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SOME INEQUALITIES OF OSTROWSKI TYPE FOR DOUBLE INTEGRAL MEAN OF ABSOLUTELY CONTINUOUS FUNCTIONS

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Abstract. In this paper we establish some Ostrowski type inequalities for double integral mean of absolutely continuous functions. An application for special means is given as well.

1. Introduction

In 1938, A. Ostrowski proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x\in[a,b]$.

Theorem 1 (Ostrowski, [12]). *Let* $f:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *such that* $f':(a,b)\to\mathbb{R}$ *is bounded on* (a,b), *i.e.,* $\|f'\|_\infty:=\sup_{t\in(a,b)}|f'(t)|<\infty$. *Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty(b-a), \tag{1.1}$$

for all $x\in[a,b]$ *and the constant* $\frac{1}{4}$ *is the best possible.*

For various Ostrowski type inequalities see the recent papers [1]-[5], [7], [9]-[13], the survey paper online [8] and the references therein.

For the integrable function $f:[a,b]\to\mathbb{C}$, we consider the *double integral mean defined* by

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{t+s}{2}\right)dtds.$$

Motivated by Ostrowski's inequality, it is thus natural to ask what is the distance between the double integral mean and the value $f(x)$, $x\in[a,b]$, in one side and the double integral mean and the integral mean in the other side ?

Some answers for the absolutely continuous functions whose derivatives are essentially bounded or p-Lebesgue integrable are provided below. An application for special means is given as well.

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2. Some Preliminary Results

We recall the function *sign* defined by

$$\operatorname{sgn}(x) := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x < 0. \end{cases}$$

We start with the following simple lemma:

Lemma 1. *We have for any $a < b$, $d \in \mathbb{R}$ and $p > 0$ that*

$$\int_a^b |x-d|^p \, dx = \frac{1}{p+1} \left[\operatorname{sgn}(b-d) |b-d|^{p+1} + \operatorname{sgn}(d-a) |d-a|^{p+1} \right] \tag{2.1}$$

$$= \frac{1}{p+1} \left[(b-d) |b-d|^p + (d-a) |d-a|^p \right].$$

Proof. If $d \leq a$, then

$$\int_a^b |x-d|^p \, dx = \int_a^b (x-d)^p \, dx = \frac{1}{p+1} \left[(b-d)^{p+1} - (a-d)^{p+1} \right]$$

$$= \left[\operatorname{sgn}(b-d) |b-d|^{p+1} + \operatorname{sgn}(d-a) |d-a|^{p+1} \right].$$

If $d \in [a,b]$, then

$$\int_a^b |x-d|^p \, dx = \int_a^d (d-x)^p \, dx + \int_d^b (x-d)^p \, dx$$

$$= \frac{1}{p+1} \left[(d-a)^{p+1} + (b-d)^{p+1} \right]$$

$$= \frac{1}{p+1} \left[\operatorname{sgn}(b-d) |b-d|^{p+1} + \operatorname{sgn}(d-a) |d-a|^{p+1} \right].$$

If $d \geq b$, then

$$\int_a^b |x-d|^p \, dx = \int_a^b (d-x)^p \, dx = \frac{1}{p+1} \left[-(d-b)^{p+1} + (d-a)^{p+1} \right]$$

$$= \frac{1}{p+1} \left[\operatorname{sgn}(b-d) |b-d|^{p+1} + \operatorname{sgn}(d-a) |d-a|^{p+1} \right]$$

and the first equality in (2.1) is thus proved.

The second part follows by the fact that

$$x = \operatorname{sgn}(x) |x| \text{ for } x \in \mathbb{R}.$$

□

Further, we have the following representation as well:

Lemma 2. *We have for any $a < b$, $s \in [a,b]$ and $p > 0$ that*

$$\int_a^b \int_a^b \left| \frac{x+y}{2} - s \right|^p dx dy \tag{2.2}$$

$$= \frac{4}{(p+1)(p+2)} \left[(b-s)^{p+2} - 2 \left| s - \frac{a+b}{2} \right|^{p+2} + (s-a)^{p+2} \right].$$

In particular, we have

$$\int_a^b \int_a^b \left(\frac{x+y}{2} - a\right)^p dxdy = \int_a^b \int_a^b \left(b - \frac{x+y}{2}\right)^p dxdy = \frac{2^{p+1}-1}{2^{p-1}(p+1)(p+2)}(b-a)^{p+2} \tag{2.3}$$

and

$$\int_a^b \int_a^b \left|\frac{x+y}{2} - \frac{a+b}{2}\right|^p dxdy = \frac{1}{2^{p-1}(p+1)(p+2)}(b-a)^{p+2}. \tag{2.4}$$

Proof. We denote

$$I_p(s) := \int_a^b \int_a^b \left|\frac{x+y}{2} - s\right|^p dxdy = \int_a^b \left(\int_a^b \left|\frac{x+y}{2} - s\right|^p dy\right) dx$$

If we make the change of variable $z = \frac{1}{2}(x+y)$, where $y \in [a,b]$, then we have

$$dz = \frac{1}{2}dy,\ z \in \left[\frac{1}{2}(x+a), \frac{1}{2}(x+b)\right]$$

and

$$I_p(s) = \int_a^b \left(\int_a^b \left|\frac{x+y}{2} - s\right|^p dy\right) dx = 2\int_a^b \left(\int_{\frac{1}{2}(x+a)}^{\frac{1}{2}(x+b)} |z-s|^p dz\right) dx. \tag{2.5}$$

Using the representation (2.1) we have

$$\int_{\frac{1}{2}(x+a)}^{\frac{1}{2}(x+b)} |z-s|^p dz = \frac{1}{p+1}\left[\left(\frac{x+b}{2} - s\right)\left|\frac{x+b}{2} - s\right|^p + \left(s - \frac{x+a}{2}\right)\left|s - \frac{x+a}{2}\right|^p\right] \tag{2.6}$$

for $s,\ x \in [a,b]$, and by (2.5) we get

$$I_p(s) = \frac{2}{p+1}\int_a^b \left[\left(\frac{x+b}{2} - s\right)\left|\frac{x+b}{2} - s\right|^p + \left(s - \frac{x+a}{2}\right)\left|s - \frac{x+a}{2}\right|^p\right] dx$$

for $s \in [a,b]$.

We consider

$$I_{1,p}(s) := \int_a^b \left|\frac{x+b}{2} - s\right|^p \left(\frac{x+b}{2} - s\right) dx$$

and

$$I_{2,p}(s) := \int_a^b \left|s - \frac{x+a}{2}\right|^p \left(s - \frac{x+a}{2}\right) dx$$

for $s \in [a,b]$.

a) For $s \in \left[a, \frac{a+b}{2}\right]$, we have

$$\frac{x+b}{2} - s \geq \frac{a+b}{2} - s \geq 0 \text{ for } x \in [a,b],$$

then

$$\begin{aligned} I_{1,p}(s) &= \int_a^b \left(\frac{x+b}{2}-s\right)^p \left(\frac{x+b}{2}-s\right) dx = \int_a^b \left(\frac{x+b}{2}-s\right)^{p+1} dx \\ &= \frac{2}{p+2}\left[(b-s)^{p+2} - \left(\frac{a+b}{2}-s\right)^{p+2}\right] \end{aligned}$$

for $s \in \left[a, \frac{a+b}{2}\right]$.

We have $s - \frac{x+a}{2} = 0$ for $x = 2s - a \in [a,b]$. Then

$$\begin{aligned} I_{2,p}(s) &= \int_a^b \left|s-\frac{x+a}{2}\right|^p \left(s-\frac{x+a}{2}\right) dx \\ &= \int_a^{2s-a} \left(s-\frac{x+a}{2}\right)^p \left(s-\frac{x+a}{2}\right) dx \\ &+ \int_{2s-a}^b \left(\frac{x+a}{2}-s\right)^p \left(s-\frac{x+a}{2}\right) dx \\ &= \int_a^{2s-a} \left(s-\frac{x+a}{2}\right)^{p+1} dx - \int_{2s-a}^b \left(\frac{x+a}{2}-s\right)^{p+1} dx \\ &= 2\frac{(s-a)^{p+2}}{p+2} - 2\frac{\left(\frac{b+a}{2}-s\right)^{p+2}}{p+2} \\ &= \frac{2}{p+2}\left[(s-a)^{p+2} - \left(\frac{b+a}{2}-s\right)^{p+2}\right] \end{aligned}$$

for $s \in \left[a, \frac{a+b}{2}\right]$.

In conclusion, for $s \in \left[a, \frac{a+b}{2}\right]$ we get

$$\begin{aligned} I_p(s) &= \frac{2}{p+1}\left[\frac{2}{p+2}\left[(b-s)^{p+2} - \left(\frac{a+b}{2}-s\right)^{p+2}\right]\right. \\ &\left. + \frac{2}{p+2}\left[(s-a)^{p+2} - \left(\frac{b+a}{2}-s\right)^{p+2}\right]\right] \\ &= \frac{4}{(p+1)(p+2)}\left[(b-s)^{p+2} - 2\left(\frac{a+b}{2}-s\right)^{p+2} + (s-a)^{p+2}\right]. \end{aligned} \tag{2.7}$$

b) Assume that $s \in \left[\frac{a+b}{2}, b\right]$. We have $\frac{x+b}{2} - s = 0$ for $x = 2s - b \in [a,b]$. Then

$$\begin{aligned} I_{1,p}(s) &= \int_a^b \left|\frac{x+b}{2}-s\right|^p \left(\frac{x+b}{2}-s\right) dx \\ &= \int_a^{2s-b} \left(s-\frac{x+b}{2}\right)^p \left(\frac{x+b}{2}-s\right) dx \\ &+ \int_{2s-b}^b \left(\frac{x+b}{2}-s\right)^p \left(\frac{x+b}{2}-s\right) dx \end{aligned}$$

$$\begin{aligned}
&= -\int_a^{2s-b}\left(s-\frac{x+b}{2}\right)^{p+1}dx+\int_{2s-b}^{b}\left(\frac{x+b}{2}-s\right)^{p+1}dx\\
&= -\frac{2}{p+2}\left(s-\frac{a+b}{2}\right)^{p+2}+\frac{2}{p+2}(b-s)^{p+2}\\
&= \frac{2}{p+2}\left[(b-s)^{p+2}-\left(s-\frac{a+b}{2}\right)^{p+2}\right]
\end{aligned}$$

for $s\in\left[\frac{a+b}{2},b\right]$.

If $s\in\left[\frac{a+b}{2},b\right]$, then we have

$$s-\frac{x+a}{2}\geq\frac{a+b}{2}-\frac{x+a}{2}=\frac{b-x}{2}\geq 0$$

for $x\in[a,b]$ and then

$$\begin{aligned}
I_{2,p}(s) &= \int_a^b\left|s-\frac{x+a}{2}\right|^p\left(s-\frac{x+a}{2}\right)dx\\
&= \int_a^b\left(s-\frac{x+a}{2}\right)^{p+1}dx=-2\frac{\left(s-\frac{b+a}{2}\right)^{p+2}}{p+2}+2\frac{(s-a)^{p+2}}{p+2}\\
&= \frac{2}{p+2}\left[(s-a)^{p+2}-\left(s-\frac{b+a}{2}\right)^{p+2}\right]
\end{aligned}$$

for $s\in\left[\frac{a+b}{2},b\right]$.

Therefore,

$$\begin{aligned}
I_p(s) &= \frac{2}{p+1}\left[\frac{2}{p+2}\left[(b-s)^{p+2}-\left(s-\frac{a+b}{2}\right)^{p+2}\right]\right.\\
&\quad\left.+\frac{2}{p+2}\left[(s-a)^{p+2}-\left(s-\frac{b+a}{2}\right)^{p+2}\right]\right]\\
&= \frac{4}{(p+1)(p+2)}\left[(b-s)^{p+2}-2\left(s-\frac{a+b}{2}\right)^{p+2}+(s-a)^{p+2}\right]
\end{aligned}\tag{2.8}$$

for $s\in\left[\frac{a+b}{2},b\right]$.

By utilising (2.7) and (2.8) we get the desired result (2.2). □

Corollary 1. *With the assumptions of Lemma 2 we have*

$$\int_a^b\int_a^b\int_a^b\left|\frac{x+y}{2}-s\right|^p dxdyds=\frac{2^{p+2}-1}{2^{p-1}(p+1)(p+2)(p+3)}(b-a)^{p+3}.\tag{2.9}$$

Proof. We observe that

$$\int_a^b(b-s)^{p+2}ds=\int_a^b(s-a)^{p+2}ds=\frac{(b-a)^{p+3}}{p+3}$$

and

$$\int_a^b\left|s-\frac{a+b}{2}\right|^{p+2}ds=2\int_{\frac{a+b}{2}}^{b}\left(s-\frac{a+b}{2}\right)^{p+2}ds=\frac{1}{2^{p+2}(p+3)}(b-a)^{p+3},$$

therefore

$$\begin{aligned}&\int_a^b \left[(b-s)^{p+2} - 2\left|s-\frac{a+b}{2}\right|^{p+2} + (s-a)^{p+2}\right] ds\\ &= \frac{2(b-a)^{p+3}}{p+3} - \frac{2}{2^{p+2}(p+3)}(b-a)^{p+3} = \frac{2^{p+2}-1}{2^{p+1}(p+3)}(b-a)^{p+3}.\end{aligned}$$

Now, by taking the integral over $s \in [a,b]$ in the identity (2.2) we get (2.9). □

Remark 1. *The case $p = 1$ is of interest in applications and produces the following equalities*

$$\int_a^b \int_a^b \left|\frac{x+y}{2} - s\right| dxdy = \frac{2}{3}\left[(b-s)^3 - 2\left|s-\frac{a+b}{2}\right|^3 + (s-a)^3\right]. \tag{2.10}$$

In particular, we have

$$\int_a^b \int_a^b \left(\frac{x+y}{2} - a\right) dxdy = \int_a^b \int_a^b \left(b - \frac{x+y}{2}\right) dxdy = \frac{1}{2}(b-a)^3, \tag{2.11}$$

$$\int_a^b \int_a^b \left|\frac{x+y}{2} - \frac{a+b}{2}\right| dxdy = \frac{1}{6}(b-a)^3 \tag{2.12}$$

and

$$\int_a^b \int_a^b \int_a^b \left|\frac{x+y}{2} - s\right| dxdyds = \frac{1}{8}(b-a)^4. \tag{2.13}$$

3. Main Results

For any $t,\ s \in [a,b]$, $s \neq t$, one has, see [6]

$$\frac{f(s)-f(t)}{s-t} = \frac{1}{s-t}\int_t^s f'(u)\,du = \int_0^1 f'[(1-\lambda)s + \lambda t]\,d\lambda,$$

showing that

$$f(s) = f(t) + (s-t)\int_0^1 f'[(1-\lambda)s + \lambda t]\,d\lambda \tag{3.1}$$

for any $t,\ s \in [a,b]$.

Now, if we take the double integral mean over t on $[a,b]$ in the identity (3.1) we get the following equality of interest

$$\begin{aligned} f(s) &= \frac{1}{(b-a)^2}\int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dxdy\\ &\quad + \frac{1}{(b-a)^2}\int_a^b \int_a^b \left(s - \frac{x+y}{2}\right)\\ &\quad \times \left(\int_0^1 f'\left[(1-\lambda)s + \lambda\left(\frac{x+y}{2}\right)\right] d\lambda\right) dxdy \end{aligned} \tag{3.2}$$

for any $s \in [a,b]$.

If we take in this equality the integral mean over s on $[a,b]$ we also get

$$\begin{aligned}\tag{3.3} \frac{1}{b-a}\int_a^b f(s)\,ds &= \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy \\ &+\frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b\left(s-\frac{x+y}{2}\right) \\ &\times\left(\int_0^1 f'\left[(1-\lambda)s+\lambda\left(\frac{x+y}{2}\right)\right]d\lambda\right)dxdyds.\end{aligned}$$

If $c<d$ and the function g is essentially bounded on $[c,d]$, namely $g\in L_\infty[c,d]$, then we use the notations

$$\|g\|_{[c,d],\infty}:=\operatorname*{essup}_{t\in[c,d]}|g(t)|<\infty \text{ and } \|g\|_{[d,c],\infty}:=-\operatorname*{essup}_{t\in[c,d]}|g(t)|>-\infty.$$

We have:

Theorem 2. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_\infty[a,b]$, then*

$$\begin{aligned}\tag{3.4} &\left|f(s)-\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy\right| \\ &\le\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|s-\frac{x+y}{2}\right|\left|\|f'\|_{\left[s,\frac{x+y}{2}\right],\infty}\right|dxdy \\ &\le\frac{2}{3(b-a)^2}\left[(b-s)^3-2\left|s-\frac{a+b}{2}\right|^3+(s-a)^3\right]\|f'\|_{[a,b],\infty}\end{aligned}$$

for any $s\in[a,b]$.

We also have

$$\begin{aligned}\tag{3.5} &\left|\frac{1}{b-a}\int_a^b f(s)\,ds-\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy\right| \\ &\le\frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b\left|s-\frac{x+y}{2}\right|\left|\|f'\|_{\left[s,\frac{x+y}{2}\right],\infty}\right|dxdyds \\ &\le\frac{1}{8}\|f'\|_{[a,b],\infty}(b-a).\end{aligned}$$

Proof. From (3.2) we have for any $s \in [a,b]$ that

$$\begin{aligned}
&\left| f(s) - \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dxdy \right| \\
&\leq \frac{1}{(b-a)^2} \int_a^b \int_a^b \left| s - \frac{x+y}{2} \right| \\
&\times \left| \int_0^1 f'\left[(1-\lambda)s + \lambda\left(\frac{x+y}{2}\right)\right] d\lambda \right| dxdy \\
&\leq \frac{1}{(b-a)^2} \int_a^b \int_a^b \left| s - \frac{x+y}{2} \right| \\
&\times \int_0^1 \left| f'\left[(1-\lambda)s + \lambda\left(\frac{x+y}{2}\right)\right] \right| d\lambda dxdy \\
&=: A(s)
\end{aligned} \tag{3.6}$$

Since $f' \in L_\infty[a,b]$, then

$$\begin{aligned}
\int_0^1 \left| f'\left[(1-\lambda)s + \lambda\left(\frac{x+y}{2}\right)\right] \right| d\lambda &\leq \int_0^1 \left| \|f'\|_{\left[s, \frac{x+y}{2}\right],\infty} \right| d\lambda \\
&\leq \left| \|f'\|_{\left[s, \frac{x+y}{2}\right],\infty} \right| \leq \|f'\|_{[a,b],\infty}
\end{aligned} \tag{3.7}$$

for any $s, x, y \in [a,b]$.

Therefore, by (3.6) we get

$$\begin{aligned}
A(s) &\leq \frac{1}{(b-a)^2} \int_a^b \int_a^b \left| s - \frac{x+y}{2} \right| \left| \|f'\|_{\left[s, \frac{x+y}{2}\right],\infty} \right| dxdy \\
&\leq \frac{1}{(b-a)^2} \|f'\|_{[a,b],\infty} \int_a^b \int_a^b \left| s - \frac{x+y}{2} \right| dxdy \\
&= \frac{2}{3(b-a)^2} \|f'\|_{[a,b],\infty} \left[(b-s)^3 - 2\left| s - \frac{a+b}{2} \right|^3 + (s-a)^3 \right],
\end{aligned}$$

which proves the inequality (3.4).

From the equality (3.3), the inequality (3.7) and the representation (2.13) we get

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(s) ds - \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dxdy \right| \\
&\leq \frac{1}{(b-a)^3} \int_a^b \int_a^b \int_a^b \left| s - \frac{x+y}{2} \right| \\
&\times \left(\int_0^1 \left| f'\left[(1-\lambda)s + \lambda\left(\frac{x+y}{2}\right)\right] \right| d\lambda \right) dxdyds
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&\leq \frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b\left|s-\frac{x+y}{2}\right|\left|\|f'\|_{\left[s,\frac{x+y}{2}\right],\infty}\right|dxdyds\\
&\leq \frac{1}{(b-a)^3}\|f'\|_{[a,b],\infty}\int_a^b\int_a^b\int_a^b\left|s-\frac{x+y}{2}\right|dxdyds\\
&= \frac{1}{(b-a)^3}\|f'\|_{[a,b],\infty}\frac{1}{8}(b-a)^4=\frac{1}{8}\|f'\|_{[a,b],\infty}(b-a),
\end{aligned}$$

which proves (3.5). □

Corollary 2. *With the assumptions of Theorem 2 we have*

$$\begin{aligned}
&\left|f(a)-\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy\right| \quad (3.9)\\
&\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b\left(\frac{x+y}{2}-a\right)\|f'\|_{\left[\frac{x+y}{2},a\right],\infty}dxdy\\
&\leq \frac{1}{2}(b-a)\|f'\|_{[a,b],\infty},
\end{aligned}$$

$$\begin{aligned}
&\left|f(b)-\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy\right| \quad (3.10)\\
&\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b\left(b-\frac{x+y}{2}\right)\|f'\|_{\left[b,\frac{x+y}{2}\right],\infty}dxdy\\
&\leq \frac{1}{2}(b-a)\|f'\|_{[a,b],\infty},
\end{aligned}$$

and

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy\right| \quad (3.11)\\
&\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b\left|\frac{a+b}{2}-\frac{x+y}{2}\right|\left|\|f'\|_{\left[\frac{a+b}{2},\frac{x+y}{2}\right],\infty}\right|dxdy\\
&\leq \frac{1}{6}(b-a)\|f'\|_{[a,b],\infty}.
\end{aligned}$$

The constant $\frac{1}{2}$ is best in both inequalities (3.9) and (3.10) while $\frac{1}{6}$ is best possible in (3.11).

The equality is realized in (3.9) for the function $f(x)=x-a$, in the equality (3.10) for $f(x)=b-x$ and in (3.11) for $f(x)=\left|x-\frac{a+b}{2}\right|$, where $x\in[a,b]$.

For an interval $[c,d]$ with $c<d$ we consider the Lebesgue p-norm with $p>1$ for $g\in L_p[c,d]$ the finite quantity

$$\|g\|_{[c,d],p}:=\left(\int_c^d|g(t)|^p\,dt\right)^{1/p}.$$

If $c>d$ then

$$\|g\|_{[c,d],p}:=\left(\int_d^c|g(t)|^p\,dt\right)^{1/p}=\left|\int_c^d|g(t)|^p\,dt\right|^{1/p}.$$

So, for the real numbers c, d we can introduce he notation

$$\left|\|g\|_{[c,d],p}\right| := \left|\int_c^d |g(t)|^p\, dt\right|^{1/p}.$$

We have the following result:

Theorem 3. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. If $f' \in L_p[a,b]$, with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned} &\left| f(s) - \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dxdy \right| \\ &\le \frac{1}{(b-a)^2} \int_a^b \int_a^b \left| s - \frac{x+y}{2} \right|^{1/q} \left| \|f'\|_{[s, \frac{x+y}{2}],p} \right| dxdy \\ &\le \frac{4}{(1/q+1)(1/q+2)(b-a)^2} \\ &\times \left[(b-s)^{1/q+2} - 2\left| s - \frac{a+b}{2} \right|^{1/q+2} + (s-a)^{1/q+2} \right] \|f'\|_{[a,b],p} \end{aligned} \tag{3.12}$$

for any $s \in [a,b]$.

We also have

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(s)\, ds - \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dxdy \right| \\ &\le \frac{1}{(b-a)^3} \int_a^b \int_a^b \int_a^b \left| s - \frac{x+y}{2} \right|^{1/q} \left| \|f'\|_{[s, \frac{x+y}{2}],p} \right| dxdyds \\ &\le \frac{2^{1/q+2} - 1}{2^{1/q-1}(1/q+1)(1/q+2)(1/q+3)} (b-a)^{1/q} \|f'\|_{[a,b],p}. \end{aligned} \tag{3.13}$$

Proof. For $p > 1$ we have the inequality

$$\begin{aligned} &\int_0^1 \left| f'\left[(1-\lambda) s + \lambda \left(\frac{x+y}{2}\right) \right] \right| d\lambda \\ &\le \left(\int_0^1 \left| f'\left[(1-\lambda) s + \lambda \left(\frac{x+y}{2}\right) \right] \right|^p d\lambda \right)^{1/p} \end{aligned} \tag{3.14}$$

for any s, x, $y \in [a,b]$.

Now, suppose that $s \neq \frac{x+y}{2}$. Then

$$\begin{aligned} \int_0^1 \left| f'\left[(1-\lambda) s + \lambda \left(\frac{x+y}{2}\right) \right] \right|^p d\lambda &= \left(s - \frac{x+y}{2} \right)^{-1} \int_{\frac{x+y}{2}}^s |f'(u)|^p \\ &= \left| s - \frac{x+y}{2} \right|^{-1} \left| \int_{\frac{x+y}{2}}^s |f'(u)|^p \right| \end{aligned}$$

namely

$$\left(\int_0^1 \left| f'\left[(1-\lambda) s + \lambda \left(\frac{x+y}{2}\right) \right] \right|^p d\lambda \right)^{1/p} = \left| s - \frac{x+y}{2} \right|^{-1/p} \left| \int_{\frac{x+y}{2}}^s |f'(u)|^p \right|^{1/p}.$$

From the inequality (3.14) we get

$$\begin{aligned}
&\left|s-\frac{x+y}{2}\right|\left|\int_0^1\left|f'\left[(1-\lambda)s+\lambda\left(\frac{x+y}{2}\right)\right]\right|\right| \\
&\leq\left|s-\frac{x+y}{2}\right|^{1-1/p}\left|\int_{\frac{x+y}{2}}^{s}\left|f'(u)\right|^p\right|^{1/p}=\left|s-\frac{x+y}{2}\right|^{1/q}\left|\int_{\frac{x+y}{2}}^{s}\left|f'(u)\right|^p\right|^{1/p}
\end{aligned}$$

for any $s,\,x,\,y\in[a,b]$.

By utilising the notations from the proof of Theorem 2 we have

$$\begin{aligned}
A(s)&\leq\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|s-\frac{x+y}{2}\right|^{1/q}\left|\int_{\frac{x+y}{2}}^{s}\left|f'(u)\right|^p\right|^{1/p}dxdy \\
&\leq\left(\int_a^b\left|f'(u)\right|^p\right)^{1/p}\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|s-\frac{x+y}{2}\right|^{1/q}
\end{aligned}$$

and, since, by Lemma 2

$$\begin{aligned}
&\int_a^b\int_a^b\left|\frac{x+y}{2}-s\right|^{1/q}dxdy \\
&=\frac{4}{(1/q+1)(1/q+2)}\left[(b-s)^{1/q+2}-2\left|s-\frac{a+b}{2}\right|^{1/q+2}+(s-a)^{1/q+2}\right],
\end{aligned}$$

hence the inequality (3.12) is proved.

By (3.8) we also have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(s)\,ds-\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy\right| \\
&\leq\frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b\left|s-\frac{x+y}{2}\right| \\
&\times\left(\int_0^1\left|f'\left[(1-\lambda)s+\lambda\left(\frac{x+y}{2}\right)\right]\right|d\lambda\right)dxdyds \\
&\leq\frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b\left|s-\frac{x+y}{2}\right|^{1/q}\left|\int_{\frac{x+y}{2}}^{s}\left|f'(u)\right|^p\right|^{1/p}dxdyds \\
&\leq\frac{1}{(b-a)^3}\left\|f'\right\|_{[a,b],p}\int_a^b\int_a^b\int_a^b\left|s-\frac{x+y}{2}\right|^{1/q}dxdyds \\
&=\frac{1}{(b-a)^3}\left\|f'\right\|_{[a,b],p}\frac{2^{1/q+2}-1}{2^{1/q-1}(1/q+1)(1/q+2)(1/q+3)}(b-a)^{1/q+3} \\
&=\frac{2^{1/q+2}-1}{2^{1/q-1}(1/q+1)(1/q+2)(1/q+3)}(b-a)^{1/q}\left\|f'\right\|_{[a,b],p},
\end{aligned}$$

which proves (3.13). □

Corollary 3. *With the assumption of Theorem 3 we have*

$$\begin{aligned}
&\left| f(a) - \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy \right| \\
&\le \frac{1}{(b-a)^2}\int_a^b\int_a^b \left(\frac{x+y}{2}-a\right)^{1/q} \|f'\|_{\left[a,\frac{x+y}{2}\right],p}\, dxdy \\
&\le \frac{2^{1/q+1}-1}{2^{1/q-1}(1/q+1)(1/q+2)}(b-a)^{1/q}\|f'\|_{[a,b],p}\,,
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
&\left| f(b) - \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy \right| \\
&\le \frac{1}{(b-a)^2}\int_a^b\int_a^b \left(b-\frac{x+y}{2}\right)^{1/q} \|f'\|_{\left[\frac{x+y}{2},b\right],p}\, dxdy \\
&\le \frac{2^{1/q+1}-1}{2^{1/q-1}(1/q+1)(1/q+2)}(b-a)^{1/q}\|f'\|_{[a,b],p}
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy \right| \\
&\le \frac{1}{(b-a)^2}\int_a^b\int_a^b \left|\frac{a+b}{2}-\frac{x+y}{2}\right|^{1/q} \left|\|f'\|_{\left[\frac{a+b}{2},\frac{x+y}{2}\right],p}\right| dxdy \\
&\le \frac{1}{2^{1/q-1}(1/q+1)(1/q+2)}(b-a)^{1/q}\|f'\|_{[a,b],p}\,.
\end{aligned} \tag{3.17}$$

4. An Application

Consider the power function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(x)=x^r$, $r\neq 0$, and consider for $r\neq -1,-2$ the double integral mean

$$\begin{aligned}
D_r(a,b) &:= \frac{1}{(b-a)^2}\int_a^b\int_a^b \left(\frac{t+s}{2}\right)^r dtds \\
&= \frac{2}{(r+1)(b-a)^2}\int_a^b \left[\left(\frac{t+b}{2}\right)^{r+1} - \left(\frac{t+a}{2}\right)^{r+1}\right]dt \\
&= \frac{4}{(r+1)(r+2)(b-a)^2}\left[b^{r+2} - 2\left(\frac{a+b}{2}\right)^{r+2} + a^{r+2}\right].
\end{aligned}$$

For $r=-1$ we define

$$\begin{aligned}
D_{-1}(a,b) &:= \frac{1}{(b-a)^2}\int_a^b\int_a^b \left(\frac{t+s}{2}\right)^{-1} dtds \\
&= \frac{4}{(b-a)^2}\left[b\ln b - 2\frac{a+b}{2}\ln\left(\frac{a+b}{2}\right) + a\ln a\right]
\end{aligned}$$

and for $r=-2$ we define

$$\begin{aligned} D_{-2}(a,b) &:= \frac{1}{(b-a)^2}\int_a^b\int_a^b \left(\frac{t+s}{2}\right)^{-2} dtds \\ &= -\frac{4}{(b-a)^2}\left[\ln b - 2\ln\left(\frac{a+b}{2}\right)+\ln a\right]. \end{aligned}$$

For $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(x)=x^r$, $r\neq 0$, we have

$$f'(x)=rx^{r-1} \text{ and } f''(x)=r(r-1)x^{r-1},\ x\in(0,\infty).$$

This shows that f' is increasing on $[a,b]$ for $r\in(-\infty,0)\cup[1,\infty)$ and decreasing for $r\in(0,1)$. Therefore

$$\Delta_r(a,b):=\|f'\|_{[a,b],\infty}=\begin{cases} rb^{r-1} & \text{if } r\in(-\infty,0)\cup[1,\infty), \\ \\ ra^{r-1} & \text{if } r\in(0,1). \end{cases}$$

Consider the sharp inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy\right|\leq \frac{1}{6}(b-a)\|f'\|_{[a,b],\infty}.$$

If we write this inequality for $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(x)=x^r$, $r\neq 0$, then we get

$$\left|\left(\frac{a+b}{2}\right)^r - D_r(a,b)\right|\leq\frac{1}{6}(b-a)\Delta_r(a,b). \tag{4.1}$$

We consider the integral mean for $r\neq 0$

$$L_r(a,b):=\frac{1}{b-a}\int_a^b t^r dt=\begin{cases} \frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)} & \text{if } r\neq -1, \\ \\ \frac{\ln b-\ln a}{b-a} & \text{if } r=-1. \end{cases}$$

Consider the inequality between means

$$\left|\frac{1}{b-a}\int_a^b f(s)\,ds-\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy\right|\leq\frac{1}{8}\|f'\|_{[a,b],\infty}(b-a).$$

If we write this inequality for $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(x)=x^r$, $r\neq 0$, then we get

$$|L_r(a,b)-D_r(a,b)|\leq\frac{1}{8}(b-a)\Delta_r(a,b). \tag{4.2}$$

The interested reader may obtain other similar inequalities by using the rest of the general inequalities above or by applying them for other functions such as $f(t)=\ln t$, $\exp t$ or the trigonometric functions.

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