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NEW APPROXIMATION OF f-DIVERGENCE MEASURES BY USING TWO POINTS TAYLOR'S TYPE REPRESENTATIONS

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Abstract. In this paper we establish some new approximations of the f-divergence measures by the use of two points Taylor's type representations with integral remainders. Some inequalities for Kullback-Leibler divergence are provided as well.

1. Introduction

One of the important issues in many applications of Probability Theory & Statistics is finding an appropriate measure of *distance* (*difference* or *discrimination*) between two probability distributions.

A number of *divergence measures* have been proposed and extensively studied by: Jeffreys 1946 [26], Kullback-Leibler 1951 [32], Rényi 1961 [39], Ali and Silvey 1966 [1], Csiszár 1967 [11], Havrda-Charvat 1967 [23], Sharma-Mittal 1977 [41], Rao 1982 [38], Burbea-Rao 1982 [8], Kapur 1984 [29], Vajda 1989 [48], Lin 1991 [33], Shioya and Da-te [42] and others, see [36]

These measures have been applied in a variety of fields such as: anthropology [38], genetics [36], finance, economics and political science [40], [45], [46], biology [37], the analysis of contingency tables [22], approximation of probability distributions [10], [30], signal processing [27], [28] and pattern recognition [7], [9].

Assume that a set Ω and the σ-finite measure μ are given. Consider the set of all probability densities on μ to be

$$\mathcal{P}:=\left\{p|p:\Omega\to\mathbb{R},\ p(x)\geq 0,\ \int_\Omega p(x)\,d\mu(x)=1\right\}.$$

The *Kullback-Leibler divergence* [32] is well known among the information divergences. It is defined for $p,\ q\in\mathcal{P}$ as follows:

$$D_{KL}(p,q):=\int_\Omega p(x)\ln\left[\frac{p(x)}{q(x)}\right]d\mu(x), \tag{1.1}$$

where ln is to base e.

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are defined for $p,\ q\in\mathcal{P}$ as follows

$$D_v(p,q):=\int_\Omega |p(x)-q(x)|\,d\mu(x),\ \textit{variation distance},$$

$$D_H(p,q):=\int_\Omega \left|\sqrt{p(x)}-\sqrt{q(x)}\right|d\mu(x),\ \textit{Hellinger distance [24]},$$

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$$D_{\chi^2}(p,q) := \int_\Omega p(x)\left[\left(\frac{q(x)}{p(x)}\right)^2 - 1\right] d\mu(x), \ \chi^2\text{-}divergence,$$

$$D_\alpha(p,q) := \frac{4}{1-\alpha^2}\left[1 - \int_\Omega [p(x)]^{\frac{1-\alpha}{2}} [q(x)]^{\frac{1+\alpha}{2}} d\mu(x)\right], \ \alpha\text{-}divergence,$$

$$D_B(p,q) := \int_\Omega \sqrt{p(x)\,q(x)}\,d\mu(x), \ \textit{Bhattacharyya distance } [6],$$

$$D_{Ha}(p,q) := \int_\Omega \frac{2p(x)\,q(x)}{p(x)+q(x)} d\mu(x), \ \textit{Harmonic distance},$$

$$D_J(p,q) := \int_\Omega [p(x) - q(x)] \ln\left[\frac{p(x)}{q(x)}\right] d\mu(x), \ \textit{Jeffrey's distance } [26],$$

and

$$D_\Delta(p,q) := \int_\Omega \frac{[p(x)-q(x)]^2}{p(x)+q(x)} d\mu(x), \ \textit{triangular discrimination } [44].$$

For other divergence measures, see the paper [29] by Kapur or the book on line [43] by Taneja.

In 1967, I. Csiszár [12] introduced the concept of *f-divergence* as follows

$$I_f(p,q) := \int_\Omega p(x)\, f\left[\frac{q(x)}{p(x)}\right] d\mu(x), \tag{1.2}$$

for p, $q \in \mathcal{P}$, where f *is convex on* $(0,\infty)$ and *normalised*, i.e. $f(1) = 0$.

Most of the above distances are particular instances of Csiszár f-divergence. There are also many others which are not in this class (see for example Taneja's book online [43]). For the basic properties of Csiszár f-divergence such as

$$I_f(p,q) \geq 0 \text{ for any } p,\ q \in \mathcal{P},$$

and

$$\mathcal{P} \times \mathcal{P} \ni (p,q) \mapsto I_f(p,q) \ \text{ is convex},$$

see [12], [13] and [48].

In the recent papers [14], [15] and [16] we obtained several reverses of Jensen's integral inequality. These applied to Csiszár f-divergence produce the following results:

Theorem 1 (Dragomir 2013, [15])**.** *Let $f : (0,\infty) \to \mathbb{R}$ be a convex function with the property that $f(1) = 0$. Assume that p, $q \in \mathcal{P}$ and there exists the constants $0 < r < 1 < R < \infty$ such that*

$$r \leq \frac{q(x)}{p(x)} \leq R \text{ for } \mu\text{-a.e. } x \in \Omega. \tag{1.3}$$

Then we have the inequalities

$$\begin{aligned} 0 \leq I_f(p,q) &\leq \frac{(R-1)(1-r)}{R-r} \sup_{t \in (r,R)} \Psi_f(t;r,R) \\ &\leq (R-1)(1-r)\frac{f'_-(R) - f'_+(r)}{R-r} \\ &\leq \frac{1}{4}(R-r)\left[f'_-(R) - f'_+(r)\right], \end{aligned} \tag{1.4}$$

and $\Psi_f(\cdot;r,R):(r,R)\to\mathbb{R}$ *is defined by*

$$\Psi_f(t;r,R)=\frac{f(R)-f(t)}{R-t}-\frac{f(t)-f(r)}{t-r}.$$

We also have the inequality

$$\begin{aligned}0&\le I_f(p,q)\le\frac{1}{4}(R-r)\frac{f(R)(1-r)+f(r)(R-1)}{(R-1)(1-r)}\\&\le\frac{1}{4}(R-r)\left[f'_-(R)-f'_+(r)\right].\end{aligned}\tag{1.5}$$

and the inequality

$$\begin{aligned}0&\le I_f(p,q)\le 2\max\left\{\frac{R-1}{R-r},\frac{1-r}{R-r}\right\}\\&\times\left[\frac{f(r)+f(R)}{2}-f\left(\frac{r+R}{2}\right)\right]\\&\le\frac{1}{2}\max\{R-1,1-r\}\left[f'_-(R)-f'_+(r)\right].\end{aligned}\tag{1.6}$$

Some bounds in terms of the variation distance are as follows:

Theorem 2 (Dragomir 2016, [16]). *With the assumptions of Theorem 1 we have*

$$\begin{aligned}0&\le I_f(p,q)\le\frac{1}{2}\left[f'_-(R)-f'_+(r)\right]D_v(p,q)\\&\le\frac{1}{2}\left[f'_-(R)-f'_+(r)\right]\left[D_{\chi^2}(p,q)\right]^{1/2}\\&\le\frac{1}{4}(R-r)\left[f'_-(R)-f'_+(r)\right].\end{aligned}\tag{1.7}$$

and

$$\begin{aligned}0&\le I_f(p,q)\le\frac{1}{2}([1,R;f]-[r,1;f])D_v(p,q)\\&\le\frac{1}{2}([1,R;f]-[r,1;f])\left[D_{\chi^2}(p,q)\right]^{1/2}\\&\le\frac{1}{4}([1,R;f]-[r,1;f])(R-r),\end{aligned}\tag{1.8}$$

where $[a,b;f]$ *is the divided difference*

$$[a,b;f]:=\frac{f(b)-f(a)}{b-a}.$$

Further bounds in terms of the Lebesgue norms of the derivative are embodied in the next theorem:

Theorem 3 (Dragomir 2013, [14]). *With the assumptions in Theorem 1 we have*

$$0\le I_f(p,q)\le B_f(r,R)\tag{1.9}$$

where

$$B_f(r,R):=\frac{(R-1)\int_r^1|f'(t)|\,dt+(1-r)\int_1^R|f'(t)|\,dt}{R-r}.\tag{1.10}$$

Moreover, we have the following bounds for $B_f(r,R)$

$$B_f(r,R) \leq \begin{cases} \left[\frac{1}{2} + \frac{\left|1-\frac{r+R}{2}\right|}{R-r}\right] \int_r^R |f'(t)|\, dt \\ \\ \frac{1}{2}\int_r^R |f'(t)|\, dt + \frac{1}{2}\left|\int_1^R |f'(t)|\, dt - \int_r^1 |f'(t)|\, dt\right|, \end{cases} \tag{1.11}$$

and

$$\begin{aligned} B_f(r,R) &\leq \frac{(1-r)(R-1)}{R-r}\left[\|f'\|_{[1,R],\infty} + \|f'\|_{[r,1],\infty}\right] \\ &\leq \frac{1}{2}(R-r)\frac{\|f'\|_{[1,R],\infty} + \|f'\|_{[r,1],\infty}}{2} \leq \frac{1}{2}(R-r)\|f'\|_{[r,R],\infty} \end{aligned} \tag{1.12}$$

and

$$\begin{aligned} B_f(r,R) &\leq \frac{1}{R-r}\left[(1-r)(R-1)^{1/q}\|f'\|_{[1,R],p} \right. \\ &\left. + (R-1)(1-r)^{1/q}\|f'\|_{[r,1],p}\right] \\ &\leq \frac{1}{R-r}\|f'\|_{[r,R],p}\left[(1-r)^q(R-1) + (R-1)^q(1-r)\right]^{1/q}, \end{aligned} \tag{1.13}$$

Motivated by the above results, in this paper we establish some new inequalities for f-divergence measures by employing two points Taylor's type expansions that are presented below. Applications for particular instances of interest are provided as well.

2. Some Preliminary Facts

The following result is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Lemma 1. *Let* $I \subset \mathbb{R}$ *be a closed interval,* $c \in I$ *and let* n *be a positive integer. If* $f : I \longrightarrow \mathbb{C}$ *is such that the* n*-derivative* $f^{(n)}$ *is absolutely continuous on* I*, then for each* $y \in I$

$$f(y) = T_n(f;c,y) + R_n(f;c,y), \tag{2.1}$$

where $T_n(f;c,y)$ *is Taylor's polynomial, i.e.,*

$$T_n(f;c,y) := \sum_{k=0}^{n} \frac{(y-c)^k}{k!} f^{(k)}(c). \tag{2.2}$$

Note that $f^{(0)} := f$ *and* $0! := 1$ *and the remainder is given by*

$$R_n(f;c,y) := \frac{1}{n!}\int_c^y (y-t)^n f^{(n+1)}(t)\, dt. \tag{2.3}$$

A simple proof of this lemma can be achieved by mathematical induction using the integration by parts formula in the Lebesgue integral.

For related results, see [2]-[5], [20]-[21], [28], [33]-[35] and [47].

The following identity can be stated:

Lemma 2. *Let $f : I \to \mathbb{C}$ be n-time differentiable function on the interior $\mathring{I}$ of the interval I and $f^{(n)}$, with $n \geq 1$, be locally absolutely continuous on $\mathring{I}$. Then for each distinct t, a, $b \in \mathring{I}$ and for any $\lambda \in \mathbb{R} \setminus \{0, 1\}$ we have the representation*

$$\begin{aligned} f(t) &= (1-\lambda) f(a) + \lambda f(b) \\ &+ \sum_{k=1}^{n} \frac{1}{k!} \left[(1-\lambda) f^{(k)}(a)(t-a)^k + (-1)^k \lambda f^{(k)}(b)(b-t)^k \right] \\ &+ S_{n,\lambda}(t,a,b), \end{aligned} \tag{2.4}$$

where the remainder $S_{n,\lambda}(t,a,b)$ is given by

$$\begin{aligned} & S_{n,\lambda}(t,a,b) \\ & := \frac{1}{n!} \left[(1-\lambda)(t-a)^{n+1} \int_0^1 f^{(n+1)}((1-s)a+st)(1-s)^n ds \right. \\ & \left. + (-1)^{n+1} \lambda (b-t)^{n+1} \int_0^1 f^{(n+1)}((1-s)t+sb) s^n ds \right]. \end{aligned} \tag{2.5}$$

Proof. Using Taylor's representation with the integral remainder (2.1) we can write the following two identities

$$f(t) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(t-a)^k + \frac{1}{n!} \int_a^t f^{(n+1)}(\tau)(t-\tau)^n d\tau \tag{2.6}$$

and

$$f(t) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(b)(b-t)^k + \frac{(-1)^{n+1}}{n!} \int_t^b f^{(n+1)}(\tau)(\tau-t)^n d\tau \tag{2.7}$$

for any t, a, $b \in \mathring{I}$.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $\tau = (1-s)c + sd$, $s \in [0,1]$ that

$$\int_c^d h(\tau) d\tau = (d-c) \int_0^1 h((1-s)c+sd) ds.$$

Therefore,

$$\begin{aligned} & \int_a^t f^{(n+1)}(\tau)(t-\tau)^n d\tau \\ & = (t-a) \int_0^1 f^{(n+1)}((1-s)a+st)(t-(1-s)a-st)^n ds \\ & = (t-a)^{n+1} \int_0^1 f^{(n+1)}((1-s)a+st)(1-s)^n ds \end{aligned}$$

and

$$\begin{aligned} & \int_t^b f^{(n+1)}(\tau)(\tau-t)^n d\tau \\ & = (b-t) \int_0^1 f^{(n+1)}((1-s)t+sb)((1-s)t+sb-t)^n ds \\ & = (b-t)^{n+1} \int_0^1 f^{(n+1)}((1-s)t+sb) s^n ds. \end{aligned}$$

The identities (2.6) and (2.7) can then be written as

$$f(t)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(a)(t-a)^{k}+\frac{1}{n!}(t-a)^{n+1}\int_{0}^{1}f^{(n+1)}((1-s)a+st)(1-s)^{n}ds \tag{2.8}$$

and

$$f(t)=\sum_{k=0}^{n}\frac{(-1)^{k}}{k!}f^{(k)}(b)(b-t)^{k}+(-1)^{n+1}\frac{(b-t)^{n+1}}{n!}\int_{0}^{1}f^{(n+1)}((1-s)t+sb)s^{n}ds. \tag{2.9}$$

Now, if we multiply (2.8) with $1-\lambda$ and (2.9) with λ and add the resulting equalities, a simple calculation yields the desired identity (2.4). □

Remark 1. *If we take in (2.4) $t=\frac{a+b}{2}$, with $a,b\in\mathring{I}$, then we have for any $\lambda\in\mathbb{R}\backslash\{0,1\}$ that*

$$f\left(\frac{a+b}{2}\right)=(1-\lambda)f(a)+\lambda f(b)+\sum_{k=1}^{n}\frac{1}{2^{k}k!}\left[(1-\lambda)f^{(k)}(a)+(-1)^{k}\lambda f^{(k)}(b)\right](b-a)^{k}+\tilde{S}_{n,\lambda}(a,b), \tag{2.10}$$

where the remainder $\tilde{S}_{n,\lambda}(a,b)$ is given by

$$\tilde{S}_{n,\lambda}(a,b):=\frac{1}{2^{n+1}n!}(b-a)^{n+1}\left[(1-\lambda)\int_{0}^{1}f^{(n+1)}\left((1-s)a+s\frac{a+b}{2}\right)(1-s)^{n}ds+(-1)^{n+1}\lambda\int_{0}^{1}f^{(n+1)}\left((1-s)\frac{a+b}{2}+sb\right)s^{n}ds\right]. \tag{2.11}$$

In particular, for $\lambda=\frac{1}{2}$ we have

$$f\left(\frac{a+b}{2}\right)=\frac{f(a)+f(b)}{2}+\sum_{k=1}^{n}\frac{1}{2^{k+1}k!}\left[f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right](b-a)^{k}+\tilde{S}_{n}(a,b), \tag{2.12}$$

where the remainder $\tilde{S}_{n}(a,b)$ is given by

$$\tilde{S}_{n}(a,b):=\frac{1}{2^{n+2}n!}(b-a)^{n+1}\left[\int_{0}^{1}f^{(n+1)}\left((1-s)a+s\frac{a+b}{2}\right)(1-s)^{n}ds+(-1)^{n+1}\int_{0}^{1}f^{(n+1)}\left((1-s)\frac{a+b}{2}+sb\right)s^{n}ds\right]. \tag{2.13}$$

Lemma 3. *With the assumptions in Lemma 2 we have for each distinct* t, a, $b \in \mathring{I}$

$$f(t) = \frac{1}{b-a}\left[(b-t) f(a) + (t-a) f(b)\right] + \frac{(b-t)(t-a)}{b-a} \times \sum_{k=1}^{n} \frac{1}{k!}\left\{(t-a)^{k-1} f^{(k)}(a) + (-1)^k (b-t)^{k-1} f^{(k)}(b)\right\} + L_n(t,a,b), \tag{2.14}$$

where

$$L_n(t,a,b) := \frac{(b-t)(t-a)}{n!(b-a)}\left[(t-a)^n \int_0^1 f^{(n+1)}((1-s)a + st)(1-s)^n ds + (-1)^{n+1}(b-t)^n \int_0^1 f^{(n+1)}((1-s)t + sb) s^n ds\right]$$

and

$$f(t) = \frac{1}{b-a}\left[(t-a) f(a) + (b-t) f(b)\right] + \frac{1}{b-a}\sum_{k=1}^{n} \frac{1}{k!}\left\{(t-a)^{k+1} f^{(k)}(a) + (-1)^k (b-t)^{k+1} f^{(k)}(b)\right\} + P_n(t,a,b), \tag{2.15}$$

where

$$P_n(t,a,b) := \frac{1}{n!(b-a)}\left[(t-a)^{n+2} \int_0^1 f^{(n+1)}((1-s)a + st)(1-s)^n ds + (-1)^{n+1}(b-t)^{n+2} \int_0^1 f^{(n+1)}((1-s)t + sb) s^n ds\right],$$

respectively.

The proof is obvious. Choose $\lambda = (t-a)/(b-a)$ and $\lambda = (b-t)/(b-a)$, respectively, in Lemma 2. The details are omitted.

Corollary 1. *With the assumption in Lemma 2 we have for each* $\lambda \in [0,1]$ *and any distinct* a, $b \in \mathring{I}$ *that*

$$f((1-\lambda)a + \lambda b) = (1-\lambda) f(a) + \lambda f(b) + \lambda(1-\lambda) \times \sum_{k=1}^{n} \frac{1}{k!}\left[\lambda^{k-1} f^{(k)}(a) + (-1)^k (1-\lambda)^{k-1} f^{(k)}(b)\right](b-a)^k + S_{n,\lambda}(a,b), \tag{2.16}$$

where the remainder $S_{n,\lambda}(a,b)$ *is given by*

$$S_{n,\lambda}(a,b) := \frac{1}{n!}(1-\lambda)\lambda(b-a)^{n+1}\left[\lambda^n \int_0^1 f^{(n+1)}((1-s\lambda)a + s\lambda b)(1-s)^n ds + (-1)^{n+1}(1-\lambda)^n \int_0^1 f^{(n+1)}((1-s-\lambda+s\lambda)a + (\lambda+s-s\lambda)b) s^n ds\right]. \tag{2.17}$$

We also have

$$f((1-\lambda)b+\lambda a)=(1-\lambda)f(a)+\lambda f(b) + \sum_{k=1}^{n}\frac{1}{k!}\left[(1-\lambda)^{k+1}f^{(k)}(a)+(-1)^{k}\lambda^{k+1}f^{(k)}(b)\right](b-a)^{k}+P_{n,\lambda}(a,b), \tag{2.18}$$

where the remainder $P_{n,\lambda}(a,b)$ *is given by*

$$\begin{aligned}P_{n,\lambda}(a,b) &:= \frac{1}{n!}(b-a)^{n+1}\left[(1-\lambda)^{n+2}\int_0^1 f^{(n+1)}((1-s+\lambda s)a+(1-\lambda)sb)(1-s)^n ds\right.\\ &\left.+(-1)^{n+1}\lambda^{n+2}\int_0^1 f^{(n+1)}((1-s)\lambda a+(1-\lambda+\lambda s)b)s^n ds\right].\end{aligned} \tag{2.19}$$

Remark 2. *The case* $n=0$*, namely when the function* f *is locally absolutely continuous on* $\mathring{I}$ *with the derivative* f' *existing almost everywhere in* $\mathring{I}$ *is important and produces the following simple identities for each distinct* $t,\ a,\ b\in\mathring{I}$ *and* $\lambda\in\mathbb{R}\setminus\{0,1\}$

$$f(t)=(1-\lambda)f(a)+\lambda f(b)+S_\lambda(t,a,b), \tag{2.20}$$

where the remainder $S_\lambda(t,a,b)$ *is given by*

$$\begin{aligned}S_\lambda(t,a,b) &:= (1-\lambda)(t-a)\int_0^1 f'((1-s)a+st)ds\\ &-\lambda(b-t)\int_0^1 f'((1-s)t+sb)ds.\end{aligned} \tag{2.21}$$

We then have for each distinct $t,\ a,\ b\in\mathring{I}$

$$f(t)=\frac{1}{b-a}\left[(b-t)f(a)+(t-a)f(b)\right]+L(t,a,b), \tag{2.22}$$

where

$$L(t,a,b) := \frac{(b-t)(t-a)}{b-a}\left[\int_0^1 f'((1-s)a+st)ds-\int_0^1 f'((1-s)t+sb)ds\right] \tag{2.23}$$

and

$$f(t)=\frac{1}{b-a}\left[(t-a)f(a)+(b-t)f(b)\right]+P(t,a,b), \tag{2.24}$$

where

$$P(t,a,b) := \frac{1}{b-a}\left[(t-a)^2\int_0^1 f'((1-s)a+st)ds-(b-t)^2\int_0^1 f'((1-s)t+sb)ds\right]. \tag{2.25}$$

3. Generalized Reverse Trapezoid Type Estimates

Assume that $p,\, q \in \mathcal{P}$ and there exists the constants $0 < r < 1 < R < \infty$ such that

$$r \leq \frac{q(x)}{p(x)} \leq R \text{ for } \mu\text{-a.e. } x \in \Omega. \tag{3.1}$$

We consider the following divergence measures

$$D_{\chi^k,r}(p,q) := \int_{\Omega.} \frac{(q(x) - rp(x))^k}{p^{k-1}(x)} d\mu(x) \geq 0 \text{ for } k \in \mathbb{N}, \tag{3.2}$$

and

$$D_{R,\chi^k}(p,q) := \int_{\Omega.} \frac{(Rp(x) - q(x))^k}{p^{k-1}(x)} d\mu(x) \geq 0 \text{ for } k \in \mathbb{N}. \tag{3.3}$$

We have the following approximation of the divergence measure using a reverse generalized trapezoid rule:

Theorem 4. *Let I be an open interval with $[r,R] \subset I$ as above, $f : I \to \mathbb{C}$ be n-time differentiable function on I and $f^{(n)}$, with $n \geq 1$, be locally absolutely continuous on I. Then for any $p,\, q \in \mathcal{P}$ satisfying the condition (3.1) we have the representation*

$$\begin{aligned}
&I_f(p,q) \\
&= \frac{(1-r)f(r) + (R-1)f(R)}{R-r} \\
&+ \frac{1}{R-r}\sum_{k=1}^{n}\frac{1}{k!}\left\{f^{(k)}(r)\,D_{\chi^{k+1},r}(p,q) + (-1)^k f^{(k)}(R)\,D_{R,\chi^{k+1}}(p,q)\right\} \\
&+ Q_{f,n}(p,q)
\end{aligned} \tag{3.4}$$

and the reminder $Q_{f,n}(p,q)$ is given by

$$\begin{aligned}
Q_{f,n}(p,q) = \frac{1}{n!\,(R-r)}\Bigg[&\int_\Omega p(x)\left(\frac{q(x)}{p(x)} - r\right)^{n+2} \\
&\times\left(\int_0^1 f^{(n+1)}\left((1-s)\,r + s\frac{q(x)}{p(x)}\right)(1-s)^n\, ds\right) d\mu(x) \\
&+ (-1)^{n+1}\int_\Omega p(x)\left(R - \frac{q(x)}{p(x)}\right)^{n+2} \\
&\times\left(\int_0^1 f^{(n+1)}\left((1-s)\frac{q(x)}{p(x)} + sR\right)s^n ds\right) d\mu(x)\Bigg].
\end{aligned} \tag{3.5}$$

Proof. From the equality (2.15) we have for $t=\frac{q(x)}{p(x)}$, $a=r$ and $b=R$ that

$$\begin{aligned}
&f\left(\frac{q\left(x\right)}{p\left(x\right)}\right) \qquad (3.6)\\
&=\frac{1}{R-r}\left[\left(\frac{q\left(x\right)}{p\left(x\right)}-r\right)f\left(r\right)+\left(R-\frac{q\left(x\right)}{p\left(x\right)}\right)f\left(R\right)\right]\\
&+\frac{1}{R-r}\\
&\times\sum_{k=1}^{n}\frac{1}{k!}\left\{\left(\frac{q\left(x\right)}{p\left(x\right)}-r\right)^{k+1}f^{(k)}\left(r\right)+\left(-1\right)^{k}\left(R-\frac{q\left(x\right)}{p\left(x\right)}\right)^{k+1}f^{(k)}\left(R\right)\right\} \qquad (3.7)\\
&+P_{n}\left(\frac{q\left(x\right)}{p\left(x\right)},r,R\right),
\end{aligned}$$

where

$$\begin{aligned}
&P_{n}\left(\frac{q\left(x\right)}{p\left(x\right)},r,R\right) \qquad (3.8)\\
&=\frac{1}{n!\left(R-r\right)}\left[\left(\frac{q\left(x\right)}{p\left(x\right)}-r\right)^{n+2}\int_{0}^{1}f^{(n+1)}\left(\left(1-s\right)r+s\frac{q\left(x\right)}{p\left(x\right)}\right)\left(1-s\right)^{n}ds\right.\\
&\left.+\left(-1\right)^{n+1}\left(R-\frac{q\left(x\right)}{p\left(x\right)}\right)^{n+2}\int_{0}^{1}f^{(n+1)}\left(\left(1-s\right)\frac{q\left(x\right)}{p\left(x\right)}+sR\right)s^{n}ds\right],
\end{aligned}$$

and $x\in\Omega$.

If we multiply (3.6) by $p(x)$ and integrate on Ω, then we get

$$\begin{aligned}
&\int_{\Omega}p\left(x\right)f\left(\frac{q\left(x\right)}{p\left(x\right)}\right)d\mu\left(x\right) \qquad (3.9)\\
&=\frac{1}{R-r}\int_{\Omega}\left[\left(q\left(x\right)-rp\left(x\right)\right)f\left(r\right)+\left(Rp\left(x\right)-q\left(x\right)\right)f\left(R\right)\right]d\mu\left(x\right)\\
&+\frac{1}{R-r}\sum_{k=1}^{n}\frac{1}{k!}\left\{f^{(k)}\left(r\right)\int_{\Omega}p\left(x\right)\left(\frac{q\left(x\right)}{p\left(x\right)}-r\right)^{k+1}d\mu\left(x\right)\right.\\
&\left.+\left(-1\right)^{k}f^{(k)}\left(R\right)\int_{\Omega}p\left(x\right)\left(R-\frac{q\left(x\right)}{p\left(x\right)}\right)^{k+1}d\mu\left(x\right)\right\}+Q_{f,n}\left(p,q\right),\\
&=\frac{\left(1-r\right)f\left(r\right)+\left(R-1\right)f\left(R\right)}{R-r}\\
&+\frac{1}{R-r}\sum_{k=1}^{n}\frac{1}{k!}\left\{f^{(k)}\left(r\right)\int_{\Omega}p\left(x\right)\left(\frac{q\left(x\right)}{p\left(x\right)}-r\right)^{k+1}d\mu\left(x\right)\right.\\
&\left.+\left(-1\right)^{k}f^{(k)}\left(R\right)\int_{\Omega}p\left(x\right)\left(R-\frac{q\left(x\right)}{p\left(x\right)}\right)^{k+1}d\mu\left(x\right)\right\}+Q_{f,n}\left(p,q\right),
\end{aligned}$$

where

$$\begin{aligned}
Q_{f,n}(p,q) &= \int_\Omega p(x)\,P_n\left(\frac{q(x)}{p(x)},r,R\right)d\mu(x)\\
&= \frac{1}{n!\,(R-r)}\left[\int_\Omega p(x)\left(\frac{q(x)}{p(x)}-r\right)^{n+2}\right.\\
&\times\left(\int_0^1 f^{(n+1)}\left((1-s)\,r+s\frac{q(x)}{p(x)}\right)(1-s)^n\,ds\right)d\mu(x)\\
&+(-1)^{n+1}\int_\Omega p(x)\left(R-\frac{q(x)}{p(x)}\right)^{n+2}\\
&\times\left.\left(\int_0^1 f^{(n+1)}\left((1-s)\frac{q(x)}{p(x)}+sR\right)s^n ds\right)d\mu(x)\right].
\end{aligned}$$

□

Corollary 2. *With the assumptions of Theorem 4 and if $f^{(n+1)}\in L_\infty[r,R]$, then we have the following bounds for the reminder*

$$\begin{aligned}
&|Q_{f,n}(p,q)| \qquad (3.10)\\
&\leq \frac{1}{(n+1)!\,(R-r)}\left[\int_\Omega p(x)\left(\frac{q(x)}{p(x)}-r\right)^{n+2}\left\|f^{(n+1)}\right\|_{\left[\frac{q(x)}{p(x)},R\right],\infty}d\mu(x)\right.\\
&+\left.\int_\Omega p(x)\left(R-\frac{q(x)}{p(x)}\right)^{n+2}\left\|f^{(n+1)}\right\|_{\left[\frac{q(x)}{p(x)},R\right],\infty}d\mu(x)\right]\\
&\leq \frac{1}{(n+1)!\,(R-r)}\left\|f^{(n+1)}\right\|_{[r,R],\infty}\left[D_{\chi^{n+2},r}(p,q)+D_{R,\chi^{n+2}}(p,q)\right]\\
&\leq \frac{2}{(n+1)!}\left\|f^{(n+1)}\right\|_{[r,R],\infty}(R-r)^{n+1}
\end{aligned}$$

Proof. From (3.5) we have

$$\begin{aligned}
|Q_{f,n}(p,q)| &\leq \frac{1}{n!\,(R-r)}\left[\int_\Omega p(x)\left(\frac{q(x)}{p(x)}-r\right)^{n+2}\right.\\
&\times\left|\int_0^1 f^{(n+1)}\left((1-s)\,r+s\frac{q(x)}{p(x)}\right)(1-s)^n\,ds\right|d\mu(x)\\
&+\int_\Omega p(x)\left(R-\frac{q(x)}{p(x)}\right)^{n+2}\\
&\times\left.\left|\int_0^1 f^{(n+1)}\left((1-s)\frac{q(x)}{p(x)}+sR\right)s^n ds\right|d\mu(x)\right]
\end{aligned} \qquad (3.11)$$

$$\leq \frac{1}{n!\,(R-r)}\left[\int_{\Omega} p(x)\left(\frac{q(x)}{p(x)}-r\right)^{n+2}\right.$$
$$\times\int_0^1\left|f^{(n+1)}\left((1-s)\,r+s\frac{q(x)}{p(x)}\right)\right|(1-s)^n\,dsd\mu(x)$$
$$+\int_{\Omega} p(x)\left(R-\frac{q(x)}{p(x)}\right)^{n+2}$$
$$\left.\times\int_0^1\left|f^{(n+1)}\left((1-s)\,\frac{q(x)}{p(x)}+sR\right)\right|s^n dsd\mu(x)\right]$$
$$=L_n(p,q)\,.$$

We also have

$$\int_0^1\left|f^{(n+1)}\left((1-s)\,r+s\frac{q(x)}{p(x)}\right)\right|(1-s)^n\,ds$$
$$\leq \operatorname*{essup}_{s\in[0,1]}\left|f^{(n+1)}\left((1-s)\,r+s\frac{q(x)}{p(x)}\right)\right|\int_0^1(1-s)^n\,ds$$
$$=\frac{1}{n+1}\left\|f^{(n+1)}\right\|_{\left[r,\frac{q(x)}{p(x)}\right],\infty}\leq\frac{1}{n+1}\left\|f^{(n+1)}\right\|_{[r,R],\infty}$$

and

$$\int_0^1\left|f^{(n+1)}\left((1-s)\,\frac{q(x)}{p(x)}+sR\right)\right|s^n ds$$
$$\leq \operatorname*{essup}_{s\in[0,1]}\left|f^{(n+1)}\left((1-s)\,\frac{q(x)}{p(x)}+sR\right)\right|\int_0^1 s^n ds$$
$$=\frac{1}{n+1}\left\|f^{(n+1)}\right\|_{\left[\frac{q(x)}{p(x)},R\right],\infty}\leq\frac{1}{n+1}\left\|f^{(n+1)}\right\|_{[r,R],\infty}$$

for $x\in\Omega$.

Therefore,

$$L_n(p,q)$$
$$\leq\frac{1}{(n+1)!\,(R-r)}\left[\int_{\Omega} p(x)\left(\frac{q(x)}{p(x)}-r\right)^{n+2}\left\|f^{(n+1)}\right\|_{\left[\frac{q(x)}{p(x)},R\right],\infty}d\mu(x)\right.$$
$$\left.+\int_{\Omega} p(x)\left(R-\frac{q(x)}{p(x)}\right)^{n+2}\left\|f^{(n+1)}\right\|_{\left[\frac{q(x)}{p(x)},R\right],\infty}d\mu(x)\right]$$
$$\leq\frac{1}{(n+1)!\,(R-r)}\left\|f^{(n+1)}\right\|_{[r,R],\infty}$$
$$\times\left[\int_{\Omega} p(x)\left(\frac{q(x)}{p(x)}-r\right)^{n+2}d\mu(x)+\int_{\Omega} p(x)\left(R-\frac{q(x)}{p(x)}\right)^{n+2}d\mu(x)\right]$$

$$
\begin{aligned}
&= \frac{1}{(n+1)!\,(R-r)} \left\| f^{(n+1)} \right\|_{[r,R],\infty} \left[D_{\chi^{n+1},r}(p,q) + D_{R,\chi^{n+2}}(p,q) \right] \\
&\le \frac{2}{(n+1)!\,(R-r)} \left\| f^{(n+1)} \right\|_{[r,R],\infty} (R-r)^{n+2} \\
&= \frac{2}{(n+1)!} \left\| f^{(n+1)} \right\|_{[r,R],\infty} (R-r)^{n+1}.
\end{aligned}
$$

By making use of (3.11) we get the desired result (3.10). □

We consider the divergence measures

$$
D_{\chi^{n+2+1/s},r}(p,q) := \int_{\Omega.} \frac{(q(x) - rp(x))^{n+2+1/s}}{p^{n+1/s}(x)} d\mu(x) \ge 0 \text{ for } n \in \mathbb{N},\ s > 1 \tag{3.12}
$$

and

$$
\begin{aligned}
&D_{R,\chi^{n+2+1/s}}(p,q) \\
&:= \int_{\Omega.} \frac{(Rp(x) - q(x))^{n+2+1/s}}{p^{n+1/s}(x)} d\mu(x) \ge 0 \text{ for } n \in \mathbb{N},\ s > 1.
\end{aligned}
\tag{3.13}
$$

Corollary 3. *With the assumptions of Theorem 4 and if $f^{(n+1)} \in L_s[r,R]$, with $s,\ q > 1$, and $\frac{1}{s} + \frac{1}{q} = 1$, then we have the following bounds for the reminder*

$$
\begin{aligned}
&|Q_{f,n}(p,q)| \\
&\le \frac{1}{(qn+1)^{1/q}\, n!\,(R-r)} \\
&\times \left[\int_\Omega p(x) \left(\frac{q(x)}{p(x)} - r \right)^{n+2+1/s} \left\| f^{(n+1)} \right\|_{\left[r,\frac{q(x)}{p(x)}\right],s} d\mu(x) \right. \\
&\left. + \int_\Omega p(x) \left(R - \frac{q(x)}{p(x)} \right)^{n+2+1/s} \left\| f^{(n+1)} \right\|_{\left[\frac{q(x)}{p(x)},R\right],s} d\mu(x) \right] \\
&\le \frac{1}{(qn+1)^{1/q}\, n!\,(R-r)} \left\| f^{(n+1)} \right\|_{[r,R],s} \\
&\times \left[D_{\chi^{n+2+1/s},r}(p,q) + D_{R,\chi^{n+2+1/s}}(p,q) \right] \\
&\le \frac{2}{(qn+1)^{1/q}\, n!} \left\| f^{(n+1)} \right\|_{[r,R],s} (R-r)^{n+1+1/s}.
\end{aligned}
\tag{3.14}
$$

Proof. Using Hölder's integral inequality for $s,\ q > 1$ and $\frac{1}{s} + \frac{1}{q} = 1$, we have

$$\begin{aligned}
&\int_0^1 \left| f^{(n+1)}\left((1-\tau) r + \tau \frac{q(x)}{p(x)}\right)\right| (1-\tau)^n \, d\tau \\
&\leq \left(\int_0^1 \left| f^{(n+1)}\left((1-\tau) r + \tau \frac{q(x)}{p(x)}\right)\right|^s ds\right)^{1/s} \left(\int_0^1 (1-\tau)^{qn} \, d\tau\right)^{1/q} \\
&= \left(\left(\frac{q(x)}{p(x)} - r\right) \int_r^{\frac{q(x)}{p(x)}} \left| f^{(n+1)}(u)\right|^s du\right)^{1/s} \left(\frac{1}{qn+1}\right)^{1/q} \\
&= \frac{1}{(qn+1)^{1/q}} \left(\frac{q(x)}{p(x)} - r\right)^{1/s} \left\| f^{(n+1)}\right\|_{\left[r, \frac{q(x)}{p(x)}\right], s} \\
&\leq \frac{1}{(qn+1)^{1/q}} \left(\frac{q(x)}{p(x)} - r\right)^{1/s} \left\| f^{(n+1)}\right\|_{[r,R], s}
\end{aligned}$$

and, similarly

$$\begin{aligned}
&\int_0^1 \left| f^{(n+1)}\left((1-\tau) \frac{q(x)}{p(x)} + \tau R\right)\right| \tau^n d\tau \\
&\qquad \leq \frac{1}{(qn+1)^{1/q}} \left(R - \frac{q(x)}{p(x)}\right)^{1/s} \left\| f^{(n+1)}\right\|_{\left[\frac{q(x)}{p(x)}, R\right], s} \\
&\qquad\qquad \leq \frac{1}{(qn+1)^{1/q}} \left(R - \frac{q(x)}{p(x)}\right)^{1/s} \left\| f^{(n+1)}\right\|_{[r,R], s}
\end{aligned}$$

for $x \in \Omega$.

Therefore

$$\begin{aligned}
L_n(p,q) \leq{}& \frac{1}{n!(R-r)} \left[\int_\Omega p(x) \left(\frac{q(x)}{p(x)} - r\right)^{n+2}\right. \\
&\times \frac{1}{(qn+1)^{1/q}} \left(\frac{q(x)}{p(x)} - r\right)^{1/s} \left\| f^{(n+1)}\right\|_{\left[r, \frac{q(x)}{p(x)}\right], s} d\mu(x) \\
&+ \int_\Omega p(x) \left(R - \frac{q(x)}{p(x)}\right)^{n+2} \\
&\left.\times \frac{1}{(qn+1)^{1/q}} \left(R - \frac{q(x)}{p(x)}\right)^{1/s} \left\| f^{(n+1)}\right\|_{\left[\frac{q(x)}{p(x)}, R\right], s} d\mu(x)\right] \\
={}& \frac{1}{(qn+1)^{1/q} n!(R-r)} \\
&\times \left[\int_\Omega p(x) \left(\frac{q(x)}{p(x)} - r\right)^{n+2+1/s} \left\| f^{(n+1)}\right\|_{\left[r, \frac{q(x)}{p(x)}\right], s} d\mu(x)\right. \\
&\left.+ \int_\Omega p(x) \left(R - \frac{q(x)}{p(x)}\right)^{n+2+1/s} \left\| f^{(n+1)}\right\|_{\left[\frac{q(x)}{p(x)}, R\right], s} d\mu(x)\right]
\end{aligned}$$

$$\leq \frac{1}{(qn+1)^{1/q} n! (R-r)} \left\| f^{(n+1)} \right\|_{[r,R],s}$$
$$\times \left[\int_\Omega p(x) \left(\frac{q(x)}{p(x)} - r \right)^{n+2+1/s} d\mu(x) + \int_\Omega p(x) \left(R - \frac{q(x)}{p(x)} \right)^{n+2+1/s} d\mu(x) \right],$$

which proves (3.14). □

4. Generalized Trapezoid Type Estimates

Assume that $p,\ q \in \mathcal{P}$ and there exists the constants $0 < r < 1 < R < \infty$ such that

$$r \leq \frac{q(x)}{p(x)} \leq R \text{ for } \mu\text{-a.e. } x \in \Omega. \tag{4.1}$$

We consider the following divergence measures

$$D_{\Phi^k,r,R}(p,q) := \int_{\Omega.} \frac{(Rp(x) - q(x))(q(x) - rp(x))^k}{p^k(x)} d\mu(x) \geq 0 \text{ for } k \in \mathbb{N}, \tag{4.2}$$

and

$$D_{\Psi^k,r,R}(p,q) := \int_{\Omega.} \frac{(Rp(x) - q(x))^k (q(x) - rp(x))}{p^k(x)} d\mu(x) \geq 0 \text{ for } k \in \mathbb{N}. \tag{4.3}$$

We have the following approximation of the divergence measure using a generalized trapezoid rule:

Theorem 5. *Let I be an open interval with $[r,R] \subset I$ as above, $f : I \to \mathbb{C}$ be n-time differentiable function on I and $f^{(n)}$, with $n \geq 1$, be locally absolutely continuous on I. Then for any $p,\ q \in \mathcal{P}$ satisfying the condition (3.1) we have the representation*

$$\begin{aligned} I_f(p,q) &= \frac{(R-1) f(r) + (1-r) f(R)}{R-r} \\ &+ \frac{1}{R-r} \sum_{k=1}^{n} \frac{1}{k!} \left[f^{(k)}(r) D_{\Phi^k,r,R}(p,q) + (-1)^k f^{(k)}(R) D_{\Psi^k,r,R}(p,q) \right] \\ &+ T_{f,n}(p,q) \end{aligned} \tag{4.4}$$

and the reminder $T_{f,n}(p,q)$ is given by

$$\begin{aligned} T_{f,n}(p,q) &= \int_\Omega p(x) L_n\left(\frac{q(x)}{p(x)}, r, R \right) d\mu(x) \\ &= \frac{1}{n!(R-r)} \left[\int_\Omega p(x) \left(R - \frac{q(x)}{p(x)} \right) \left(\frac{q(x)}{p(x)} - r \right)^{n+1} \right. \\ &\times \left(\int_0^1 f^{(n+1)}\left((1-s) r + s\frac{q(x)}{p(x)} \right) (1-s)^n ds \right) d\mu(x) \\ &+ (-1)^{n+1} \int_\Omega p(x) \left(R - \frac{q(x)}{p(x)} \right)^{n+1} \left(\frac{q(x)}{p(x)} - r \right) \\ &\times \left. \left(\int_0^1 f^{(n+1)}\left((1-s) \frac{q(x)}{p(x)} + sR \right) s^n ds \right) d\mu(x) \right]. \end{aligned} \tag{4.5}$$

Proof. We use the identity 2.14 in Lemma 3 in the following form

$$\begin{aligned} f(t) &= \frac{1}{b-a}\left[(b-t) f(a) + (t-a) f(b)\right] \\ &+ \frac{1}{b-a}\sum_{k=1}^{n}\frac{1}{k!}\left\{(b-t)(t-a)^k f^{(k)}(a) + (-1)^k (b-t)^k (t-a) f^{(k)}(b)\right\} \\ &+ L_n(t,a,b), \end{aligned}$$

where

$$\begin{aligned} L_n(t,a,b) := \frac{1}{n!\,(b-a)} &\left[(b-t)(t-a)^{n+1}\int_0^1 f^{(n+1)}((1-s)a+st)(1-s)^n\,ds\right. \\ &\left. + (-1)^{n+1}(b-t)^{n+1}(t-a)\int_0^1 f^{(n+1)}((1-s)t+sb)\,s^n ds\right] \end{aligned}$$

If we take in these equalities $t = \frac{q(x)}{p(x)}$, $a = r$ and $b = R$, then we get

$$\begin{aligned} & f\left(\frac{q(x)}{p(x)}\right) \\ &= \frac{1}{R-r}\left[\left(R - \frac{q(x)}{p(x)}\right) f(r) + \left(\frac{q(x)}{p(x)} - r\right) f(R)\right] \\ &+ \frac{1}{R-r}\sum_{k=1}^{n}\frac{1}{k!}\left[\left(R - \frac{q(x)}{p(x)}\right)\left(\frac{q(x)}{p(x)} - r\right)^k f^{(k)}(r)\right. \\ &\left. + (-1)^k\left(R - \frac{q(x)}{p(x)}\right)^k\left(\frac{q(x)}{p(x)} - r\right) f^{(k)}(R)\right] + L_n\left(\frac{q(x)}{p(x)}, r, R\right), \end{aligned}$$

where

$$\begin{aligned} & L_n\left(\frac{q(x)}{p(x)}, r, R\right) \\ &:= \frac{1}{n!\,(R-r)}\left[\left(R - \frac{q(x)}{p(x)}\right)\left(\frac{q(x)}{p(x)} - r\right)^{n+1}\right. \\ &\times \int_0^1 f^{(n+1)}\left((1-s)r + s\frac{q(x)}{p(x)}\right)(1-s)^n\,ds \\ &+ (-1)^{n+1}\left(R - \frac{q(x)}{p(x)}\right)^{n+1}\left(\frac{q(x)}{p(x)} - r\right) \\ &\left. \times \int_0^1 f^{(n+1)}\left((1-s)\frac{q(x)}{p(x)} + sR\right)s^n ds\right] \end{aligned}$$

and $x \in \Omega$.

If we multiply (3.6) by $p(x)$ and integrate on Ω, then we get

$$\begin{aligned}
&\int_\Omega p(x) f\left(\frac{q(x)}{p(x)}\right) d\mu(x) \\
&= \frac{1}{R-r}\int_\Omega \left[(Rp(x)-q(x)) f(r) + (q(x)-rp(x)) f(R)\right] d\mu(x) \\
&+ \frac{1}{R-r}\sum_{k=1}^{n}\frac{1}{k!}\left[f^{(k)}(r)\int_\Omega p(x)\left(R-\frac{q(x)}{p(x)}\right)\left(\frac{q(x)}{p(x)}-r\right)^k d\mu(x)\right. \\
&\left.+ (-1)^k f^{(k)}(R)\int_\Omega p(x)\left(R-\frac{q(x)}{p(x)}\right)^k\left(\frac{q(x)}{p(x)}-r\right) d\mu(x)\right] \\
&+ T_{f,n}(p,q) \\
&= \frac{(R-1) f(r) + (1-r) f(R)}{R-r} \\
&+ \frac{1}{R-r}\sum_{k=1}^{n}\frac{1}{k!}\left[f^{(k)}(r) D_{\Phi^k,r,R}(p,q) + (-1)^k f^{(k)}(R) D_{\Psi^k,r,R}(p,q)\right] \\
&+ T_{f,n}(p,q)
\end{aligned}$$

where

$$\begin{aligned}
T_{f,n}(p,q) &= \int_\Omega p(x) L_n\left(\frac{q(x)}{p(x)}, r, R\right) d\mu(x) \\
&= \frac{1}{n!(R-r)}\left[\int_\Omega p(x)\left(R-\frac{q(x)}{p(x)}\right)\left(\frac{q(x)}{p(x)}-r\right)^{n+1}\right. \\
&\times \left(\int_0^1 f^{(n+1)}\left((1-s) r + s\frac{q(x)}{p(x)}\right)(1-s)^n ds\right) d\mu(x) \\
&+ (-1)^{n+1}\int_\Omega p(x)\left(R-\frac{q(x)}{p(x)}\right)^{n+1}\left(\frac{q(x)}{p(x)}-r\right) \\
&\left.\times \left(\int_0^1 f^{(n+1)}\left((1-s)\frac{q(x)}{p(x)} + sR\right) s^n ds\right) d\mu(x)\right],
\end{aligned}$$

which proves the theorem. □

Corollary 4. *With the assumptions of Theorem 4 and if $f^{(n+1)} \in L_\infty[r,R]$, then we have the following bounds for the reminder*

$$\begin{aligned}
&|T_{f,n}(p,q)| \\
&\le \frac{1}{(n+1)!(R-r)} \\
&\times\left[\int_\Omega p(x)\left(R-\frac{q(x)}{p(x)}\right)\left(\frac{q(x)}{p(x)}-r\right)^{n+1}\left\|f^{(n+1)}\right\|_{\left[r,\frac{q(x)}{p(x)}\right],\infty}\right. \\
&\left.+\int_\Omega p(x)\left(R-\frac{q(x)}{p(x)}\right)^{n+1}\left(\frac{q(x)}{p(x)}-r\right)\left\|f^{(n+1)}\right\|_{\left[\frac{q(x)}{p(x)},R\right],\infty}\right]
\end{aligned} \tag{4.6}$$

$$
\begin{aligned}
&\leq \frac{1}{(n+1)!\,(R-r)}\left\|f^{(n+1)}\right\|_{[r,R],\infty}\left[D_{\Phi^{n+1},r,R}(p,q)+D_{\Psi^{n+1},r,R}(p,q)\right]\\
&\leq \frac{1}{4\,(n+1)!}(R-r)\left\|f^{(n+1)}\right\|_{[r,R],\infty}\left[D_{\chi^{n},r}(p,q)+D_{R,\chi^{n}}(p,q)\right]\\
&\leq \frac{1}{2\,(n+1)!}(R-r)^{n+1}\left\|f^{(n+1)}\right\|_{[r,R],\infty}.
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
&\left|T_{f,n}(p,q)\right|\\
&\leq \frac{1}{n!\,(R-r)}\left[\int_{\Omega}p(x)\left(R-\frac{q(x)}{p(x)}\right)\left(\frac{q(x)}{p(x)}-r\right)^{n+1}\right.\\
&\times\left|\int_0^1 f^{(n+1)}\left((1-s)\,r+s\frac{q(x)}{p(x)}\right)(1-s)^n\,ds\right|d\mu(x)\\
&+\int_{\Omega}p(x)\left(R-\frac{q(x)}{p(x)}\right)^{n+1}\left(\frac{q(x)}{p(x)}-r\right)\\
&\left.\times\left|\int_0^1 f^{(n+1)}\left((1-s)\frac{q(x)}{p(x)}+sR\right)s^n ds\right|d\mu(x)\right]\\
&\leq \frac{1}{(n+1)!\,(R-r)}\\
&\times\left[\int_{\Omega}p(x)\left(R-\frac{q(x)}{p(x)}\right)\left(\frac{q(x)}{p(x)}-r\right)^{n+1}\left\|f^{(n+1)}\right\|_{\left[r,\frac{q(x)}{p(x)}\right],\infty}\right.\\
&\left.+\int_{\Omega}p(x)\left(R-\frac{q(x)}{p(x)}\right)^{n+1}\left(\frac{q(x)}{p(x)}-r\right)\left\|f^{(n+1)}\right\|_{\left[\frac{q(x)}{p(x)},R\right],\infty}\right]\\
&\leq \frac{1}{(n+1)!\,(R-r)}\left\|f^{(n+1)}\right\|_{[r,R],\infty}\left[D_{\Phi^{n+1},r,R}(p,q)+D_{\Psi^{n+1},r,R}(p,q)\right].
\end{aligned}
$$

Further, by using the elementary inequality

$$
\alpha\beta\leq\frac{1}{4}(\beta-\alpha)^2\,,\ \alpha,\ \beta\geq 0
$$

we have

$$
\begin{aligned}
D_{\Phi^{n+1},r,R}(p,q) &= \int_{\Omega.}p(x)\left(R-\frac{q(x)}{p(x)}\right)\left(\frac{q(x)}{p(x)}-r\right)^{n+1}d\mu(x)\\
&= \int_{\Omega.}p(x)\left(R-\frac{q(x)}{p(x)}\right)\left(\frac{q(x)}{p(x)}-r\right)\left(\frac{q(x)}{p(x)}-r\right)^{n}d\mu(x)\\
&\leq \frac{1}{4}(R-r)^2\int_{\Omega.}p(x)\left(\frac{q(x)}{p(x)}-r\right)^{n}d\mu(x)\\
&= \frac{1}{4}(R-r)^2 D_{\chi^{n},r}(p,q)
\end{aligned}
$$

and

$$\begin{aligned}
D_{\Psi^{n+1},r,R}(p,q) &= \int_{\Omega.} p(x)\left(R-\frac{q(x)}{p(x)}\right)^{n+1}\left(\frac{q(x)}{p(x)}-r\right)d\mu(x) \\
&= \int_{\Omega.} p(x)\left(R-\frac{q(x)}{p(x)}\right)\left(\frac{q(x)}{p(x)}-r\right)\left(R-\frac{q(x)}{p(x)}\right)^{n} d\mu(x) \\
&\leq \frac{1}{4}(R-r)^2\int_{\Omega.} p(x)\left(R-\frac{q(x)}{p(x)}\right)^{n} d\mu(x) \\
&= \frac{1}{4}(R-r)^2 D_{R,\chi^n}(p,q),
\end{aligned}$$

which completes the proof. □

5. Application for Kullback-Leibler Divergence

Consider the logarithmic function $f(t)=-\ln t$, $t>0$. Then

$$I_f(p,q) = -\int_{\Omega} p(x)\ln\left[\frac{q(x)}{p(x)}\right]d\mu(x) = D_{KL}(p,q)$$

for p, $q\in\mathcal{P}$.

We have

$$f^{(k)}(t)=\frac{(-1)^k(k-1)!}{t^k},\ \ k\in\mathbb{N},\ \ k\geq 1$$

and for $[a,b]\subset(0,\infty)$,

$$\left\|f^{(n+1)}\right\|_{[a,b],\infty} := \sup_{t\in[a,b]}\left|f^{(n+1)}(t)\right| = n!\sup_{t\in[a,b]}\left\{\frac{1}{t^{n+1}}\right\} = \frac{n!}{a^{n+1}};$$

and for $\alpha\geq 1$

$$\begin{aligned}
\left\|f^{(n+1)}\right\|_{[a,b],\alpha} &:= \left(\int_a^b\left|f^{(n+1)}(t)\right|^{\alpha}dt\right)^{\frac{1}{\alpha}} = n!\left[\int_a^b\frac{dt}{t^{(n+1)\alpha}}\right]^{\frac{1}{\alpha}} \\
&= n!\left[\frac{b^{(n+1)\alpha-1}-a^{(n+1)\alpha-1}}{[(n+1)\alpha-1]\,b^{(n+1)\alpha-1}a^{(n+1)\alpha-1}}\right]^{\frac{1}{\alpha}}.
\end{aligned}$$

Assume that p, $q\in\mathcal{P}$ and there exists the constants $0<r<1<R<\infty$ such that

$$r\leq\frac{q(x)}{p(x)}\leq R \text{ for } \mu\text{-a.e. } x\in\Omega.$$

Using the identity (3.4) we get

$$\begin{aligned}
D_{KL}(p,q) &= \ln\left[r^{-(1-r)}R^{-(R-1)}\right] \\
&+ \frac{1}{R-r}\sum_{k=1}^{n}\frac{1}{k}\left\{\frac{(-1)^k}{r^k}D_{\chi^{k+1},r}(p,q)+\frac{1}{R^k}D_{R,\chi^{k+1}}(p,q)\right\} \\
&+ Q_{f,n}(p,q)
\end{aligned} \tag{5.1}$$

and the remainder satisfies the inequality (by (3.10))

$$
\begin{aligned}
(5.2)\qquad \left|Q_n(p,q)\right| &\leq \frac{1}{(n+1)\,r^{n+1}(R-r)}\left[D_{\chi^{n+2},r}(p,q)+D_{R,\chi^{n+2}}(p,q)\right] \\
&\leq \frac{2}{(n+1)}\left(\frac{R}{r}-1\right)^{n+1}
\end{aligned}
$$

and, by (3.14), the bound

$$
\begin{aligned}
(5.3)\qquad & \left|Q_n(p,q)\right| \\
&\leq \frac{1}{(qn+1)^{1/q}(R-r)}\left[\frac{R^{(n+1)s-1}-r^{(n+1)s-1}}{[(n+1)s-1]\,R^{(n+1)s-1}r^{(n+1)s-1}}\right]^{\frac{1}{s}} \\
&\quad\times\left[D_{\chi^{n+2+1/s},r}(p,q)+D_{R,\chi^{n+2+1/s}}(p,q)\right] \\
&\leq \frac{2}{(qn+1)^{1/q}}\left[\frac{R^{(n+1)s-1}-r^{(n+1)s-1}}{[(n+1)s-1]\,R^{(n+1)s-1}r^{(n+1)s-1}}\right]^{\frac{1}{s}}(R-r)^{n+1+1/s},
\end{aligned}
$$

where $s,\, q>1$ with $\frac{1}{s}+\frac{1}{q}=1$.

Using the identity (4.4) we have

$$
\begin{aligned}
(5.4)\qquad D_{KL}(p,q) &= \ln\left[r^{-(R-1)}R^{-(1-r)}\right] \\
&\quad+\frac{1}{R-r}\sum_{k=1}^{n}\frac{1}{k}\left[\frac{(-1)^k}{r^k}D_{\Phi^k,r,R}(p,q)+\frac{1}{R^k}D_{\Psi^k,r,R}(p,q)\right] \\
&\quad+T_n(p,q)
\end{aligned}
$$

and the remainder satisfies the inequality (see (4.6))

$$
\begin{aligned}
(5.5)\qquad & \left|T_n(p,q)\right| \\
&\leq \frac{1}{(n+1)\,r^{n+1}(R-r)}\left[D_{\Phi^{n+1},r,R}(p,q)+D_{\Psi^{n+1},r,R}(p,q)\right] \\
&\leq \frac{1}{4(n+1)\,r^{n+1}}(R-r)\left\|f^{(n+1)}\right\|_{[r,R],\infty}\left[D_{\chi^n,r}(p,q)+D_{R,\chi^n}(p,q)\right] \\
&\leq \frac{1}{2(n+1)}\left(\frac{R}{r}-1\right)^{n+1}.
\end{aligned}
$$

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