\leq \frac{(b-a)^2}{4} \left(\frac{1}{10}\right)^{1/2} \left(\frac{r}{r+1}\right)^{1/2} \\ &\times \left\{ \left[\left(\frac{1}{2}\right)^{1+1/r} \left| f^{''}(a) \right|^2 + \left(1 - \left(\frac{1}{2}\right)^{1+1/r} \right) \left| f^{''}(b) \right|^2 \right]^{1/2} \right. \\ &+ \left. \left[\left(\frac{1}{2}\right)^{1+1/r} \left| f^{''}(b) \right|^2 + \left(1 - \left(\frac{1}{2}\right)^{1+1/r} \right) \left| f^{''}(a) \right|^2 \right]^{1/2} \right\}. \end{aligned}$$

Theorem 8. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ be a positive twice differentable function on I° such that $f'' \in L[a,b]$ where $a, b \in I$ with $a < b$. If $\left|f''\right|$ is r-convex on $[a,b]$ for $r > q > 1$, $\frac{1}{q} + \frac{1}{p} = 1$, then the following inequalitiy holds:*

$$\begin{aligned}
\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq\ & \frac{(b-a)^2}{2\times 24^{1/p}}\left[\left(\left(\frac{1}{2}\right)^{3+1/r}\frac{r}{3r+1}\left|f''(a)\right|^q\right.\right. \\
& \left. + \frac{3r^3 - \left(\frac{1}{2}\right)^{3+1/r}(14r^3+7r^2+r)}{(r+1)(2r+1)(3r+1)}\left|f''(b)\right|^q\right)^{1/q} \\
& + \left(\left(\frac{1}{2}\right)^{3+1/r}\frac{r}{3r+1}\left|f''(b)\right|^q\right. \\
& \left.\left. + \frac{3r^3 - \left(\frac{1}{2}\right)^{3+1/r}(14r^3+7r^2+r)}{(r+1)(2r+1)(3r+1)}\left|f''(a)\right|^q\right)^{1/q}\right].
\end{aligned}$$

Proof. From Lemma 2, using properties of modulus and applying Power Mean inequality we have

$$\begin{aligned}
A &= \left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)^2}{2}\left[\left(\int_0^{1/2} t^2 dt\right)^{1/p}\left(\int_0^{1/2} t^2\left|f''(ta+(1-t)b)\right|^q dt\right)^{1/q}\right. \\
&\quad + \left(\int_{1/2}^1 (1-t)^2 dt\right)^{1/p}\left(\int_{1/2}^1 (1-t)^2\left|f''(ta+(1-t)b)\right|^q dt\right)^{1/q} \\
&\quad + \left(\int_0^{1/2} t^2 dt\right)^{1/p}\left(\int_0^{1/2} t^2\left|f''(tb+(1-t)a)\right|^q dt\right)^{1/q} \\
&\quad \left. + \left(\int_{1/2}^1 (1-t)^2 dt\right)^{1/p}\left(\int_{1/2}^1 (1-t)^2\left|f''(tb+(1-t)a)\right|^q dt\right)^{1/q}\right].
\end{aligned}$$

If we choose $v = 1-t$, use r-convexity of $\left|f'\right|$ and $\sum_{i=1}^{n}(a_i+b_i)^k \leq \sum_{i=1}^{n} a_i^k + \sum_{i=1}^{n} b_i^k$ for $0<k<1$; $a_1, a_2, ..., a_n \geq 0$ and $b_1, b_2, ..., b_n \geq 0$, then we have following inequality:

$$\begin{aligned} A \leq \; & \frac{(b-a)^2}{2}\left[\left(\int_0^{1/2} t^2 dt\right)^{1/p}\left(\int_0^{1/2} t^2\left[t^{1/r}\left|f''(a)\right|^q + (1-t)^{1/r}\left|f''(b)\right|^q\right]dt\right)^{1/q}\right. \\ & + \left(\int_0^{1/2} v^2 dv\right)^{1/p}\left(\int_{1/2}^{1} (1-t)^2\left[t^{1/r}\left|f''(a)\right|^q + (1-t)^{1/r}\left|f''(b)\right|^q\right]dt\right)^{1/q} \\ & + \left(\int_0^{1/2} t^2 dt\right)^{1/p}\left(\int_0^{1/2} t^2\left[t^{1/r}\left|f''(b)\right|^q + (1-t)^{1/r}\left|f''(a)\right|^q\right]dt\right)^{1/q} \\ & \left. + \left(\int_0^{1/2} v^2 dv\right)^{1/p}\left(\int_{1/2}^{1} (1-t)^2\left[t^{1/r}\left|f''(b)\right|^q + (1-t)^{1/r}\left|f''(a)\right|^q\right]dt\right)^{1/q}\right]. \end{aligned}$$

If we calculate above integrals, then the proof is completed. □

Corollary 11. *If choose $q = 2$ in Theorem 8, then we have*

$$\begin{aligned} \left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \; & \frac{(b-a)^2}{2\times 24^{1/2}}\left[\left(\left(\frac{1}{2}\right)^{3+1/r}\frac{r}{3r+1}\left|f''(a)\right|^2\right.\right. \\ & \left. + \frac{3r^3 - \left(\frac{1}{2}\right)^{3+1/r}(14r^3+7r^2+r)}{(r+1)(2r+1)(3r+1)}\left|f''(b)\right|^2\right)^{1/2} \\ & + \left(\left(\frac{1}{2}\right)^{3+1/r}\frac{r}{3r+1}\left|f''(b)\right|^2\right. \\ & \left.\left. + \frac{3r^3 - \left(\frac{1}{2}\right)^{3+1/r}(14r^3+7r^2+r)}{(r+1)(2r+1)(3r+1)}\left|f''(a)\right|^2\right)^{1/2}\right]. \end{aligned}$$

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