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FURTHER INEQUALITIES FOR TRIGONOMETRICALLY ρ-CONVEX FUNCTIONS AND APPLICATIONS

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Abstract. In this paper we establish some new inequalities for trigonometrically ρ-convex functions. Applications for discrete inequalities of Jensen's type are also provided.

1. Introduction

Let I be a finite or infinite open interval of real numbers and $\rho > 0$.

In the following we present the basic definitions and results concerning the class of trigonometrically ρ-convex function, see for example [13], [14] and [3], [5], [6], [12], [15], [17] and [18].

Following [1], we say that a function $f : I \to \mathbb{R}$ is *trigonometrically ρ-convex* on I if for any closed subinterval $[a, b]$ of I with $0 < b - a < \frac{\pi}{\rho}$ we have

$$f(x) \leq \frac{\sin[\rho(b-x)]}{\sin[\rho(b-a)]} f(a) + \frac{\sin[\rho(x-a)]}{\sin[\rho(b-a)]} f(b) \tag{1.1}$$

for all $x \in [a, b]$.

If the inequality (1.1) holds with " $\geq$ ", then the function will be called *trigonometrically ρ-concave* on I.

Geometrically speaking, this means that the graph of f on $[a, b]$ lies nowhere above the ρ-trigonometric function determined by the equation

$$H(x) = H(x; a, b, f) := A\cos(\rho x) + B\sin(\rho x)$$

where A and B are chosen such that $H(a) = f(a)$ and $H(b) = f(b)$.

If we take $x = (1-t)a + tb \in [a, b]$, $t \in [0, 1]$, then the condition (1.1) becomes

$$f((1-t)a + tb) \leq \frac{\sin[\rho(1-t)(b-a)]}{\sin[\rho(b-a)]} f(a) + \frac{\sin[\rho t(b-a)]}{\sin[\rho(b-a)]} f(b) \tag{1.2}$$

for any $t \in [0, 1]$.

We have the following properties of trigonometrically ρ-convex on I, [1].

(i) A trigonometrically ρ-convex function $f : I \to \mathbb{R}$ has finite right and left derivatives $f'_+(x)$ and $f'_-(x)$ at every point $x \in I$ and $f'_-(x) \leq f'_+(x)$. The function f is differentiable on I with the exception of an at most countable set.

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(ii) A necessary and sufficient condition for the function $f : I \to \mathbb{R}$ to be trigonometrically ρ-convex function on I is that it satisfies the *gradient inequality*

$$f(y) \geq f(x)\cos[\rho(y-x)] + K_{x,f}\sin[\rho(y-x)] \tag{1.3}$$

for any x, $y \in I$ where $K_{x,f} \in \left[f'_{-}(x), f'_{+}(x)\right]$. If f is differentiable at the point x then $K_{x,f} = f'(x)$.

(iii) A necessary and sufficient condition for the function f to be a trigonometrically ρ-convex in I, is that the function

$$\varphi(x) = f'(x) + \rho^2 \int_a^x f(t)\,dt$$

is nondecreasing on I, where $a \in I$.

(iv) Let $f : I \to \mathbb{R}$ be a two times continuously differentiable function on I. Then f is trigonometrically ρ-convex on I if and only if for all $x \in I$ we have

$$f''(x) + \rho^2 f(x) \geq 0. \tag{1.4}$$

For other properties of trigonometrically ρ-convex functions, see [1].

As general examples of trigonometrically ρ-convex functions we can give the indicator function

$$h_F(\theta) := \limsup_{r\to\infty} \frac{\log\left|F\left(re^{i\theta}\right)\right|}{r^\rho},\ \theta \in (\alpha,\beta),$$

where F is an entire function of order $\rho \in (0,\infty)$.

If $0 < \beta - \alpha < \frac{\pi}{\rho}$, then, it was shown in 1908 by Phragmén and Lindelöf, see [13], that h_F is trigonometrically ρ-convex on (α,β).

Using the condition (1.4) one can also observe that any nonnegative twice differentiable and convex function on I is also trigonometrically ρ-convex on I for any $\rho > 0$.

There exists also concave functions on an interval that are trigonometrically ρ-convex on that interval for some $\rho > 0$.

Consider for example $f(x) = \cos x$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then

$$f''(x) + \rho^2 f(x) = -\cos x + \rho^2\cos x = \left(\rho^2 - 1\right)\cos x,$$

which shows that it is trigonometrically ρ-convex on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for all $\rho > 1$ and trigonometrically ρ-concave for $\rho \in (0,1)$.

Consider the function $f : (0,\infty) \to (0,\infty)$, $f(x) = x^p$ with $p \in \mathbb{R}\backslash\{0\}$. If $p \in (-\infty,0)\cup[1,\infty)$ the function is convex and therefore trigonometrically ρ-convex for any $\rho > 0$. If $p \in (0,1)$ then the function is concave and

$$f''(x) + \rho^2 f(x) = \rho^2 x^p - p(1-p)x^{p-2} = \rho^2 x^{p-2}\left(x^2 - \frac{p(1-p)}{\rho^2}\right),\ x > 0.$$

This shows that for $p \in (0,1)$ and $\rho > 0$ the function $f(x) = x^p$ is trigonometrically ρ-convex on $\left(\frac{1}{\rho}\sqrt{p(1-p)}, \infty\right)$ and trigonometrically ρ-concave on $\left(0, \frac{1}{\rho}\sqrt{p(1-p)}\right)$.

Consider the concave function $f : (0,\infty) \to \mathbb{R}$, $f(x) = \ln x$. We observe that

$$g(x) := f''(x) + \rho^2 f(x) = \rho^2\ln x - \frac{1}{x^2},\ x > 0.$$

We have $g'(x) = \frac{2+\rho^2 x^2}{x^2} > 0$ for $x > 0$ and $\lim_{x\to 0+} g(x) = -\infty$, $\lim_{x\to\infty} g(x) = \infty$, showing that the function g is strictly increasing on $(0,\infty)$ and the equation $g(x) = 0$ has a unique solution. Therefore $g(x) < 0$ for $x \in (0, x_\rho)$ and $g(x) > 0$ for $x \in (x_\rho, \infty)$, where x_ρ is the unique solution of the equation $\ln x = \frac{1}{\rho^2 x^2}$. We observe that $x_\rho > 1$.

In conclusion, if $\rho > 0$, then the function $f(x) = \ln x$ is trigonometrically ρ-concave on $(0, x_\rho)$ and trigonometrically ρ-convex on (x_ρ, ∞).

The following Hermite-Hadamard type inequality that was obtained in 2013 in [2].

Theorem 1. *Assume that the function $f : I \to \mathbb{R}$ is trigonometrically ρ-convex on I. Then for any a, $b \in I$ with $0 < b - a < \frac{\pi}{\rho}$ we have*

$$\frac{2}{\rho} f\left(\frac{a+b}{2}\right) \sin\left[\frac{\rho(b-a)}{2}\right] \le \int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{\rho} \tan\left[\frac{\rho(b-a)}{2}\right]. \tag{1.5}$$

The inequality (1.5) for $\rho = 1$ was obtained in 2004 by M. Bessenyei in his Ph.D. Thesis [4, Corollary 2.13] in the context of Chebyshev system $(\cos, \sin)$. For a simpler proof than provided in [2] and the following related results, see [11]:

Theorem 2. *Assume that the function $f : I \to \mathbb{R}$ is trigonometrically ρ-convex on I. Then for any a, $b \in I$ with $0 < b - a < \frac{\pi}{\rho}$ we have*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x) \sec\left[\rho\left(x - \frac{a+b}{2}\right)\right] dx \le \frac{f(a)+f(b)}{2} \tag{1.6}$$

and

$$\begin{aligned} \frac{1}{2}\left[b - a + \frac{1}{\rho}\sin[\rho(b-a)]\right] f\left(\frac{a+b}{2}\right) &\le \int_a^b f(x) \cos\left[\rho\left(x - \frac{a+b}{2}\right)\right] dx \\ &\le \frac{b - a + \frac{1}{\rho}\sin[\rho(b-a)]}{2\cos\left[\frac{\rho(b-a)}{2}\right]} \frac{f(a)+f(b)}{2}. \end{aligned} \tag{1.7}$$

Motivated by the above results, in this paper we establish some new inequalities for trigonometrically ρ-convex functions. Applications for discrete inequalities of Jensen's type are also provided.

2. One Variable Inequalities

The following upper and lower bounds for the function f holds:

Theorem 3. *Let $f : I \to \mathbb{R}$ be a twice differentiable and ρ-trigonometrically convex function on I and $[a,b] \subset I$ with $0 < b - a < \frac{\pi}{\rho}$. Then for any $x \in [a,b]$ we have*

$$\begin{aligned} & f(a) + f'_+(a)(x-a) - \rho^2 \int_a^x (x-t) f(t)\,dt \\ & \le f(x) \\ & \le f(a) + f'_-(b)(x-a) + \rho^2 \left[(x-a)\int_x^b f(s)\,ds + \int_a^x (t-a) f(t)\,dt\right] \end{aligned} \tag{2.1}$$

and

$$f(b) - f'_-(b)(b-x) + \rho^2 \int_x^b (x-t) f(t)\,dt \tag{2.2}$$

$$\leq f(x)$$

$$\leq f(b) - f'_+(a)(b-x) + \rho^2 \left[(b-x) \int_a^x f(s)\,ds + \int_x^b (b-t) f(t)\,dt \right].$$

Proof. Let $[a,b] \subset I$ and $t \in [a,b]$. Assume that f is differentiable at t (and this happens in all the points of $[a,b]$ except at most a countable number of them, see (ii)). Using the monotonicity property from Introduction, (iii) we get that

$$f'(t) + \rho^2 \int_a^t f(s)\,ds \geq f'_+(a),$$

which is equivalent to

$$f'(t) \geq f'(a) - \rho^2 \int_a^t f(s)\,ds$$

for $t \in [a,b]$ with the above property.

Using the monotonicity property from (iii) we also get that

$$f'_-(b) \geq f'(t) + \rho^2 \int_b^t f(s)\,ds,$$

which is equivalent to

$$f'_-(b) + \rho^2 \int_t^b f(s)\,ds \geq f'(t),$$

for $t \in [a,b]$ with the above property.

Therefore we have the following upper and lower bounds for the derivative

$$f'_+(a) - \rho^2 \int_a^t f(s)\,ds \leq f'(t) \leq f'_-(b) + \rho^2 \int_t^b f(s)\,ds \tag{2.3}$$

for $t \in [a,b]$ with the above property.

Let $x \in [a,b]$ and integrate the inequality (2.3) over t on $[a,x]$ to get

$$\begin{aligned} f'_+(a)(x-a) - \rho^2 \int_a^x \left(\int_a^t f(s)\,ds \right) dt &\leq f(x) - f(a) \\ &\leq f'_-(b)(x-a) + \rho^2 \int_a^x \left(\int_t^b f(s)\,ds \right) dt. \end{aligned} \tag{2.4}$$

Integrating by parts, we have

$$\begin{aligned} \int_a^x \left(\int_a^t f(s)\,ds \right) dt &= t \int_a^t f(s)\,ds \bigg|_a^x - \int_a^x t f(t)\,dt \\ &= x \int_a^x f(s)\,ds - \int_a^x t f(t)\,dt = \int_a^x (x-t) f(t)\,dt \end{aligned}$$

and

$$\int_a^x \left(\int_t^b f(s)\,ds \right) dt = \int_a^x \left(\int_a^b f(s)\,ds - \int_a^t f(s)\,ds \right) dt$$
$$= (x-a) \int_a^b f(s)\,ds - \int_a^x (x-t) f(t)\,dt$$

and by (2.4) we get

$$f'_+(a)(x-a) - \rho^2 \int_a^x (x-t) f(t)\,dt \le f(x) - f(a) \tag{2.5}$$
$$\le f'_-(b)(x-a) + \rho^2 (x-a) \int_a^b f(s)\,ds - \rho^2 \int_a^x (x-t) f(t)\,dt$$
$$= f'_-(b)(x-a) + \rho^2 (x-a) \int_a^x f(s)\,ds$$
$$+ \rho^2 (x-a) \int_x^b f(s)\,ds - \rho^2 \int_a^x (x-t) f(t)\,dt$$
$$= f'_-(b)(x-a) + \rho^2 (x-a) \int_x^b f(s)\,ds + \rho^2 \int_a^x (t-a) f(t)\,dt,$$

for any $x \in [a,b]$, which proves (2.1).

Let $x \in [a,b]$ and integrate the inequality (2.3) over t on $[x,b]$ to get

$$f'_+(a)(b-x) - \rho^2 \int_x^b \left(\int_a^t f(s)\,ds \right) dt \le f(b) - f(x) \tag{2.6}$$
$$\le f'_-(b)(b-x) + \rho^2 \int_x^b \left(\int_t^b f(s)\,ds \right) dt$$

for any $x \in [a,b]$.

Integrating by parts, we have

$$\int_x^b \left(\int_a^t f(s)\,ds \right) dt = t \int_a^t f(s)\,ds \Big|_x^b - \int_x^b t f(t)\,dt$$
$$= b \int_a^b f(s)\,ds - x \int_a^x f(s)\,ds - \int_x^b t f(t)\,dt$$
$$= b \int_a^x f(s)\,ds + b \int_x^b f(s)\,ds - x \int_a^x f(s)\,ds - \int_x^b t f(t)\,dt$$
$$= (b-x) \int_a^x f(s)\,ds + \int_x^b (b-t) f(t)\,dt$$

and

$$\int_x^b \left(\int_t^b f(s)\,ds \right) dt = \int_x^b \left(\int_a^b f(s)\,ds - \int_a^t f(s)\,ds \right) dt$$
$$= (b-x) \int_a^b f(s)\,ds - (b-x) \int_a^x f(s)\,ds - \int_x^b (b-t) f(t)\,dt$$
$$= (b-x) \int_x^b f(s)\,ds - \int_x^b (b-t) f(t)\,dt = \int_x^b (t-x) f(t)\,dt$$

and by (2.6) we get

$$f'_{+}(a)(b-x)-\rho^{2}(b-x)\int_{a}^{x}f(s)\,ds-\rho^{2}\int_{x}^{b}(b-t)f(t)\,dt$$
$$\leq f(b)-f(x)\leq f'_{-}(b)(b-x)+\rho^{2}\int_{x}^{b}(t-x)f(t)\,dt$$

for any $x\in[a,b]$, which is equivalent to the desired result (2.1). □

Remark 1. *If we take in (2.1) and (2.2) $x=\frac{a+b}{2}$, then we get*

$$f(a)+\frac{b-a}{2}f'_{+}(a)-\rho^{2}\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)f(t)\,dt \tag{2.7}$$
$$\leq f\left(\frac{a+b}{2}\right)$$
$$\leq f(a)+\frac{b-a}{2}f'_{-}(b)+\rho^{2}\left[\frac{b-a}{2}\int_{\frac{a+b}{2}}^{b}f(s)\,ds+\int_{a}^{\frac{a+b}{2}}(t-a)f(t)\,dt\right]$$

and

$$f(b)-\frac{b-a}{2}f'_{-}(b)+\rho^{2}\int_{\frac{a+b}{2}}^{b}\left(\frac{a+b}{2}-t\right)f(t)\,dt \tag{2.8}$$
$$\leq f\left(\frac{a+b}{2}\right)$$
$$\leq f(b)-\frac{b-a}{2}f'_{+}(a)+\rho^{2}\left[\frac{b-a}{2}\int_{a}^{\frac{a+b}{2}}f(s)\,ds+\int_{\frac{a+b}{2}}^{b}(b-t)f(t)\,dt\right].$$

If in (2.1) we put instead of a, $\frac{a+b}{2}$, then we get for $x\in\left[\frac{a+b}{2},b\right]$ that

$$f\left(\frac{a+b}{2}\right)+f'_{+}\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)-\rho^{2}\int_{\frac{a+b}{2}}^{x}(x-t)f(t)\,dt \tag{2.9}$$
$$\leq f(x)$$
$$\leq f\left(\frac{a+b}{2}\right)+f'_{-}(b)\left(x-\frac{a+b}{2}\right)$$
$$+\rho^{2}\left[\left(x-\frac{a+b}{2}\right)\int_{x}^{b}f(s)\,ds+\int_{\frac{a+b}{2}}^{x}\left(t-\frac{a+b}{2}\right)f(t)\,dt\right].$$

If in (2.2) we put instead of b, $\frac{a+b}{2}$, then we get for $x\in\left[a,\frac{a+b}{2}\right]$ that

$$f\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}-x\right)+\rho^{2}\int_{x}^{\frac{a+b}{2}}(x-t)f(t)\,dt \tag{2.10}$$
$$\leq f(x)$$
$$\leq f\left(\frac{a+b}{2}\right)-f'_{+}(a)\left(\frac{a+b}{2}-x\right)$$
$$+\rho^{2}\left[\left(\frac{a+b}{2}-x\right)\int_{a}^{x}f(s)\,ds+\int_{x}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)f(t)\,dt\right].$$

Corollary 1. *With the assumptions of Theorem 3 and for $\lambda \in [0,1]$ we have*

$$
\begin{aligned}
(2.11)\quad & (1-\lambda) f'_+(a)(x-a) - \lambda f'_-(b)(b-x) \\
& + \rho^2 \left[\lambda \int_x^b (x-t) f(t)\,dt - (1-\lambda) \int_a^x (x-t) f(t)\,dt\right] \\
& \le f(x) - (1-\lambda) f(a) - \lambda f(b) \\
& \le (1-\lambda) f'_-(b)(x-a) - \lambda f'_+(a)(b-x) \\
& + (1-\lambda)\rho^2 \left[(x-a)\int_x^b f(s)\,ds + \int_a^x (t-a) f(t)\,dt\right] \\
& + \lambda\rho^2 \left[(b-x)\int_a^x f(s)\,ds + \int_x^b (b-t) f(t)\,dt\right]
\end{aligned}
$$

for any $x \in [a,b]$.

In particular, for $\lambda = \frac{1}{2}$, we get

$$
\begin{aligned}
(2.12)\quad & \frac{1}{2}\left[f'_+(a)(x-a) - f'_-(b)(b-x)\right] \\
& + \frac{1}{2}\rho^2 \left[\int_x^b (x-t) f(t)\,dt - \int_a^x (x-t) f(t)\,dt\right] \\
& \le f(x) - \frac{f(a)+f(b)}{2} \\
& \le \frac{1}{2}\left[f'_-(b)(x-a) - f'_+(a)(b-x)\right] \\
& + \frac{1}{2}\rho^2 \left[(x-a)\int_x^b f(s)\,ds + \int_a^x (t-a) f(t)\,dt\right] \\
& + \frac{1}{2}\rho^2 \left[(b-x)\int_a^x f(s)\,ds + \int_x^b (b-t) f(t)\,dt\right]
\end{aligned}
$$

for any $x \in [a,b]$.

3. Two Variable Inequalities

In this section we provide a double inequality for two independent variables that can be used to obtain Jensen's type inequalities:

Theorem 4. *Let $f : I \to \mathbb{R}$ be a twice differentiable and ρ-trigonometrically convex function on I and $[a,b] \subset I$ with $0 < b - a < \frac{\pi}{\rho}$. Then for any $x,\ y \in (a,b)$ we have*

$$(x-y) f'_{-}(x) + \rho^2 \left(\int_a^y (y-t) f(t)\, dt + \int_a^x (t-y) f(t)\, dt \right) \tag{3.1}$$
$$\geq f(x) - f(y)$$
$$\geq (x-y) f'_{+}(y) - \rho^2 \left(\int_a^x (x-t) f(t)\, dt + \int_a^y (t-x) f(t)\, dt \right).$$

Proof. Since f is ρ-trigonometrically convex function on I then by property (iii) from Introduction, we have that $f' + \rho^2 \int_a f$ is increasing and therefore the function

$$F(x) := \int_a^x \left[f'(t) + \rho^2 \int_a^t f(s)\, ds \right] dt$$

is convex on $[a,b]$.

Integrating by parts, we have

$$\begin{aligned} F(x) &= f(x) - f(a) + \rho^2 \int_a^x \left(\int_a^t f(s)\, ds \right) dt \\ &= f(x) - f(a) + \rho^2 \left[t \int_a^t f(s)\, ds \Big|_a^x - \int_a^x t f(t)\, dt \right] \\ &= f(x) - f(a) + \rho^2 \left[x \int_a^x f(s)\, ds - \int_a^x t f(t)\, dt \right], \end{aligned}$$

for any $x \in [a,b]$.

Similarly,

$$F(y) = f(y) - f(a) + \rho^2 \left[y \int_a^y f(s)\, ds - \int_a^y t f(t)\, dt \right],$$

for any $y \in [a,b]$.

Also,

$$F_{+}(y) = f'_{+}(y) + \rho^2 \int_a^y f(s)\, ds$$

for $y \in [a,b)$.

Since F is convex, then for any x, $y \in (a,b)$ we have

$$\begin{aligned}
0 &\le F(x) - F(y) - (x-y) F'_+(y) \\
&= f(x) - f(a) + \rho^2 x \int_a^x f(s)\,ds - \rho^2 \int_a^x t f(t)\,dt \\
&- f(y) + f(a) - \rho^2 y \int_a^y f(s)\,ds + \rho^2 \int_a^y t f(t)\,dt \\
&- (x-y)\left(f'_+(y) + \rho^2 \int_a^y f(s)\,ds\right) \\
&= f(x) - f(y) - (x-y) f'_+(y) + \rho^2 \int_a^x (x-t) f(t)\,dt \\
&+ \rho^2 \int_a^y [t - y - (x-y)] f(t)\,dt \\
&= f(x) - f(y) - (x-y) f'_+(y) + \rho^2 \int_a^x (x-t) f(t)\,dt \\
&+ \rho^2 \int_a^y (t-x) f(t)\,dt.
\end{aligned}$$

In a similar manner, we get

$$f(x) - f(y) \le (x-y) f'_-(x) + \rho^2 \int_a^y (y-t) f(t)\,dt + \rho^2 \int_a^x (t-y) f(t)\,dt,$$

which proves the first inequality in (3.1). □

Remark 2. *If we take $y = \frac{a+b}{2}$ in (3.1), then we get*

$$\begin{aligned}
&\left(x - \frac{a+b}{2}\right) f'_-(x) \\
&\quad + \rho^2 \left(\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right) f(t)\,dt + \int_a^x \left(t - \frac{a+b}{2}\right) f(t)\,dt\right) \\
&\qquad \ge f(x) - f\left(\frac{a+b}{2}\right) \\
&\ge \left(x - \frac{a+b}{2}\right) f'_+\left(\frac{a+b}{2}\right) - \rho^2 \left(\int_a^x (x-t) f(t)\,dt + \int_a^{\frac{a+b}{2}} (t-x) f(t)\,dt\right)
\end{aligned} \tag{3.2}$$

for any $x \in (a,b)$.

4. Applications for Jensen's Discrete Inequalities

Let $f : I \to \mathbb{R}$ be a twice differentiable and ρ-trigonometrically convex function on I and $[a,b] \subset I$ with $0 < b - a < \frac{\pi}{\rho}$. Assume that $x_i \in [a,b]$, $p_i \ge 0$ for $i \in \{1,...,n\}$ with $\sum_{i=1}^n p_i = 1$ and let $\bar{x}_p := \sum_{i=1}^n p_i x_i \in [a,b]$, then by (2.1) and (2.2) on replacing x with x_i, multiplying with $p_i \ge 0$ and summing over $i \in \{1,...,n\}$

we get

$$f(a)+f'_{+}(a)(\bar{x}_p-a)-\rho^2\sum_{i=1}^{n}p_i\int_a^{x_i}(x_i-t)f(t)\,dt \tag{4.1}$$

$$\leq\sum_{i=1}^{n}p_if(x_i)$$

$$\leq f(a)+f'_{-}(b)(\bar{x}_p-a) + \rho^2\left[\sum_{i=1}^{n}p_i(x_i-a)\int_{x_i}^{b}f(s)\,ds+\sum_{i=1}^{n}p_i\int_a^{x_i}(t-a)f(t)\,dt\right]$$

and

$$f(b)-f'_{-}(b)(b-\bar{x}_p)+\rho^2\sum_{i=1}^{n}p_i\int_{x_i}^{b}(x_i-t)f(t)\,dt \tag{4.2}$$

$$\leq\sum_{i=1}^{n}p_if(x_i)$$

$$\leq f(b)-f'_{+}(a)(b-\bar{x}_p) + \rho^2\left[\sum_{i=1}^{n}p_i(b-x_i)\int_a^{x_i}f(s)\,ds+\sum_{i=1}^{n}p_i\int_{x_i}^{b}(b-t)f(t)\,dt\right].$$

From (2.11) we get in a similar way

$$(1-\lambda)f'_{+}(a)(\bar{x}_p-a)-\lambda f'_{-}(b)(b-\bar{x}_p) + \rho^2\left[\lambda\sum_{i=1}^{n}p_i\int_{x_i}^{b}(x_i-t)f(t)\,dt-(1-\lambda)\sum_{i=1}^{n}p_i\int_a^{x_i}(x_i-t)f(t)\,dt\right] \tag{4.3}$$

$$\leq\sum_{i=1}^{n}p_if(x_i)-(1-\lambda)f(a)-\lambda f(b)$$

$$\leq(1-\lambda)f'_{-}(b)(\bar{x}_p-a)-\lambda f'_{+}(a)(b-\bar{x}_p) + (1-\lambda)\rho^2\left[\sum_{i=1}^{n}p_i(x_i-a)\int_{x_i}^{b}f(s)\,ds+\sum_{i=1}^{n}p_i\int_a^{x_i}(t-a)f(t)\,dt\right] + \lambda\rho^2\left[\sum_{i=1}^{n}p_i(b-x_i)\int_a^{x_i}f(s)\,ds+\sum_{i=1}^{n}p_i\int_{x_i}^{b}(b-t)f(t)\,dt\right]$$

for any $\lambda\in[0,1]$.

In particular, for $\lambda = \frac{1}{2}$, we get

$$\begin{aligned}
(4.4)\quad & \frac{1}{2}\left[f'_+(a)(\bar{x}_p - a) - f'_-(b)(b - \bar{x}_p)\right] \\
& + \frac{1}{2}\rho^2\left[\sum_{i=1}^n p_i \int_{x_i}^b (x_i - t) f(t)\,dt - \sum_{i=1}^n p_i \int_a^{x_i} (x_i - t) f(t)\,dt\right] \\
& \leq \sum_{i=1}^n p_i f(x_i) - \frac{f(a) + f(b)}{2} \\
& \leq \frac{1}{2}\left[f'_-(b)(\bar{x}_p - a) - f'_+(a)(b - \bar{x}_p)\right] \\
& + \frac{1}{2}\rho^2\left[\sum_{i=1}^n p_i (x_i - a)\int_{x_i}^b f(s)\,ds + \sum_{i=1}^n p_i \int_a^{x_i} (t - a) f(t)\,dt\right] \\
& + \frac{1}{2}\rho^2\left[\sum_{i=1}^n p_i (b - x_i)\int_a^{x_i} f(s)\,ds + \sum_{i=1}^n p_i \int_{x_i}^b (b - t) f(t)\,dt\right].
\end{aligned}$$

From (3.2) we also have

$$\begin{aligned}
(4.5)\quad & \sum_{i=1}^n p_i\left(x_i - \frac{a+b}{2}\right) f'_-(x_i) \\
& + \rho^2\left(\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2} - t\right) f(t)\,dt + \sum_{i=1}^n p_i \int_a^{x_i}\left(t - \frac{a+b}{2}\right) f(t)\,dt\right) \\
& \geq \sum_{i=1}^n p_i f(x_i) - f\left(\frac{a+b}{2}\right) \\
& \geq \left(\bar{x}_p - \frac{a+b}{2}\right) f'_+\left(\frac{a+b}{2}\right) \\
& - \rho^2\left(\sum_{i=1}^n p_i \int_a^{x_i} (x_i - t) f(t)\,dt + \int_a^{\frac{a+b}{2}} (t - \bar{x}_p) f(t)\,dt\right).
\end{aligned}$$

Further, on replacing in (3.1) y by $\bar{x}_p$, x by x_i, multiplying with $p_i \geq 0$ and summing over $i \in \{1, ..., n\}$, we get the following Jensen's type discrete inequality

$$\sum_{i=1}^{n} p_i (x_i - \bar{x}_p) f'_-(x_i) + \rho^2 \left(\sum_{i=1}^{n} p_i \int_a^{x_i} (t - \bar{x}_p) f(t)\, dt - \int_a^{\bar{x}_p} (t - \bar{x}_p) f(t)\, dt \right) \geq \sum_{i=1}^{n} p_i f(x_i) - f(\bar{x}_p) \geq \rho^2 \left(\sum_{i=1}^{n} p_i \int_a^{x_i} (t - x_i) f(t)\, dt - \int_a^{\bar{x}_p} (t - \bar{x}_p) f(t)\, dt \right). \tag{4.6}$$

Under the above assumptions for a, b, consider the function

$$g(x) := \int_a^x (t - x) f(t)\, dt = \int_a^x t f(t)\, dt - x \int_a^x f(t)\, dt.$$

Then

$$g'(x) = -\int_a^x f(t)\, dt \text{ and } g''(x) = -f(x), \ x \in [a, b].$$

So, if $f(x) \leq 0$, $x \in [a, b]$, then g is convex and by Jensen's inequality for g we have

$$\sum_{i=1}^{n} p_i \int_a^{x_i} (t - x_i) f(t)\, dt \geq \int_a^{\bar{x}_p} (t - \bar{x}_p) f(t)\, dt$$

where $x_i \in [a, b]$, $p_i \geq 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$.

In conclusion, if $f : I \to \mathbb{R}$ is a twice differentiable and ρ-trigonometrically convex function on I and $[a, b] \subset I$ with $0 < b - a < \frac{\pi}{\rho}$ and $f(x) \leq 0$, $x \in [a, b]$, then we have the refinement of Jensen's inequality

$$\sum_{i=1}^{n} p_i f(x_i) - f(\bar{x}_p) \geq \rho^2 \left(\sum_{i=1}^{n} p_i \int_a^{x_i} (t - x_i) f(t)\, dt - \int_a^{\bar{x}_p} (t - \bar{x}_p) f(t)\, dt \right) \geq 0 \tag{4.7}$$

where $x_i \in [a, b]$, $p_i \geq 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$.

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