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SOME INEQUALITIES OF TRAPEZOID TYPE FOR DOUBLE INTEGRAL MEAN OF ABSOLUTELY CONTINUOUS FUNCTIONS

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Abstract. In this paper we establish some Ostrowski type inequalities for double integral mean of absolutely continuous functions.

1. Introduction

In 1999, Cerone & Dragomir [10] obtained the following *generalized trapezoid inequality* for absolutely continuous functions $f:[a,b]\to\mathbb{C}$

$$\left|\left(\frac{x-a}{b-a}\right)f(a)+\left(\frac{b-x}{b-a}\right)f(b)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.1}$$

$$\leq\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_{[a,b],\infty} \text{ if } f'\in L_\infty[a,b],\\[2ex] \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/q}(b-a)\|f'\|_{[a,b],p}\\ \text{if } f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right](b-a)\|f'\|_{[a,b],1}\end{cases}$$

where the *Lebesgue norms* $\|\cdot\|_{[a,b],p}$ are defined by

$$\|g\|_{[a,b],p}:=\begin{cases}\operatorname{essup}_{t\in[a,b]}|g(t)| \text{ if } f'\in L_\infty[a,b],\\[2ex] \left(\int_a^b|g(t)|^p\,dt\right)^{1/p} \text{ if } f'\in L_p[a,b],\ p\geq 1.\end{cases}$$

The best inequality one can get from (1.1) is for $x=\frac{a+b}{2}$, namely

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.2}$$

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$$\leq \begin{cases} \frac{1}{4}(b-a)\|f'\|_{[a,b],\infty} & \text{if } f' \in L_\infty[a,b], \\ \\ \frac{1}{2^{1/q}(q+1)^{1/q}}(b-a)^{1/q}\|f'\|_{[a,b],p} & \text{if } f' \in L_p[a,b], \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \frac{1}{2}\|f'\|_{[a,b],1} \end{cases}$$

with $\frac{1}{4}$, $\frac{1}{2^{1/q}}$ and $\frac{1}{2}$ as the best possible constants.

For some related trapezoid type inequalities see [1]-[9], [11]-[15] and [17]-[24].

For the integrable function $f:[a,b]\to\mathbb{C}$, we consider the *double integral mean defined* by

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{t+s}{2}\right)dtds.$$

Motivated by the trapezoid inequality above, it is thus natural to ask what is the distance between the double integral mean and the convex combination of two values, $f(x)$ and $f(y)$ with $x, y \in [a,b]$, in general and $f(a)$ and $f(b)$ in particular ?

Some answers for the absolutely continuous functions whose derivatives are essentially bounded or p-Lebesgue integrable are provided below.

2. Some Preliminary Facts

We need the following preliminary facts:

Lemma 1. *If $f:I\to\mathbb{C}$ is locally absolutely continuous on $\mathring{I}$ then for each distinct $z, x, y \in \mathring{I}$ and $\lambda\in\mathbb{R}\setminus\{0,1\}$ we have*

$$f(z) = (1-\lambda)f(x) + \lambda f(y) + S_\lambda(z,x,y), \tag{2.1}$$

where the remainder $S_\lambda(z,x,y)$ is given by

$$S_\lambda(z,x,y) := (1-\lambda)(z-x)\int_0^1 f'((1-s)x+sz)\,ds - \lambda(y-z)\int_0^1 f'((1-s)z+sy)\,ds. \tag{2.2}$$

Proof. For any $t, s\in I$, $s\neq t$, one has, see [13]

$$\frac{f(s)-f(t)}{s-t} = \frac{1}{s-t}\int_t^s f'(u)\,du = \int_0^1 f'[(1-\lambda)s+\lambda t]\,d\lambda,$$

showing that

$$f(s) = f(t) + (s-t)\int_0^1 f'[(1-\lambda)s+\lambda t]\,d\lambda \tag{2.3}$$

for all $t, s\in I$.

Then by (2.3) we have

$$f(z) = f(x) + (z-x)\int_0^1 f'[(1-\lambda)z+\lambda x]\,d\lambda \tag{2.4}$$

and

$$f(z) = f(y) + (z-y)\int_0^1 f'[(1-\lambda)z+\lambda y]\,d\lambda. \tag{2.5}$$

If we multiply (2.4) by $1-\lambda$ and (2.5) by λ and add the obtained equalities, then we get the equality (2.1), where the remainder is given by (2.2). □

The above identity (2.1) produces the following simple representations of interest. We have for each distinct $z,\ x,\ y \in \mathring{I}$ that

$$f(z)=\frac{1}{y-x}\left[(y-z)f(x)+(z-x)f(y)\right]+L(z,x,y), \tag{2.6}$$

where

$$L(z,x,y):=\frac{(y-z)(z-x)}{y-x}\left[\int_0^1 f'((1-s)x+sz)\,ds-\int_0^1 f'((1-s)z+sy)\,ds\right] \tag{2.7}$$

and

$$f(z)=\frac{1}{y-x}\left[(z-x)f(x)+(y-z)f(y)\right]+P(z,x,y), \tag{2.8}$$

where

$$P(z,x,y):=\frac{1}{y-x}\left[(z-x)^2\int_0^1 f'((1-s)x+sz)\,ds-(y-z)^2\int_0^1 f'((1-s)z+sy)\,ds\right]. \tag{2.9}$$

We also have

$$f((1-\lambda)x+\lambda y)=(1-\lambda)f(x)+\lambda f(y)+S_\lambda(x,y), \tag{2.10}$$

where the remainder $S_\lambda(x,y)$ is given by

$$S_\lambda(x,y):=(1-\lambda)\lambda(y-x)\left[\int_0^1 f'((1-s\lambda)x+s\lambda y)\,ds-\int_0^1 f'((1-s-\lambda+s\lambda)x+(\lambda+s-s\lambda)y)\,ds\right] \tag{2.11}$$

and

$$f((1-\lambda)y+\lambda x)=(1-\lambda)f(x)+\lambda f(y)+P_\lambda(x,y), \tag{2.12}$$

where the remainder $P_\lambda(x,y)$ is given by

$$P_\lambda(x,y):=(y-x)\left[(1-\lambda)^2\int_0^1 f'((1-s+\lambda s)x+(1-\lambda)sy)\,ds-\lambda^2\int_0^1 f'((1-s)\lambda x+(1-\lambda+\lambda s)y)\,ds\right]. \tag{2.13}$$

Moreover, if we take in (2.1) $z=\frac{x+y}{2}$ for each distinct $x,\ y\in\mathring{I}$ and $\lambda\in\mathbb{R}\setminus\{0,1\}$, then we have

$$f\left(\frac{x+y}{2}\right)=(1-\lambda)f(x)+\lambda f(y)+S_\lambda(x,y), \tag{2.14}$$

where the remainder $S_\lambda(x,y)$ is given by

$$S_\lambda(x,y) := \frac{1}{2}(y-x)\left[(1-\lambda)\int_0^1 f'\left((1-s)x+s\frac{x+y}{2}\right)ds \right. \\ \left. -\lambda\int_0^1 f'\left((1-s)\frac{x+y}{2}+sy\right)ds\right]. \tag{2.15}$$

In particular, for $\lambda=\frac{1}{2}$ we have

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}+S(x,y), \tag{2.16}$$

where

$$S(x,y) := \frac{1}{4}(y-x)\left[\int_0^1 f'\left((1-s)x+s\frac{x+y}{2}\right)ds \right. \\ \left. -\int_0^1 f'\left((1-s)\frac{x+y}{2}+sy\right)ds\right]. \tag{2.17}$$

We also need some technical identities that are used to obtain the main results. They have been obtained in [16], however, for the sake of completeness, we provide here their proofs.

We recall the function *sign* defined by

$$\operatorname{sgn}(x) := \begin{cases} -1 & \text{if } x<0, \\ 0 & \text{if } x=0, \\ 1 & \text{if } x<0. \end{cases}$$

The following simple lemma holds:

Lemma 2. *We have for any $a<b$, $d\in\mathbb{R}$ and $p>0$ that*

$$\int_a^b |x-d|^p\,dx = \frac{1}{p+1}\left[\operatorname{sgn}(b-d)|b-d|^{p+1}+\operatorname{sgn}(d-a)|d-a|^{p+1}\right] \\ = \frac{1}{p+1}\left[(b-d)|b-d|^p+(d-a)|d-a|^p\right]. \tag{2.18}$$

Proof. If $d\le a$, then

$$\int_a^b |x-d|^p\,dx = \int_a^b (x-d)^p\,dx = \frac{1}{p+1}\left[(b-d)^{p+1}-(a-d)^{p+1}\right] \\ = \left[\operatorname{sgn}(b-d)|b-d|^{p+1}+\operatorname{sgn}(d-a)|d-a|^{p+1}\right].$$

If $d\in[a,b]$, then

$$\int_a^b |x-d|^p\,dx = \int_a^d (d-x)^p\,dx+\int_d^b (x-d)^p\,dx \\ = \frac{1}{p+1}\left[(d-a)^{p+1}+(b-d)^{p+1}\right] \\ = \frac{1}{p+1}\left[\operatorname{sgn}(b-d)|b-d|^{p+1}+\operatorname{sgn}(d-a)|d-a|^{p+1}\right].$$

If $d \geq b$, then

$$\begin{aligned}\int_a^b |x-d|^p\, dx &= \int_a^b (d-x)^p\, dx = \frac{1}{p+1}\left[-(d-b)^{p+1} + (d-a)^{p+1}\right] \\ &= \frac{1}{p+1}\left[\operatorname{sgn}(b-d)\,|b-d|^{p+1} + \operatorname{sgn}(d-a)\,|d-a|^{p+1}\right]\end{aligned}$$

and the first equality in (2.18) is thus proved.

The second part follows by the fact that

$$x = \operatorname{sgn}(x)\,|x| \text{ for } x \in \mathbb{R}.$$

□

Further, we have the following representation as well:

Lemma 3. *We have for any $a < b$, $s \in [a,b]$ and $p > 0$ that*

$$\begin{aligned}&\int_a^b \int_a^b \left|\frac{x+y}{2} - s\right|^p dxdy \\ &= \frac{4}{(p+1)(p+2)}\left[(b-s)^{p+2} - 2\left|s - \frac{a+b}{2}\right|^{p+2} + (s-a)^{p+2}\right].\end{aligned} \tag{2.19}$$

In particular, we have

$$\begin{aligned}\int_a^b \int_a^b \left(\frac{x+y}{2} - a\right)^p dxdy &= \int_a^b \int_a^b \left(b - \frac{x+y}{2}\right)^p dxdy \\ &= \frac{2^{p+1}-1}{2^{p-1}(p+1)(p+2)}(b-a)^{p+2}\end{aligned} \tag{2.20}$$

and

$$\int_a^b \int_a^b \left|\frac{x+y}{2} - \frac{a+b}{2}\right|^p dxdy = \frac{1}{2^{p-1}(p+1)(p+2)}(b-a)^{p+2}. \tag{2.21}$$

Proof. We denote

$$I_p(s) := \int_a^b \int_a^b \left|\frac{x+y}{2} - s\right|^p dxdy = \int_a^b \left(\int_a^b \left|\frac{x+y}{2} - s\right|^p dy\right) dx.$$

If we make the change of variable $z = \frac{1}{2}(x+y)$, where $y \in [a,b]$, then we have

$$dz = \frac{1}{2}dy,\ z \in \left[\frac{1}{2}(x+a), \frac{1}{2}(x+b)\right]$$

and

$$I_p(s) = \int_a^b \left(\int_a^b \left|\frac{x+y}{2} - s\right|^p dy\right) dx = 2\int_a^b \left(\int_{\frac{1}{2}(x+a)}^{\frac{1}{2}(x+b)} |z-s|^p\, dz\right) dx. \tag{2.22}$$

Using the representation (2.18) we have

$$\begin{aligned}&\int_{\frac{1}{2}(x+a)}^{\frac{1}{2}(x+b)} |z-s|^p\, dz \\ &= \frac{1}{p+1}\left[\left(\frac{x+b}{2} - s\right)\left|\frac{x+b}{2} - s\right|^p + \left(s - \frac{x+a}{2}\right)\left|s - \frac{x+a}{2}\right|^p\right]\end{aligned} \tag{2.23}$$

for $s,\, x\in[a,b]$, and by (2.22) we get

$$I_p(s)=\frac{2}{p+1}\int_a^b\left[\left(\frac{x+b}{2}-s\right)\left|\frac{x+b}{2}-s\right|^p+\left(s-\frac{x+a}{2}\right)\left|s-\frac{x+a}{2}\right|^p\right]dx$$

for $s\in[a,b]$.

We consider

$$I_{1,p}(s):=\int_a^b\left|\frac{x+b}{2}-s\right|^p\left(\frac{x+b}{2}-s\right)dx$$

and

$$I_{2,p}(s):=\int_a^b\left|s-\frac{x+a}{2}\right|^p\left(s-\frac{x+a}{2}\right)dx$$

for $s\in[a,b]$.

a) For $s\in\left[a,\frac{a+b}{2}\right]$, we have

$$\frac{x+b}{2}-s\geq\frac{a+b}{2}-s\geq 0\text{ for }x\in[a,b],$$

then

$$\begin{aligned}I_{1,p}(s)&=\int_a^b\left(\frac{x+b}{2}-s\right)^p\left(\frac{x+b}{2}-s\right)dx=\int_a^b\left(\frac{x+b}{2}-s\right)^{p+1}dx\\&=\frac{2}{p+2}\left[(b-s)^{p+2}-\left(\frac{a+b}{2}-s\right)^{p+2}\right]\end{aligned}$$

for $s\in\left[a,\frac{a+b}{2}\right]$.

We have $s-\frac{x+a}{2}=0$ for $x=2s-a\in[a,b]$. Then

$$\begin{aligned}I_{2,p}(s)&=\int_a^b\left|s-\frac{x+a}{2}\right|^p\left(s-\frac{x+a}{2}\right)dx\\&=\int_a^{2s-a}\left(s-\frac{x+a}{2}\right)^p\left(s-\frac{x+a}{2}\right)dx\\&+\int_{2s-a}^b\left(\frac{x+a}{2}-s\right)^p\left(s-\frac{x+a}{2}\right)dx\\&=\int_a^{2s-a}\left(s-\frac{x+a}{2}\right)^{p+1}dx-\int_{2s-a}^b\left(\frac{x+a}{2}-s\right)^{p+1}dx\\&=2\frac{(s-a)^{p+2}}{p+2}-2\frac{\left(\frac{b+a}{2}-s\right)^{p+2}}{p+2}\\&=\frac{2}{p+2}\left[(s-a)^{p+2}-\left(\frac{b+a}{2}-s\right)^{p+2}\right]\end{aligned}$$

for $s\in\left[a,\frac{a+b}{2}\right]$.

In conclusion, for $s \in \left[a, \frac{a+b}{2}\right]$ we get

$$
\begin{aligned}
I_p(s) &= \frac{2}{p+1}\left[\frac{2}{p+2}\left[(b-s)^{p+2}-\left(\frac{a+b}{2}-s\right)^{p+2}\right]\right. \\
&\left.+\frac{2}{p+2}\left[(s-a)^{p+2}-\left(\frac{b+a}{2}-s\right)^{p+2}\right]\right] \\
&= \frac{4}{(p+1)(p+2)}\left[(b-s)^{p+2}-2\left(\frac{a+b}{2}-s\right)^{p+2}+(s-a)^{p+2}\right].
\end{aligned} \tag{2.24}
$$

b) Assume that $s \in \left[\frac{a+b}{2}, b\right]$. We have $\frac{x+b}{2} - s = 0$ for $x = 2s - b \in [a, b]$. Then

$$
\begin{aligned}
I_{1,p}(s) &= \int_a^b \left|\frac{x+b}{2}-s\right|^p \left(\frac{x+b}{2}-s\right) dx \\
&= \int_a^{2s-b} \left(s-\frac{x+b}{2}\right)^p \left(\frac{x+b}{2}-s\right) dx \\
&+ \int_{2s-b}^b \left(\frac{x+b}{2}-s\right)^p \left(\frac{x+b}{2}-s\right) dx
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_a^{2s-b} \left(s-\frac{x+b}{2}\right)^{p+1} dx + \int_{2s-b}^b \left(\frac{x+b}{2}-s\right)^{p+1} dx \\
&= -\frac{2}{p+2}\left(s-\frac{a+b}{2}\right)^{p+2} + \frac{2}{p+2}(b-s)^{p+2} \\
&= \frac{2}{p+2}\left[(b-s)^{p+2}-\left(s-\frac{a+b}{2}\right)^{p+2}\right]
\end{aligned}
$$

for $s \in \left[\frac{a+b}{2}, b\right]$.

If $s \in \left[\frac{a+b}{2}, b\right]$, then we have

$$
s - \frac{x+a}{2} \geq \frac{a+b}{2} - \frac{x+a}{2} = \frac{b-x}{2} \geq 0
$$

for $x \in [a, b]$ and then

$$
\begin{aligned}
I_{2,p}(s) &= \int_a^b \left|s-\frac{x+a}{2}\right|^p \left(s-\frac{x+a}{2}\right) dx \\
&= \int_a^b \left(s-\frac{x+a}{2}\right)^{p+1} dx = -2\frac{\left(s-\frac{b+a}{2}\right)^{p+2}}{p+2} + 2\frac{(s-a)^{p+2}}{p+2} \\
&= \frac{2}{p+2}\left[(s-a)^{p+2}-\left(s-\frac{b+a}{2}\right)^{p+2}\right]
\end{aligned}
$$

for $s \in \left[\frac{a+b}{2}, b\right]$.

Therefore,

$$I_p(s) = \frac{2}{p+1}\left[\frac{2}{p+2}\left[(b-s)^{p+2} - \left(s - \frac{a+b}{2}\right)^{p+2}\right]\right.$$
$$\left.+\frac{2}{p+2}\left[(s-a)^{p+2} - \left(s - \frac{b+a}{2}\right)^{p+2}\right]\right]$$
$$= \frac{4}{(p+1)(p+2)}\left[(b-s)^{p+2} - 2\left(s - \frac{a+b}{2}\right)^{p+2} + (s-a)^{p+2}\right] \tag{2.25}$$

for $s \in \left[\frac{a+b}{2}, b\right]$.

By utilising (2.24) and (2.25) we get the desired result (2.19). □

Corollary 1. *With the assumptions of Lemma 3 we have*

$$\int_a^b \int_a^b \int_a^b \left|\frac{x+y}{2} - s\right|^p dxdyds = \frac{2^{p+2}-1}{2^{p-1}(p+1)(p+2)(p+3)}(b-a)^{p+3}. \tag{2.26}$$

Proof. We observe that

$$\int_a^b (b-s)^{p+2}\, ds = \int_a^b (s-a)^{p+2}\, ds = \frac{(b-a)^{p+3}}{p+3}$$

and

$$\int_a^b \left|s - \frac{a+b}{2}\right|^{p+2} ds = 2\int_{\frac{a+b}{2}}^b \left(s - \frac{a+b}{2}\right)^{p+2} ds = \frac{1}{2^{p+2}(p+3)}(b-a)^{p+3},$$

therefore

$$\int_a^b \left[(b-s)^{p+2} - 2\left|s - \frac{a+b}{2}\right|^{p+2} + (s-a)^{p+2}\right] ds$$
$$= \frac{2(b-a)^{p+3}}{p+3} - \frac{2}{2^{p+2}(p+3)}(b-a)^{p+3} = \frac{2^{p+2}-1}{2^{p+1}(p+3)}(b-a)^{p+3}.$$

Now, by taking the integral over $s \in [a,b]$ in the identity (2.19) we get (2.26). □

Remark 1. *The case $p = 1$ is of interest in applications and produces the following equalities*

$$\int_a^b \int_a^b \left|\frac{x+y}{2} - s\right| dxdy = \frac{2}{3}\left[(b-s)^3 - 2\left|s - \frac{a+b}{2}\right|^3 + (s-a)^3\right]. \tag{2.27}$$

In particular, we have

$$\int_a^b \int_a^b \left(\frac{x+y}{2} - a\right) dxdy = \int_a^b \int_a^b \left(b - \frac{x+y}{2}\right) dxdy = \frac{1}{2}(b-a)^3, \tag{2.28}$$

$$\int_a^b \int_a^b \left|\frac{x+y}{2} - \frac{a+b}{2}\right| dxdy = \frac{1}{6}(b-a)^3 \tag{2.29}$$

and

$$\int_a^b \int_a^b \int_a^b \left|\frac{x+y}{2} - s\right| dxdyds = \frac{1}{8}(b-a)^4. \tag{2.30}$$

3. Two Points Inequalities for Integral Mean

We have:

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. Then for any $x,\ y\in[a,b]$ and $\lambda\in[0,1]$ we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-(1-\lambda)f(x)-\lambda f(y)\right| \leq (1-\lambda)\frac{1}{b-a}\int_a^b |t-x|\,\nu_{f'}(x,t)\,dt+\lambda\frac{1}{b-a}\int_a^b |y-t|\,\nu_{f'}(y,t)\,dt \tag{3.1}$$

where

$$0\leq \nu_{f'}(z,t):=\int_0^1 |f'((1-s)z+st)|\,ds,\ \ and\ z,\ t\in[a,b].$$

In particular, we have the λ-weighted trapezoid inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-(1-\lambda)f(a)-\lambda f(b)\right| \leq (1-\lambda)\frac{1}{b-a}\int_a^b |t-a|\,\nu_{f'}(a,t)\,dt+\lambda\frac{1}{b-a}\int_a^b |b-t|\,\nu_{f'}(b,t)\,dt \tag{3.2}$$

for any $\lambda\in[0,1]$ and the Ostrowski's type inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f(x)\right|\leq \frac{1}{b-a}\int_a^b |t-x|\,\nu_{f'}(x,t)\,dt \tag{3.3}$$

for any $x\in[a,b]$.

Proof. From Lemma 1 we have

$$f(t)=(1-\lambda)f(x)+\lambda f(y)+(1-\lambda)(t-x)\int_0^1 f'((1-s)x+st)\,ds -\lambda(y-t)\int_0^1 f'((1-s)t+sy)\,ds \tag{3.4}$$

for any $x,\ y\in[a,b]$ and $\lambda\in[0,1]$.

Taking the integral mean in (3.4) we get the following identity of interest

$$\frac{1}{b-a}\int_a^b f(t)\,dt=(1-\lambda)f(x)+\lambda f(y) +(1-\lambda)\frac{1}{b-a}\int_a^b (t-x)\left(\int_0^1 f'((1-s)x+st)\,ds\right)dt -\lambda\frac{1}{b-a}\int_a^b (y-t)\left(\int_0^1 f'((1-s)t+sy)\,ds\right)dt \tag{3.5}$$

for any $x,\ y\in[a,b]$ and $\lambda\in[0,1]$.

This implies that

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-(1-\lambda)f(x)-\lambda f(y)\right| \tag{3.6}$$
$$\leq(1-\lambda)\frac{1}{b-a}\left|\int_a^b(t-x)\left(\int_0^1 f'((1-s)x+st)\,ds\right)dt\right|$$
$$+\lambda\frac{1}{b-a}\left|\int_a^b(y-t)\left(\int_0^1 f'((1-s)t+sy)\,ds\right)dt\right|$$
$$\leq(1-\lambda)\frac{1}{b-a}\int_a^b\left|(t-x)\left(\int_0^1 f'((1-s)x+st)\,ds\right)\right|dt$$
$$+\lambda\frac{1}{b-a}\int_a^b\left|(y-t)\left(\int_0^1 f'((1-s)t+sy)\,ds\right)\right|dt$$
$$\leq(1-\lambda)\frac{1}{b-a}\int_a^b|t-x|\left(\int_0^1|f'((1-s)x+st)|\,ds\right)dt$$
$$+\lambda\frac{1}{b-a}\int_a^b|y-t|\left(\int_0^1|f'((1-s)t+sy)|\,ds\right)dt$$

$$=(1-\lambda)\frac{1}{b-a}\int_a^b|t-x|\left(\int_0^1|f'((1-s)x+st)|\,ds\right)dt$$
$$+\lambda\frac{1}{b-a}\int_a^b|y-t|\left(\int_0^1|f'((1-s)y+st)|\,ds\right)dt$$
$$=(1-\lambda)\frac{1}{b-a}\int_a^b|t-x|\,\nu_{f'}(x,t)\,dt$$
$$+\lambda\frac{1}{b-a}\int_a^b|y-t|\,\nu_{f'}(x,t)\,dt=:B_\lambda(x,y)$$

for any $x,\,y\in[a,b]$ and $\lambda\in[0,1]$.

This proves the desired inequality (3.1). □

Remark 2. *We observe that for any bound $M(\cdot,\cdot)$ for the function $\nu_{f'}(\cdot,\cdot)$ on the square $[a,b]^2$ one can obtain the corresponding two point Ostrowski inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-(1-\lambda)f(x)-\lambda f(y)\right| \tag{3.7}$$
$$\leq(1-\lambda)\frac{1}{b-a}\int_a^b|t-x|\,M(x,t)\,dt+\lambda\frac{1}{b-a}\int_a^b|y-t|\,M(y,t)\,dt$$

for any $x,\,y\in[a,b]$ and $\lambda\in[0,1]$.

In the following, we give only a few examples of such inequalities. The interested reader may obtain many more by choosing other bounds for the function $\nu_{f'}(\cdot,\cdot)$ on the square $[a,b]^2$.

For $g \in L_\infty [a,b]$ and any $x,\, y \in [a,b]$ we use the notation

$$\left| \|g\|_{[x,y],\infty} \right| := \begin{cases} \|g\|_{[x,y],\infty} & \text{if } x < y, \\ \\ \|g\|_{[y,x],\infty} & \text{if } y < x. \end{cases}$$

Corollary 2. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. Assume that $f' \in L_\infty [a,b]$, then for any $x,\, y \in [a,b]$ and $\lambda \in [0,1]$ we have*

$$\begin{aligned} (3.8) \quad & \left| \frac{1}{b-a} \int_a^b f(t)\, dt - (1-\lambda) f(x) - \lambda f(y) \right| \\ & \leq (1-\lambda) \frac{1}{b-a} \int_a^b |t-x| \left| \|f'\|_{[x,t],\infty} \right| dt + \lambda \frac{1}{b-a} \int_a^b |y-t| \left| \|f'\|_{[y,t],\infty} \right| dt \\ & \leq \left[\frac{1}{4} + (1-\lambda) \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 + \lambda \left(\frac{y - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_{[a,b],\infty} (b-a). \end{aligned}$$

In particular, we have the λ-weighted trapezoid inequality

$$\begin{aligned} (3.9) \quad & \left| \frac{1}{b-a} \int_a^b f(t)\, dt - (1-\lambda) f(a) - \lambda f(b) \right| \\ & \leq (1-\lambda) \frac{1}{b-a} \int_a^b (t-a) \|f'\|_{[t,a],\infty}\, dt + \lambda \frac{1}{b-a} \int_a^b (b-t) \|f'\|_{[t,b],\infty}\, dt \\ & \leq \frac{1}{2} \|f'\|_{[a,b],\infty} (b-a) \end{aligned}$$

for any $\lambda \in [0,1]$ and the refinement of Ostrowski inequality

$$\begin{aligned} (3.10) \quad & \left| \frac{1}{b-a} \int_a^b f(t)\, dt - f(x) \right| \\ & \leq \frac{1}{b-a} \int_a^b |t-x| \left| \|f'\|_{[x,t],\infty} \right| dt \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_{[a,b],\infty} (b-a). \end{aligned}$$

Proof. Since $f' \in L_\infty [a,b]$, then by using the notation above, we have

$$|f'((1-s)z + st)| \leq \left| \|f'\|_{[z,t],\infty} \right| \text{ for any } s \in [0,1],$$

which implies that

$$\nu_{f'}(x,t) \leq \left| \|f'\|_{[x,t],\infty} \right| \text{ and } \nu_{f'}(y,t) \leq \left| \|f'\|_{[y,t],\infty} \right|$$

for any $t,\, x,\, y \in [a,b]$.

On using (3.1) we have

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-(1-\lambda)f(x)-\lambda f(y)\right| \tag{3.11}$$
$$\leq(1-\lambda)\frac{1}{b-a}\int_a^b|t-x|\,\nu_{f'}(x,t)\,dt+\lambda\frac{1}{b-a}\int_a^b|y-t|\,\nu_{f'}(y,t)\,dt$$
$$\leq(1-\lambda)\frac{1}{b-a}\int_a^b|t-x|\left|\|f'\|_{[x,t],\infty}\right|dt$$
$$+\lambda\frac{1}{b-a}\int_a^b|y-t|\left|\|f'\|_{[y,t],\infty}\right|dt:=C_\lambda(x,y)$$

for any $x,\ y\in[a,b]$ and $\lambda\in[0,1]$.

Since

$$\left|\|f'\|_{[x,t],\infty}\right|,\ \ \left|\|f'\|_{[y,t],\infty}\right|\leq\|f'\|_{[a,b],\infty}\ \text{ for any } x,\ y\in[a,b],$$

then

$$\int_a^b|t-x|\left|\|f'\|_{[x,t],\infty}\right|dt\leq\|f'\|_{[a,b],\infty}\int_a^b|t-x|\,dt=\frac{(b-x)^2+(x-a)^2}{2}\|f'\|_{[a,b],\infty}$$
$$=\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_{[a,b],\infty}$$

and

$$\int_a^b|y-t|\left|\|f'\|_{[y,t],\infty}\right|dt\leq\left[\frac{1}{4}(b-a)^2+\left(y-\frac{a+b}{2}\right)^2\right]\|f'\|_{[a,b],\infty}$$

for any $x,\ y\in[a,b]$.

Therefore

$$C(x,y)\leq(1-\lambda)\frac{1}{b-a}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_{[a,b],\infty}$$
$$+\lambda\frac{1}{b-a}\left[\frac{1}{4}(b-a)^2+\left(y-\frac{a+b}{2}\right)^2\right]\|f'\|_{[a,b],\infty}$$
$$=(1-\lambda)\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_{[a,b],\infty}(b-a)$$
$$+\lambda\left[\frac{1}{4}+\left(\frac{y-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_{[a,b],\infty}(b-a)$$
$$=\left[\frac{1}{4}+(1-\lambda)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2+\lambda\left(\frac{y-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_{[a,b],\infty}(b-a),$$

for any $x,\ y\in[a,b]$ and $\lambda\in[0,1]$, which proves the last part of (3.8). □

Remark 3. *With the assumptions of Theorem 1, we have the midpoint inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \le \frac{1}{b-a}\int_a^b \left|t-\frac{a+b}{2}\right| \left|\|f'\|_{\left[\frac{a+b}{2},t\right],\infty}\right| dt \le \frac{1}{4}\|f'\|_{[a,b],\infty}(b-a). \tag{3.12}$$

We have:

Theorem 2. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_p[a,b]$, with $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then for any $x,\ y\in[a,b]$ and $\lambda\in[0,1]$ we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-(1-\lambda)f(x)-\lambda f(y)\right| \le (1-\lambda)\frac{1}{b-a}\int_a^b |t-x|\,\nu_{f',p}(x,t)\,dt+\lambda\frac{1}{b-a}\int_a^b |y-t|\,\nu_{f',p}(y,t)\,dt \tag{3.13}$$

where

$$0\le \nu_{f',p}(z,t):=\left(\int_0^1 |f'((1-s)z+st)|^p\,ds\right)^{1/p},\ \text{ and } z,\ t\in[a,b].$$

In particular, we have the λ-weighted trapezoid inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-(1-\lambda)f(a)-\lambda f(b)\right| \le (1-\lambda)\frac{1}{b-a}\int_a^b |t-a|\,\nu_{f',p}(a,t)\,dt+\lambda\frac{1}{b-a}\int_a^b |b-t|\,\nu_{f',p}(b,t)\,dt \tag{3.14}$$

for any $\lambda\in[0,1]$ and the Ostrowski's type inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f(x)\right|\le\frac{1}{b-a}\int_a^b |t-x|\,\nu_{f',p}(x,t)\,dt \tag{3.15}$$

for any $x\in[a,b]$.

Proof. For $p>1$ we have

$$\int_0^1 |f'((1-s)z+st)|\,ds\le\left(\int_0^1 |f'((1-s)z+st)|^p\,ds\right)^{1/p}$$

for any $z,\ t\in[a,b]$.

From (3.6) we obtain

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-(1-\lambda)f(x)-\lambda f(y)\right| \tag{3.16}$$
$$\leq (1-\lambda)\frac{1}{b-a}\int_a^b |t-x|\left(\int_0^1 |f'((1-s)x+st)|\,ds\right)dt$$
$$+\lambda\frac{1}{b-a}\int_a^b |y-t|\left(\int_0^1 |f'((1-s)t+sy)|\,ds\right)dt$$
$$\leq (1-\lambda)\frac{1}{b-a}\int_a^b |t-x|\left(\int_0^1 |f'((1-s)x+st)|^p\,ds\right)^{1/p}dt$$
$$+\lambda\frac{1}{b-a}\int_a^b |y-t|\left(\int_0^1 |f'((1-s)y+st)|^p\,ds\right)^{1/p}dt$$
$$=(1-\lambda)\frac{1}{b-a}\int_a^b |t-x|\,\nu_{f',p}(x,t)\,dt+\lambda\frac{1}{b-a}\int_a^b |y-t|\,\nu_{f',p}(y,t)\,dt$$

for any $x,\ y\in[a,b]$ and $\lambda\in[0,1]$.

This proves the desired result (3.13). □

Corollary 3. *With the assumptions of Theorem 2, we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-(1-\lambda)f(x)-\lambda f(y)\right| \tag{3.17}$$
$$\leq (1-\lambda)\frac{1}{b-a}\int_a^b |t-x|^{1/q}\left|\int_x^t |f'(\tau)|^p\,d\tau\right|^{1/p}dt$$
$$+\lambda\frac{1}{b-a}\int_a^b |t-y|^{1/q}\left|\int_y^t |f'(\tau)|^p\,d\tau\right|^{1/p}dt$$
$$\leq\frac{\|f'\|_{[a,b],p}}{(1/q+1)(b-a)}$$
$$\times\left\{(1-\lambda)\left[(b-x)^{1/q+1}+(x-a)^{1/q+1}\right]+\lambda\left[(b-y)^{1/q+1}+(y-a)^{1/q+1}\right]\right\}$$

for any $x,\ y\in[a,b]$ and $\lambda\in[0,1]$.

In particular, we have the λ-weighted trapezoid inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-(1-\lambda)f(a)-\lambda f(b)\right| \tag{3.18}$$
$$\leq (1-\lambda)\frac{1}{b-a}\int_a^b (t-a)^{1/q}\left(\int_a^t |f'(\tau)|^p\,d\tau\right)^{1/p}dt$$
$$+\lambda\frac{1}{b-a}\int_a^b (b-t)^{1/q}\left(\int_t^b |f'(\tau)|^p\,d\tau\right)^{1/p}dt\leq\frac{\|f'\|_{[a,b],p}}{(1/q+1)}(b-a)$$

for any $\lambda \in [0,1]$ *and the Ostrowski's type inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(x)\right| \le \frac{1}{b-a}\int_a^b |t-x|^{1/q}\left|\int_x^t |f'(\tau)|^p\,d\tau\right|^{1/p} dt \tag{3.19}$$
$$\le \frac{\|f'\|_{[a,b],p}}{(1/q+1)(b-a)}\left[(b-x)^{1/q+1} + (x-a)^{1/q+1}\right]$$

for any $x \in [a,b]$.

Proof. If $z \ne t$, we have

$$\int_0^1 |f'((1-s)z+st)|^p\,ds = \frac{1}{t-z}\int_z^t |f'(\tau)|^p\,d\tau,$$

which implies that

$$\begin{aligned}|t-z|\left(\int_0^1 |f'((1-s)z+st)|^p\,ds\right)^{1/p} &= |t-z|^{1-1/p}\left|\int_z^t |f'(\tau)|^p\,d\tau\right|^{1/p}\\ &= |t-z|^{1/q}\left|\int_z^t |f'(\tau)|^p\,d\tau\right|^{1/p}.\end{aligned}$$

Therefore

$$\int_0^1 |f'((1-s)z+st)|\,ds \le |t-z|^{1/q}\left|\int_z^t |f'(\tau)|^p\,d\tau\right|^{1/p}$$

for any $t,\ z \in [a,b]$.

By utilising (3.16) we get

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - (1-\lambda)f(x) - \lambda f(y)\right|\\ &\le (1-\lambda)\frac{1}{b-a}\int_a^b |t-x|\left(\int_0^1 |f'((1-s)x+st)|\,ds\right)dt\\ &\quad + \lambda\frac{1}{b-a}\int_a^b |y-t|\left(\int_0^1 |f'((1-s)t+sy)|\,ds\right)dt\\ &\le (1-\lambda)\frac{1}{b-a}\int_a^b |t-x|^{1/q}\left|\int_x^t |f'(\tau)|^p\,d\tau\right|^{1/p} dt\\ &\quad + \lambda\frac{1}{b-a}\int_a^b |t-y|^{1/q}\left|\int_y^t |f'(\tau)|^p\,d\tau\right|^{1/p} dt =: D_\lambda(x,y)\end{aligned}$$

for any $x,\ y \in [a,b]$ and $\lambda \in [0,1]$.

This proves the first inequality in (3.17).

Since

$$\left|\int_x^t |f'(\tau)|^p\,d\tau\right|^{1/p},\ \left|\int_y^t |f'(\tau)|^p\,d\tau\right|^{1/p} \le \|f'\|_{[a,b],p}$$

for any t, $x \in [a, b]$ and

$$\int_a^b |t-x|^{1/q}\, dt = \frac{(b-x)^{1/q+1} + (x-a)^{1/q+1}}{1/q+1},$$
$$\int_a^b |t-y|^{1/q}\, dt = \frac{(b-y)^{1/q+1} + (y-a)^{1/q+1}}{1/q+1},$$

then

$$\begin{aligned} D(x,y) &\le (1-\lambda)\|f'\|_{[a,b],p}\left[\frac{(b-x)^{1/q+1}+(x-a)^{1/q+1}}{(1/q+1)(b-a)}\right] \\ &\quad + \lambda\|f'\|_{[a,b],p}\left[\frac{(b-y)^{1/q+1}+(y-a)^{1/q+1}}{(1/q+1)(b-a)}\right] \\ &= \frac{\|f'\|_{[a,b],p}}{(1/q+1)(b-a)}\Big[(1-\lambda)\left[(b-x)^{1/q+1}+(x-a)^{1/q+1}\right] \\ &\qquad +\lambda\left[(b-y)^{1/q+1}+(y-a)^{1/q+1}\right]\Big] \end{aligned}$$

for any x, $y \in [a, b]$ and $\lambda \in [0, 1]$.

This proves the last part of (3.17). □

Remark 4. *With the assumptions of Theorem 2, we have the midpoint inequality*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \\ &\le \frac{1}{b-a}\int_a^b \left|t-\frac{a+b}{2}\right|^{1/q}\left|\int_{\frac{a+b}{2}}^t |f'(\tau)|^p\, d\tau\right|^{1/p} dt \\ &\le \frac{\|f'\|_{[a,b],p}}{2^{1/q}(1/q+1)}(b-a)^{1/q}. \end{aligned} \tag{3.20}$$

4. Two Points Inequalities for Double Integral Mean

We also have:

Theorem 3. *Let $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on $[a, b]$. Then for any x, $y \in [a, b]$ and $\lambda \in [0, 1]$ we have*

$$\begin{aligned} &\left|\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{u+v}{2}\right) dudv - (1-\lambda) f(x) - \lambda f(y)\right| \\ &\le (1-\lambda)\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|\frac{u+v}{2}-x\right|\nu_{f'}\left(x,\frac{u+v}{2}\right) dudv \\ &\quad + \lambda\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|y-\frac{u+v}{2}\right|\nu_{f'}\left(y,\frac{u+v}{2}\right) dudv \end{aligned} \tag{4.1}$$

where, as above,

$$0 \le \nu_{f'}(z,t) := \int_0^1 |f'((1-s)z+st)|\, ds, \text{ and } z,\ t \in [a,b].$$

In particular, we have the λ-weighted trapezoid inequality

$$\begin{aligned}(4.2)\quad & \left|\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{u+v}{2}\right)dudv-(1-\lambda)f(a)-\lambda f(b)\right| \\ & \le (1-\lambda)\frac{1}{(b-a)^2}\int_a^b\int_a^b\left(\frac{u+v}{2}-a\right)\nu_{f'}\left(a,\frac{u+v}{2}\right)dudv \\ & \quad +\lambda\frac{1}{(b-a)^2}\int_a^b\int_a^b\left(b-\frac{u+v}{2}\right)\nu_{f'}\left(b,\frac{u+v}{2}\right)dudv\end{aligned}$$

for any $\lambda\in[0,1]$ and the Ostrowski's type inequality

$$\begin{aligned}(4.3)\quad & \left|\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{u+v}{2}\right)dudv-f(x)\right| \\ & \le\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|\frac{u+v}{2}-x\right|\nu_{f'}\left(x,\frac{u+v}{2}\right)dudv\end{aligned}$$

for any $x\in[a,b]$.

Proof. From Lemma 1 we have

$$\begin{aligned}(4.4)\quad f\left(\frac{u+v}{2}\right) & =(1-\lambda)f(x)+\lambda f(y) \\ & +(1-\lambda)\left(\frac{u+v}{2}-x\right)\int_0^1 f'\left((1-s)x+s\frac{u+v}{2}\right)ds \\ & -\lambda\left(y-\frac{u+v}{2}\right)\int_0^1 f'\left((1-s)\frac{u+v}{2}+sy\right)ds\end{aligned}$$

for any $x,\ y,\ u,\ v\in[a,b]$ and $\lambda\in[0,1]$.

Taking the double integral mean in (4.4) we have

$$\begin{aligned}&\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{u+v}{2}\right)dudv=(1-\lambda)f(x)+\lambda f(y) \\ & +(1-\lambda)\frac{1}{(b-a)^2}\int_a^b\int_a^b\left(\frac{u+v}{2}-x\right)\left(\int_0^1 f'\left((1-s)x+s\frac{u+v}{2}\right)ds\right)dudv \\ & -\lambda\frac{1}{(b-a)^2}\int_a^b\int_a^b\left(y-\frac{u+v}{2}\right)\left(\int_0^1 f'\left((1-s)\frac{u+v}{2}+sy\right)ds\right)dudv\end{aligned}$$

for any $x,\ y\in[a,b]$ and $\lambda\in[0,1]$.

This implies that

$$
\begin{aligned}
(4.5)\quad & \left|\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{u+v}{2}\right)dudv-(1-\lambda)f(x)-\lambda f(y)\right| \\
&\leq (1-\lambda)\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|\frac{u+v}{2}-x\right|\left|\int_0^1 f'\left((1-s)x+s\frac{u+v}{2}\right)ds\right|dudv \\
&\quad +\lambda\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|y-\frac{u+v}{2}\right|\left|\int_0^1 f'\left((1-s)\frac{u+v}{2}+sy\right)ds\right|dudv \\
&\leq (1-\lambda)\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|\frac{u+v}{2}-x\right|\int_0^1\left|f'\left((1-s)x+s\frac{u+v}{2}\right)\right|dsdudv \\
&\quad +\lambda\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|y-\frac{u+v}{2}\right|\int_0^1\left|f'\left((1-s)\frac{u+v}{2}+sy\right)\right|dsdudv \\
&= (1-\lambda)\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|\frac{u+v}{2}-x\right|\nu_{f'}\left(x,\frac{u+v}{2}\right)dsdudv \\
&\quad +\lambda\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|y-\frac{u+v}{2}\right|\nu_{f'}\left(y,\frac{u+v}{2}\right)dudv
\end{aligned}
$$

for any $x,\, y\in[a,b]$ and $\lambda\in[0,1]$. □

Remark 5. *We observe that for any bound $M(\cdot,\cdot)$ for the function $\nu_{f'}(\cdot,\cdot)$ on the square $[a,b]^2$ one can obtain the corresponding two point Ostrowski inequality*

$$
\begin{aligned}
(4.6)\quad & \left|\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{u+v}{2}\right)dudv-(1-\lambda)f(x)-\lambda f(y)\right| \\
&\leq (1-\lambda)\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|\frac{u+v}{2}-x\right|M\left(x,\frac{u+v}{2}\right)dudv \\
&\quad +\lambda\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|y-\frac{u+v}{2}\right|M\left(y,\frac{u+v}{2}\right)dudv
\end{aligned}
$$

for any $x,\, y\in[a,b]$ and $\lambda\in[0,1]$.

In the following, we give only a few examples of such inequalities. The interested reader may obtain many more by choosing other bounds for the function $\nu_{f'}(\cdot,\cdot)$ on the square $[a,b]^2$.

Corollary 4. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. Assume that $f' \in L_\infty [a,b]$, then for any x, $y \in [a,b]$ and $\lambda \in [0,1]$ we have*

$$
\begin{aligned}
(4.7)\quad & \left| \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{u+v}{2}\right) dudv - (1-\lambda) f(x) - \lambda f(y) \right| \\
&\leq (1-\lambda) \frac{1}{(b-a)^2} \int_a^b \int_a^b \left| \frac{u+v}{2} - x \right| \left| \|f'\|_{\left[x, \frac{u+v}{2}\right], \infty} \right| dudv \\
&\quad + \lambda \frac{1}{(b-a)^2} \int_a^b \int_a^b \left| y - \frac{u+v}{2} \right| \left| \|f'\|_{\left[y, \frac{u+v}{2}\right], \infty} \right| dudv \\
&\leq \frac{2}{3(b-a)^2} (1-\lambda) \left[(b-x)^3 - 2\left| x - \frac{a+b}{2} \right|^3 + (x-a)^3 \right] \|f'\|_{[a,b],\infty} \\
&\quad + \frac{2}{3(b-a)^2} \lambda \left[(b-y)^3 - 2\left| y - \frac{a+b}{2} \right|^3 + (y-a)^3 \right] \|f'\|_{[a,b],\infty}.
\end{aligned}
$$

In particular, we have the λ-weighted trapezoid inequality

$$
\begin{aligned}
(4.8)\quad & \left| \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{u+v}{2}\right) dudv - (1-\lambda) f(a) - \lambda f(b) \right| \\
&\leq (1-\lambda) \frac{1}{(b-a)^2} \int_a^b \int_a^b \left(\frac{u+v}{2} - a \right) \|f'\|_{\left[a, \frac{u+v}{2}\right], \infty} dudv \\
&+ \lambda \frac{1}{(b-a)^2} \int_a^b \int_a^b \left(b - \frac{u+v}{2} \right) \|f'\|_{\left[\frac{u+v}{2}, b\right], \infty} dudv \leq \frac{1}{2} \|f'\|_{[a,b],\infty} (b-a)
\end{aligned}
$$

for any $\lambda \in [0,1]$ and the refinement of Ostrowski inequality

$$
\begin{aligned}
(4.9)\quad & \left| \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{u+v}{2}\right) dudv - f(x) \right| \\
&\leq \frac{1}{(b-a)^2} \int_a^b \int_a^b \left| \frac{u+v}{2} - x \right| \left| \|f'\|_{\left[x, \frac{u+v}{2}\right], \infty} \right| dudv \\
&\leq \frac{2}{3(b-a)^2} \left[(b-x)^3 - 2\left| x - \frac{a+b}{2} \right|^3 + (x-a)^3 \right] \|f'\|_{[a,b],\infty}
\end{aligned}
$$

for any $x \in [a,b]$.

Remark 6. *With the assumption of Theorem 3, we have the midpoint inequality*

$$
\begin{aligned}
(4.10)\quad & \left| \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{u+v}{2}\right) dudv - f\left(\frac{a+b}{2}\right) \right| \\
&\leq \frac{1}{(b-a)^2} \int_a^b \int_a^b \left| \frac{u+v}{2} - \frac{a+b}{2} \right| \left| \|f'\|_{\left[\frac{a+b}{2}, \frac{u+v}{2}\right], \infty} \right| dudv \\
&\leq \frac{1}{6} (b-a)^3 \|f'\|_{[a,b],\infty}.
\end{aligned}
$$

Further, we have:

Theorem 4. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_p[a,b]$, with $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then for any x, $y\in[a,b]$ and $\lambda\in[0,1]$ we have*

$$\left|\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{u+v}{2}\right)dudv-(1-\lambda)f(x)-\lambda f(y)\right| \tag{4.11}$$
$$\leq(1-\lambda)\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|\frac{u+v}{2}-x\right|\nu_{f',p}\left(x,\frac{u+v}{2}\right)dudv$$
$$+\lambda\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|y-\frac{u+v}{2}\right|\nu_{f',p}\left(y,\frac{u+v}{2}\right)dudv$$

where, as above,

$$0\leq\nu_{f',p}(z,t):=\left(\int_0^1|f'((1-s)z+st)|^p ds\right)^{1/p},\ \text{and } z,\ t\in[a,b].$$

In particular, we have the λ-weighted trapezoid inequality

$$\left|\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{u+v}{2}\right)dudv-(1-\lambda)f(a)-\lambda f(b)\right| \tag{4.12}$$
$$\leq(1-\lambda)\frac{1}{(b-a)^2}\int_a^b\int_a^b\left(\frac{u+v}{2}-a\right)\nu_{f',p}\left(a,\frac{u+v}{2}\right)dudv$$
$$+\lambda\frac{1}{(b-a)^2}\int_a^b\int_a^b\left(b-\frac{u+v}{2}\right)\nu_{f',p}\left(b,\frac{u+v}{2}\right)dudv$$

for any $\lambda\in[0,1]$ and the Ostrowski's type inequality

$$\left|\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{u+v}{2}\right)dudv-f(x)\right| \tag{4.13}$$
$$\leq\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|\frac{u+v}{2}-x\right|\nu_{f',p}\left(x,\frac{u+v}{2}\right)dudv$$

for any $x\in[a,b]$.

Proof. For $p>1$ we have the inequality

$$\int_0^1\left|f'\left[(1-s)z+s\left(\frac{u+v}{2}\right)\right]\right|ds\leq\left(\int_0^1\left|f'\left[(1-s)z+s\left(\frac{u+v}{2}\right)\right]\right|^p ds\right)^{1/p}$$

for any z, x, $y\in[a,b]$.

From (4.5) we have

$$\begin{aligned}
&\left|\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{u+v}{2}\right)dudv-(1-\lambda)f(x)-\lambda f(y)\right| \\
&\quad\leq (1-\lambda)\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|\frac{u+v}{2}-x\right|\int_0^1\left|f'\left((1-s)x+s\frac{u+v}{2}\right)\right|dsdudv \\
&\qquad+\lambda\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|y-\frac{u+v}{2}\right|\int_0^1\left|f'\left((1-s)\frac{u+v}{2}+sy\right)\right|dsdudv \\
&\leq (1-\lambda)\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|\frac{u+v}{2}-x\right|\left(\int_0^1\left|f'\left[(1-s)x+s\left(\frac{u+v}{2}\right)\right]\right|^p ds\right)^{1/p}dsdudv \\
&+\lambda\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|y-\frac{u+v}{2}\right|\left(\int_0^1\left|f'\left[(1-s)y+s\left(\frac{u+v}{2}\right)\right]\right|^p ds\right)^{1/p}dsdudv \\
&\qquad\qquad =: E_\lambda(x,y)
\end{aligned}$$

which proves (4.11). □

Corollary 5. *With the assumptions of Theorem 4, we have*

$$\begin{aligned}
&\left|\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{u+v}{2}\right)dudv-(1-\lambda)f(x)-\lambda f(y)\right| \\
&\leq (1-\lambda)\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|x-\frac{u+v}{2}\right|^{1/q}\left|\int_{\frac{u+v}{2}}^x|f'(\tau)|^p d\tau\right|^{1/p}dudv \\
&\quad+\lambda\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|y-\frac{u+v}{2}\right|^{1/q}\left|\int_{\frac{u+v}{2}}^y|f'(\tau)|^p d\tau\right|^{1/p}dudv \\
&\qquad\leq \frac{4}{(1/q+1)(1/q+2)(b-a)^2}\|f'\|_{[a,b],p} \\
&\quad\times\left\{(1-\lambda)\left[(b-x)^{1/q+2}-2\left|x-\frac{a+b}{2}\right|^{1/q+2}+(x-a)^{1/q+2}\right]\right. \\
&\qquad\left.+\lambda\left[(b-y)^{1/q+2}-2\left|y-\frac{a+b}{2}\right|^{1/q+2}+(y-a)^{1/q+2}\right]\right\}
\end{aligned} \tag{4.14}$$

for any $x,\ y\in[a,b]$ *and* $\lambda\in[0,1]$.

In particular, we have the λ-weighted trapezoid inequality

$$
\begin{aligned}
(4.15)\quad & \left|\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{u+v}{2}\right)dudv-(1-\lambda)f(a)-\lambda f(b)\right| \\
&\leq (1-\lambda)\frac{1}{(b-a)^2}\int_a^b\int_a^b\left(\frac{u+v}{2}-a\right)^{1/q}\left|\int_{\frac{u+v}{2}}^a|f'(\tau)|^p d\tau\right|^{1/p}dudv \\
&\quad+\lambda\frac{1}{(b-a)^2}\int_a^b\int_a^b\left(b-\frac{u+v}{2}\right)^{1/q}\left|\int_{\frac{u+v}{2}}^y|f'(\tau)|^p d\tau\right|^{1/p}dudv \\
&\leq \frac{2^{1/q+1}-1}{2^{1/q-1}(1/q+1)(1/q+2)}\|f'\|_{[a,b],p}(b-a)^{1/q}
\end{aligned}
$$

for any $\lambda\in[0,1]$ and the Ostrowski's type inequality

$$
\begin{aligned}
(4.16)\quad & \left|\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{u+v}{2}\right)dudv-f(x)\right| \\
&\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b\left|x-\frac{u+v}{2}\right|^{1/q}\left|\int_{\frac{u+v}{2}}^x|f'(\tau)|^p d\tau\right|^{1/p}dudv \\
&\leq \frac{4}{(1/q+1)(1/q+2)(b-a)^2} \\
&\quad\times\left[(b-x)^{1/q+2}-2\left|x-\frac{a+b}{2}\right|^{1/q+2}+(x-a)^{1/q+2}\right]\|f'\|_{[a,b],p}
\end{aligned}
$$

for any $x\in[a,b]$.

Proof. For $p>1$ we have

$$
\int_0^1\left|f'\left[(1-s)z+s\left(\frac{u+v}{2}\right)\right]\right|ds\leq\left(\int_0^1\left|f'\left[(1-s)z+s\left(\frac{u+v}{2}\right)\right]\right|^p ds\right)^{1/p}
$$

for any $z,\ u,\ v\in[a,b]$.

Now, suppose that $z\neq\frac{u+v}{2}$. Then

$$
\begin{aligned}
\int_0^1\left|f'\left[(1-s)z+s\left(\frac{u+v}{2}\right)\right]\right|^p ds &= \left(z-\frac{u+v}{2}\right)^{-1}\int_{\frac{u+v}{2}}^z|f'(\tau)|^p d\tau \\
&= \left|z-\frac{u+v}{2}\right|^{-1}\left|\int_{\frac{u+v}{2}}^z|f'(\tau)|^p d\tau\right|
\end{aligned}
$$

namely

$$
\left(\int_0^1\left|f'\left[(1-s)z+s\left(\frac{u+v}{2}\right)\right]\right|^p ds\right)^{1/p}=\left|z-\frac{u+v}{2}\right|^{-1/p}\left|\int_{\frac{u+v}{2}}^z|f'(\tau)|^p d\tau\right|^{1/p}.
$$

This implies that

$$\left|z-\frac{u+v}{2}\right|\left(\int_0^1\left|f'\left[(1-s)\,z+s\left(\frac{u+v}{2}\right)\right]\right|^p ds\right)^{1/p}$$
$$=\left|z-\frac{u+v}{2}\right|^{1-1/p}\left|\int_{\frac{u+v}{2}}^{z}\left|f'(\tau)\right|^p d\tau\right|^{1/p}=\left|z-\frac{u+v}{2}\right|^{1/q}\left|\int_{\frac{u+v}{2}}^{z}\left|f'(\tau)\right|^p d\tau\right|^{1/p}$$

for any $z,\,u,\,v\in[a,b]$.

Using the notation from the proof of Theorem 4, we get

$$E_\lambda(x,y)\le(1-\lambda)\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|x-\frac{u+v}{2}\right|^{1/q}\left|\int_{\frac{u+v}{2}}^{x}\left|f'(\tau)\right|^p d\tau\right|^{1/p}dudv$$
$$+\lambda\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|y-\frac{u+v}{2}\right|^{1/q}\left|\int_{\frac{u+v}{2}}^{y}\left|f'(\tau)\right|^p d\tau\right|^{1/p}dudv,$$

which proves the first inequality in (4.14).

Since

$$\left|\int_{\frac{u+v}{2}}^{x}\left|f'(\tau)\right|^p d\tau\right|^{1/p},\quad\left|\int_{\frac{u+v}{2}}^{y}\left|f'(\tau)\right|^p d\tau\right|^{1/p}\le\|f'\|_{[a,b],p}$$

for any $x,\,y,\,u,\,v\in[a,b]$, then

$$\int_a^b\int_a^b\left|x-\frac{u+v}{2}\right|^{1/q}\left|\int_{\frac{u+v}{2}}^{x}\left|f'(\tau)\right|^p d\tau\right|^{1/p}dudv$$
$$\le\|f'\|_{[a,b],p}\int_a^b\int_a^b\left|x-\frac{u+v}{2}\right|^{1/q}dudv$$
$$=\frac{4}{(1/q+1)(1/q+2)}$$
$$\times\|f'\|_{[a,b],p}\left[(b-x)^{1/q+2}-2\left|x-\frac{a+b}{2}\right|^{1/q+2}+(x-a)^{1/q+2}\right]$$

and

$$\int_a^b\int_a^b\left|y-\frac{u+v}{2}\right|^{1/q}\left|\int_{\frac{u+v}{2}}^{y}\left|f'(\tau)\right|^p d\tau\right|^{1/p}dudv$$
$$\le\frac{4}{(1/q+1)(1/q+2)}\|f'\|_{[a,b],p}$$
$$\times\left[(b-y)^{1/q+2}-2\left|y-\frac{a+b}{2}\right|^{1/q+2}+(y-a)^{1/q+2}\right]$$

for any $x,\,y\in[a,b]$.

This proves the last part of (4.14). □

Remark 7. *With the assumption of Theorem 4, we have*

$$\left|\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{u+v}{2}\right)dudv - f\left(\frac{a+b}{2}\right)\right| \tag{4.17}$$
$$\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b \left|\frac{a+b}{2}-\frac{u+v}{2}\right|^{1/q}\left|\int_{\frac{u+v}{2}}^{\frac{a+b}{2}}|f'(\tau)|^p d\tau\right|^{1/p} dudv$$
$$\leq \frac{1}{2^{1/q-1}(1/q+1)(1/q+2)}(b-a)^{1/q}\|f'\|_{[a,b],p}.$$

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