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INEQUALITIES FOR THE FINITE HILBERT TRANSFORM OF CONVEX FUNCTIONS

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ABSTRACT. In this paper we obtain some new inequalities for the finite Hilbert transform of convex functions. Applications for some particular functions of interest are provided as well.

1. INTRODUCTION

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x,\ y \in \mathring{I}$ and $x < y$, then $f'_{-}(x) \leq f'_{+}(x) \leq f'_{-}(y) \leq f'_{+}(y)$ which shows that both f'_{-} and f'_{+} are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \geq f(a) + (x-a)\varphi(a) \text{ for any } x,\ a \in I. \tag{1.1}$$

It is also well known that if f is convex on I, then ∂f is nonempty, $f'_{-},\ f'_{+} \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_{-}(x) \leq \varphi(x) \leq f'_{+}(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function. If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

Allover this paper, we consider the finite Hilbert transform on the open interval (a,b) defined by

$$(Tf)(a,b;t) := \frac{1}{\pi} PV \int_a^b \frac{f(\tau)}{\tau - t} d\tau := \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b\right] \frac{f(\tau)}{\pi(\tau - t)} d\tau$$

for $t \in (a,b)$ and for various classes of functions f for which the above Cauchy Principal Value integral exists, see [12, Section 3.2] or [16, Lemma II.1.1].

For several recent papers devoted to inequalities for the finite Hilbert transform (Tf), see [2]-[10], [13]-[15] and [17]-[18].

Now, if we assume that the mapping $f : (a,b) \to \mathbb{R}$ is convex on (a,b), then it is locally Lipschitzian on (a,b) and then the finite Hilbert transform of f exists in every point $t \in (a,b)$.

The following result concerning upper and lower bounds for the finite Hilbert transform of a convex function holds.

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Theorem 1 (Dragomir et al., 2001 [1])**.** *Let $f:(a,b)\to\mathbb{R}$ be a convex function on (a,b) and $t\in(a,b)$. Then we have*

$$\begin{aligned}(1.2)\quad & \frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right)+f(t)-f(a)+\varphi(t)(b-t)\right]\\ &\leq (Tf)(a,b;t)\\ &\leq \frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right)+f(b)-f(t)+\varphi(t)(t-a)\right],\end{aligned}$$

where $\varphi(t)\in\left[f'_{-}(t),f'_{+}(t)\right]$, $t\in(a,b)$.

Corollary 1. *Let $f:(a,b)\to\mathbb{R}$ be a differentiable convex function on (a,b) and $t\in(a,b)$. Then we have*

$$\begin{aligned}(1.3)\quad & \frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right)+f(t)-f(a)+f'(t)(b-t)\right]\\ &\leq (Tf)(a,b;t)\\ &\leq \frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right)+f(b)-f(t)+f'(t)(t-a)\right].\end{aligned}$$

We observe that if we take $t=\frac{a+b}{2}$, then we get from (1.3) that

$$\begin{aligned}(1.4)\quad & \frac{1}{\pi}\left[f\left(\frac{a+b}{2}\right)-f(a)+\frac{1}{2}f'\left(\frac{a+b}{2}\right)(b-a)\right]\\ &\leq (Tf)\left(a,b;\frac{a+b}{2}\right)\\ &\leq \frac{1}{\pi}\left[f(b)-f\left(\frac{a+b}{2}\right)+\frac{1}{2}f'\left(\frac{a+b}{2}\right)(b-a)\right].\end{aligned}$$

In this paper we obtain some new inequalities for the finite Hilbert transform of convex functions. Applications for some particular functions of interest are provided as well.

2. Inequalities for Convex Functions

We can prove the following slightly more general result than Theorem 1.

Theorem 2. *Let $f:(a,b)\to\mathbb{R}$ be a convex mapping on (a,b). Then for $t\in(a,b)$ and $\varphi(t),\ \psi(t)\in\left[f'_{-}(t),f'_{+}(t)\right]$ we have*

$$\begin{aligned}(2.1)\quad & \frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right)+f(t)-f(a)+\varphi(t)(b-t)\right]\\ &\leq (Tf)(a,b;t)\\ &\leq \frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right)+f(b)-f(t)+\psi(t)(t-a)\right].\end{aligned}$$

In particular, we have

$$\begin{aligned}(2.2)\qquad &\frac{1}{\pi}\left[f\left(\frac{a+b}{2}\right)-f(a)+\frac{1}{2}\varphi\left(\frac{a+b}{2}\right)(b-a)\right]\\ &\le (Tf)\left(a,b;\frac{a+b}{2}\right)\\ &\le \frac{1}{\pi}\left[f(b)-f\left(\frac{a+b}{2}\right)+\frac{1}{2}\psi\left(\frac{a+b}{2}\right)(b-a)\right].\end{aligned}$$

Proof. The proof is similar to the one from [1]. For the sake of completeness we provide a proof here.

As for the mapping $f:(a,b)\to\mathbb{R}$, $f(t)=1$, $t\in(a,b)$, we have

$$\begin{aligned}(Tf)(a,b;t) &= \frac{1}{\pi}PV\int_a^b \frac{1}{\tau-t}d\tau\\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}\frac{1}{\tau-t}d\tau+\int_{t+\varepsilon}^b\frac{1}{\tau-t}d\tau\right]\\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\ln|\tau-t|\big|_a^{t-\varepsilon}+\ln(\tau-t)\big|_{t+\varepsilon}^b\right]\\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\ln\varepsilon-\ln(t-a)+\ln(b-t)-\ln\varepsilon\right]\\ &= \frac{1}{\pi}\ln\left(\frac{b-t}{t-a}\right),\quad t\in(a,b).\end{aligned}$$

Then, obviously

$$\begin{aligned}(Tf)(a,b;t) &= \frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)+f(t)}{\tau-t}d\tau\\ &= \frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau+\frac{f(t)}{\pi}PV\int_a^b\frac{1}{\tau-t}d\tau\end{aligned}$$

from where we get the equality

$$(2.3)\qquad (Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau$$

for all $t\in(a,b)$.

By the convexity of f we can state that for all $a\le c<d\le b$ we have

$$(2.4)\qquad \frac{f(d)-f(c)}{d-c}\ge\varphi(c),$$

where $\varphi(c)\in\left[f'_-(c),f'_+(c)\right]$.

Using (2.5), we have

$$(2.5)\qquad \int_a^{t-\varepsilon}\frac{f(t)-f(\tau)}{t-\tau}d\tau\ge\int_a^{t-\varepsilon}\varphi(\tau)\,d\tau$$

and

$$(2.6)\qquad \int_{t+\varepsilon}^b\frac{f(\tau)-f(t)}{\tau-t}d\tau\ge\int_{t+\varepsilon}^b l(t)\,d\tau=\varphi(t)(b-t-\varepsilon)$$

and then, by adding (2.5) and (2.6), we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}\frac{f(t)-f(\tau)}{t-\tau}d\tau+\int_{t+\varepsilon}^{b}\frac{f(\tau)-f(t)}{\tau-t}d\tau\right]\\
&\geq \lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}\varphi(\tau)\,d\tau+\varphi(t)(b-t-\varepsilon)\right]\\
&=\int_a^{t}\varphi(\tau)\,d\tau+\varphi(t)(b-t)=f(t)-f(a)+\varphi(t)(b-t)\,.
\end{aligned}$$

Consequently, we have

$$PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau\geq f(t)-f(a)+\varphi(t)(b-t)$$

and by the identity (2.3), we deduce the first inequality in (2.1).

Similarly, by the convexity of f we have for $a\leq c<d\leq b$

$$\psi(d)\geq\frac{f(d)-f(c)}{d-c},\tag{2.7}$$

where $\psi(c)\in\left[f'_{-}(c),f'_{+}(c)\right]$.

Using (2.7) we may state

$$\int_a^{t-\varepsilon}\frac{f(t)-f(\tau)}{t-\tau}d\tau\leq\int_a^{t-\varepsilon}\psi(t)\,d\tau=\psi(t)(t-\varepsilon-a)$$

and

$$\int_{t+\varepsilon}^{b}\frac{f(\tau)-f(t)}{\tau-t}d\tau\leq\int_{t+\varepsilon}^{b}\psi(\tau)\,d\tau=f(b)-f(t+\varepsilon)\,.$$

By adding these inequalities and taking the limit, we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}\frac{f(t)-f(\tau)}{t-\tau}d\tau+\int_{t+\varepsilon}^{b}\frac{f(\tau)-f(t)}{\tau-t}d\tau\right]\\
&\leq\lim_{\varepsilon\to 0+}\left[\psi(t)(t-\varepsilon-a)+f(b)-f(t+\varepsilon)\right]\\
&=\psi(t)(t-a)+f(b)-f(t)\,,
\end{aligned}$$

namely

$$PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau\leq\psi(t)(t-a)+f(b)-f(t)$$

and by the identity (2.3), we deduce the second inequality in (2.1). □

Remark 1. *We observe that for $\psi=\varphi\in\partial f$ we recapture the inequality (2.1). If f is differentiable on (a,b) then we also get (1.3).*

Corollary 2. *Let $f:(a,b)\to\mathbb{R}$ be a convex mapping on (a,b). Then*

$$\begin{aligned}
&\frac{2}{\pi}\left(\frac{1}{b-a}\int_a^b f(t)\,dt-f(a)\right)\\
&\leq\frac{1}{b-a}\int_a^b(Tf)(a,b;t)\,dt-\frac{1}{\pi}\frac{1}{b-a}\int_a^b f(t)\ln\left(\frac{b-t}{t-a}\right)dt\\
&\leq\frac{2}{\pi}\left[f(b)-\frac{1}{b-a}\int_a^b f(t)\,dt\right].
\end{aligned}\tag{2.8}$$

Proof. If we take the integral mean in (2.1), we get

$$\begin{aligned}
(2.9)\quad & \frac{1}{\pi}\left[\frac{1}{b-a}\int_a^b f(t)\ln\left(\frac{b-t}{t-a}\right)+\frac{1}{b-a}\int_a^b\left[f(t)-f(a)+\varphi(t)(b-t)\right]dt\right] \\
& \leq \frac{1}{b-a}\int_a^b (Tf)(a,b;t)\,dt \\
& \leq \frac{1}{\pi}\left[\frac{1}{b-a}\int_a^b f(t)\ln\left(\frac{b-t}{t-a}\right)+\frac{1}{b-a}\int_a^b\left[f(b)-f(t)+\psi(t)(t-a)\right]\right].
\end{aligned}$$

Observe that

$$\begin{aligned}
& \int_a^b\left[f(t)-f(a)+\varphi(t)(b-t)\right]dt \\
& = \int_a^b f(t)\,dt - f(a)(b-a)+\int_a^b f'(t)(b-t)\,dt \\
& = 2\left(\int_a^b f(t)\,dt - f(a)(b-a)\right)
\end{aligned}$$

and

$$\begin{aligned}
& \int_a^b\left[f(b)-f(t)+\psi(t)(t-a)\right] \\
& = f(b)(b-a)-\int_a^b f(t)\,dt+\int_a^b f'(t)(t-a)\,dt \\
& = 2\left(f(b)(b-a)-\int_a^b f(t)\,dt\right)
\end{aligned}$$

and by (2.9) we get the desired result □

We have:

Theorem 3. *Let $f:(a,b)\to\mathbb{R}$ be a convex mapping on (a,b) with finite lateral derivatives $f'_+(a)$ and $f_-(b)$. Then for $t\in(a,b)$ we have*

$$\begin{aligned}
(2.10)\quad \frac{1}{\pi}(b-a)f'_+(a) & \leq \frac{1}{\pi}(b-a)\frac{f(t)-f(a)}{t-a} \\
& \leq (Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) \\
& \leq \frac{1}{\pi}(b-a)\frac{f(b)-f(t)}{b-t}\leq\frac{1}{\pi}(b-a)f_-(b).
\end{aligned}$$

In particular,

$$\begin{aligned}
(2.11)\quad \frac{1}{\pi}(b-a)f'_+(a) & \leq \frac{2}{\pi}(b-a)\frac{f\left(\frac{a+b}{2}\right)-f(a)}{b-a} \\
& \leq (Tf)\left(a,b;\frac{a+b}{2}\right) \\
& \leq \frac{2}{\pi}(b-a)\frac{f(b)-f\left(\frac{a+b}{2}\right)}{b-a}\leq\frac{1}{\pi}(b-a)f_-(b).
\end{aligned}$$

Proof. We recall that if $\Phi : I \to \mathbb{R}$ is a continuous convex function on the interval of real numbers I and $\alpha \in I$ then the divided difference function $\Phi_\alpha : I \setminus \{\alpha\} \to \mathbb{R}$,

$$\Phi_\alpha (t) := [\alpha, t; \Phi] := \frac{\Phi (t) - \Phi (\alpha)}{t - \alpha}$$

is monotonic nondecreasing on $I \setminus \{\alpha\}$.

Using this property for the function $f : (a, b) \to \mathbb{R}$, we have for $t \in (a, b)$ that

$$\frac{f (a) - f (t)}{a - t} \leq \frac{f (\tau) - f (t)}{\tau - t} \leq \frac{f (b) - f (t)}{b - t}$$

for any $\tau \in (a, b)$, $\tau \neq t$.

By the gradient inequality for the convex function f we also have

$$\frac{f (t) - f (a)}{t - a} \geq f'_+ (a) \text{ for } t \in (a, b)$$

and

$$\frac{f (b) - f (t)}{b - t} \leq f_- (b) \text{ for } t \in (a, b).$$

Therefore we have the following inequality

$$f'_+ (a) \leq \frac{f (t) - f (a)}{t - a} \leq \frac{f (\tau) - f (t)}{\tau - t} \leq \frac{f (b) - f (t)}{b - t} \leq f_- (b) \tag{2.12}$$

for t, $\tau \in (a, b)$ and $\tau \neq t$.

If we tale the PV in (2.12), then we get

$$\begin{aligned} f'_+ (a) (b - a) &\leq \frac{f (t) - f (a)}{t - a} (b - a) \\ &\leq PV \int_a^b \frac{f (\tau) - f (t)}{\tau - t} d\tau \\ &\leq \frac{f (b) - f (t)}{b - t} (b - a) \leq f_- (b) (b - a) \end{aligned} \tag{2.13}$$

for $t \in (a, b)$.

Using the equality (2.3) we deduce the desired result (2.10). $\square$

Corollary 3. *With the assumptions in Theorem 3 we have*

$$\begin{aligned} \frac{1}{\pi} (b - a) f'_+ (a) &\leq \frac{1}{\pi} \int_a^b \frac{f (t) - f (a)}{t - a} dt \\ &\leq \frac{1}{b - a} \int_a^b (Tf) (a, b; t) \, dt - \frac{1}{b - a} \int_a^b \frac{f (t)}{\pi} \ln \left(\frac{b - t}{t - a} \right) dt \\ &\leq \frac{1}{\pi} \int_a^b \frac{f (b) - f (t)}{b - t} dt \leq \frac{1}{\pi} (b - a) f_- (b). \end{aligned} \tag{2.14}$$

The proof follows by (2.10) on taking the integral mean over t on $[a, b]$.

Proposition 1. *With the assumptions in Theorem 3, the inequality (2.8) is better than the inequality (2.14). In fact, we have the chain of inequalities*

$$\begin{aligned}(2.15)\quad \frac{1}{\pi}(b-a) f'_{+}(a) &\leq \frac{1}{\pi}\int_a^b \frac{f(t)-f(a)}{t-a}dt \\ &\leq \frac{2}{\pi}\left(\frac{1}{b-a}\int_a^b f(t)\,dt - f(a)\right) \\ &\leq \frac{1}{b-a}\int_a^b (Tf)(a,b;t)\,dt - \frac{1}{b-a}\int_a^b \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)dt \\ &\leq \frac{2}{\pi}\left[f(b) - \frac{1}{b-a}\int_a^b f(t)\,dt\right] \\ &\leq \frac{1}{\pi}\int_a^b \frac{f(b)-f(t)}{b-t} \leq \frac{1}{\pi}(b-a) f_{-}(b).\end{aligned}$$

Proof. We use the following Čebyšev's inequality which states that, if g, h have the same monotonicity (opposite monotonicity) then

$$(2.16)\qquad \frac{1}{b-a}\int_a^b g(t)\,h(t)\,dt \geq (\leq)\,\frac{1}{b-a}\int_a^b g(t)\,dt\frac{1}{b-a}\int_a^b h(t)\,dt.$$

Now, since $g(t) = \frac{f(b)-f(t)}{b-t}$ is nondecreasing on (a,b) and $h(t) = b-t$ is decreasing on $[a,b]$, then by (2.16) we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b \frac{f(b)-f(t)}{b-t}(b-t)\,dt &\leq \frac{1}{b-a}\int_a^b \frac{f(b)-f(t)}{b-t}dt\frac{1}{b-a}\int_a^b (b-t)\,dt \\ &= \frac{1}{2}\int_a^b \frac{f(b)-f(t)}{b-t}dt,\end{aligned}$$

which is equivalent to

$$2\left[f(b) - \frac{1}{b-a}\int_a^b f(t)\,dt\right] \leq \int_a^b \frac{f(b)-f(t)}{b-t}dt,$$

which proves the fifth inequality in (2.15).

Also, since $g(t) = \frac{f(t)-f(a)}{t-a}$ is nondecreasing on (a,b) and $h(t) = t-a$ is increasing on $[a,b]$, then by (2.16) we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b \frac{f(t)-f(a)}{t-a}(t-a)\,dt &\leq \frac{1}{b-a}\int_a^b \frac{f(t)-f(a)}{t-a}dt\frac{1}{b-a}\int_a^b (t-a)\,dt \\ &= \frac{1}{2}\int_a^b \frac{f(t)-f(a)}{t-a}dt,\end{aligned}$$

which proves the second inequality in (2.15). $\square$

We also have:

Theorem 4. *Let $f:(a,b)\to\mathbb{R}$ be a convex mapping on (a,b). Then for $t\in(a,b)$*

$$\begin{aligned}(2.17)\quad & \left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\right. \\ & \qquad\left.-\frac{2}{\pi}\left(\frac{1}{b-t}\int_t^b f(\tau)\,d\tau-\frac{1}{t-a}\int_a^t f(\tau)\,d\tau\right)\right| \\ & \leq\frac{1}{2\pi}(t-a)\left[f_-'(t)-\frac{f(t)-f(a)}{t-a}\right]+\frac{1}{2\pi}(b-t)\left[\frac{f(b)-f(t)}{b-t}-f_+'(t)\right].\end{aligned}$$

In particular,

$$\begin{aligned}(2.18)\quad & \left|(Tf)\left(a,b;\frac{a+b}{2}\right)-\frac{4}{\pi}\left(\frac{1}{b-a}\int_{\frac{a+b}{2}}^b f(\tau)\,d\tau-\frac{1}{b-a}\int_a^{\frac{a+b}{2}} f(\tau)\,d\tau\right)\right| \\ & \leq\frac{1}{4\pi}(b-a)\left[4\frac{\frac{f(b)+f(a)}{2}-f\left(\frac{a+b}{2}\right)}{b-a}-\left(f_+'\left(\frac{a+b}{2}\right)-f_-'\left(\frac{a+b}{2}\right)\right)\right] \\ & \leq\frac{1}{\pi}\left[\frac{f(b)+f(a)}{2}-f\left(\frac{a+b}{2}\right)\right].\end{aligned}$$

Proof. We use Grüss' inequality for integrable functions g, h

$$\begin{aligned}(2.19)\quad & \left|\frac{1}{b-a}\int_a^b g(t)h(t)\,dt-\frac{1}{b-a}\int_a^b g(t)\,dt\frac{1}{b-a}\int_a^b h(t)\,dt\right| \\ & \leq\frac{1}{4}(M-m)(N-n),\end{aligned}$$

provided $m\leq g(t)\leq M$, $n\leq h(t)\leq N$ for almost every $t\in[a,b]$.

Using Grüss' inequality for increasing functions, we have

$$\begin{aligned}(2.20)\quad & \left|\int_a^{t-\varepsilon}\frac{f(\tau)-f(t)}{\tau-t}(\tau-t)\,d\tau\right. \\ & \qquad\left.-\int_a^{t-\varepsilon}\frac{f(\tau)-f(t)}{\tau-t}d\tau\frac{1}{t-\varepsilon-a}\int_a^{t-\varepsilon}(\tau-t)\,d\tau\right| \\ & \qquad\qquad\leq\frac{1}{4}(t-\varepsilon-a)(t-\varepsilon-a)\left[\frac{f(t-\varepsilon)-f(t)}{t-\varepsilon-t}-\frac{f(a)-f(t)}{a-t}\right]\end{aligned}$$

and

$$\begin{aligned}(2.21)\quad & \left|\int_{t+\varepsilon}^b\frac{f(\tau)-f(t)}{\tau-t}(\tau-t)\,d\tau\right. \\ & \qquad\left.-\int_{t+\varepsilon}^b\frac{f(\tau)-f(t)}{\tau-t}d\tau\frac{1}{b-t-\varepsilon}\int_{t+\varepsilon}^b(\tau-t)\,d\tau\right| \\ & \qquad\qquad\leq\frac{1}{4}(b-t-\varepsilon)(b-t-\varepsilon)\left[\frac{f(b)-f(t)}{b-t}-\frac{f(t+\varepsilon)-f(t)}{t+\varepsilon-t}\right]\end{aligned}$$

where $t\in(a,b)$ and for small $\varepsilon>0$.

We have

$$\int_a^{t-\varepsilon} \frac{f(\tau)-f(t)}{\tau-t}(\tau-t)\,d\tau = \int_a^{t-\varepsilon}(f(\tau)-f(t))\,d\tau = \int_a^{t-\varepsilon} f(\tau)\,d\tau - f(t)(t-\varepsilon-a)$$

and

$$\int_{t+\varepsilon}^{b} \frac{f(\tau)-f(t)}{\tau-t}(\tau-t)\,d\tau = \int_{t+\varepsilon}^{b}(f(\tau)-f(t))\,d\tau = \int_{t+\varepsilon}^{b} f(\tau)\,d\tau - f(t)(b-t-\varepsilon).$$

Also

$$\frac{1}{t-\varepsilon-a}\int_a^{t-\varepsilon}(\tau-t)\,d\tau = \frac{\varepsilon^2-(a-t)^2}{2(t-\varepsilon-a)} = -\frac{(t-a+\varepsilon)}{2}$$

and

$$\frac{1}{b-t-\varepsilon}\int_{t+\varepsilon}^{b}(\tau-t)\,d\tau = \frac{(b-t)^2-\varepsilon^2}{2(b-t-\varepsilon)} = \frac{(b-t+\varepsilon)}{2}.$$

From (2.20) we get

$$\left|\int_a^{t-\varepsilon} f(\tau)\,d\tau - f(t)(t-\varepsilon-a) + \frac{t-a+\varepsilon}{2}\int_a^{t-\varepsilon}\frac{f(\tau)-f(t)}{\tau-t}d\tau\right| \le \frac{1}{4}(t-\varepsilon-a)(t-\varepsilon-a)\left[\frac{f(t)-f(t-\varepsilon)}{\varepsilon} - \frac{f(a)-f(t)}{a-t}\right] \tag{2.22}$$

while from (2.21) we get

$$\left|\int_{t+\varepsilon}^{b} f(\tau)\,d\tau - f(t)(b-t-\varepsilon) - \frac{b-t+\varepsilon}{2}\int_{t+\varepsilon}^{b}\frac{f(\tau)-f(t)}{\tau-t}d\tau\right| \le \frac{1}{4}(b-t-\varepsilon)(b-t-\varepsilon)\left[\frac{f(b)-f(t)}{b-t} - \frac{f(t+\varepsilon)-f(t)}{\varepsilon}\right] \tag{2.23}$$

for $t \in (a,b)$ and small $\varepsilon > 0$.

For $t-a > \varepsilon > 0$ we get from (2.22) that

$$\left|\frac{1}{2}\int_a^{t-\varepsilon}\frac{f(\tau)-f(t)}{\tau-t}d\tau + \frac{1}{t-a+\varepsilon}\int_a^{t-\varepsilon} f(\tau)\,d\tau - f(t)\right| \le \frac{1}{4}(t-\varepsilon-a)\left[\frac{f(t)-f(t-\varepsilon)}{\varepsilon} - \frac{f(a)-f(t)}{a-t}\right] \tag{2.24}$$

and from (2.23) for $b-t > \varepsilon > 0$ that

$$\left|\frac{1}{b-t+\varepsilon}\int_{t+\varepsilon}^{b} f(\tau)\,d\tau - f(t)\frac{b-t-\varepsilon}{b-t+\varepsilon} - \frac{1}{2}\int_{t+\varepsilon}^{b}\frac{f(\tau)-f(t)}{\tau-t}d\tau\right| \le \frac{1}{4}\frac{(b-t-\varepsilon)(b-t-\varepsilon)}{b-t+\varepsilon}\left[\frac{f(b)-f(t)}{b-t} - \frac{f(t+\varepsilon)-f(t)}{\varepsilon}\right]$$

or, that

$$\left|\frac{1}{2}\int_{t+\varepsilon}^{b}\frac{f(\tau)-f(t)}{\tau-t}d\tau-\frac{1}{b-t+\varepsilon}\int_{t+\varepsilon}^{b}f(\tau)\,d\tau+f(t)\,\frac{b-t-\varepsilon}{b-t+\varepsilon}\right| \tag{2.25}$$
$$\leq\frac{1}{4}\frac{(b-t-\varepsilon)(b-t-\varepsilon)}{b-t+\varepsilon}\left[\frac{f(b)-f(t)}{b-t}-\frac{f(t+\varepsilon)-f(t)}{\varepsilon}\right].$$

If we add (2.24) and (2.25) and use the triangle inequality, then we get

$$\begin{aligned}&\left|\frac{1}{2}\int_{a}^{t-\varepsilon}\frac{f(\tau)-f(t)}{\tau-t}d\tau+\frac{1}{2}\int_{t+\varepsilon}^{b}\frac{f(\tau)-f(t)}{\tau-t}d\tau\right.\\&\left.+\frac{1}{t-a+\varepsilon}\int_{a}^{t-\varepsilon}f(\tau)\,d\tau-f(t)-\frac{1}{b-t+\varepsilon}\int_{t+\varepsilon}^{b}f(\tau)\,d\tau+f(t)\,\frac{b-t-\varepsilon}{b-t+\varepsilon}\right|\\&\leq\frac{1}{4}(t-\varepsilon-a)\left[\frac{f(t)-f(t-\varepsilon)}{\varepsilon}-\frac{f(a)-f(t)}{a-t}\right]\\&+\frac{1}{4}\frac{(b-t-\varepsilon)(b-t-\varepsilon)}{b-t+\varepsilon}\left[\frac{f(b)-f(t)}{b-t}-\frac{f(t+\varepsilon)-f(t)}{\varepsilon}\right]\end{aligned}$$

for $t\in(a,b)$ and $\min\{t-a,b-t\}>\varepsilon>0$.

Taking the limit over $\varepsilon\to0+$ we get

$$\left|\frac{1}{2}PV\int_{a}^{b}\frac{f(\tau)-f(t)}{\tau-t}d\tau+\frac{1}{t-a}\int_{a}^{t}f(\tau)\,d\tau-\frac{1}{b-t}\int_{t}^{b}f(\tau)\,d\tau\right| \tag{2.26}$$
$$\leq\frac{1}{4}(t-a)\left[f_{-}'(t)-\frac{f(t)-f(a)}{t-a}\right]+\frac{1}{4}(b-t)\left[\frac{f(b)-f(t)}{b-t}-f_{+}'(t)\right]$$

for $t\in(a,b)$.

Using the identity (2.3) we get from (2.26) the desired result (2.17). □

3. Some Examples

If we consider the function $\exp t=e^{t}$, $t\in(a,b)$ a real interval, then

$$(T\exp)(a,b;t)=\frac{\exp(t)}{\pi}\left[E_i(b-t)-E_i(a-t)\right], \tag{3.1}$$

where E_i is defined by

$$E_i(x):=PV\int_{-\infty}^{x}\frac{\exp(s)}{s}ds,\quad x\in\mathbb{R}.$$

Indeed, we have

$$\begin{aligned}E_i(b-t)-E_i(a-t)&=PV\int_{a-t}^{b-t}\frac{\exp(s)}{s}ds=PV\int_{a}^{b}\frac{\exp(\tau-t)}{\tau-t}ds\\&=\exp(-t)\,\pi\,(T\exp)(a,b;t)\end{aligned}$$

and the equality (3.1) is proved.

Now, if we use the inequality (1.3) for the convex function exp on an interval of real numbers (a,b), then we get

$$\begin{aligned}
&\frac{1}{\pi}\left[\exp t\ln\left(\frac{b-t}{t-a}\right)+\exp t-\exp a+(b-t)\exp t\right] \\
&\leq \frac{\exp t}{\pi}\left[E_i(b-t)-E_i(a-t)\right] \\
&\leq \frac{1}{\pi}\left[\exp t\ln\left(\frac{b-t}{t-a}\right)+\exp b-\exp t+(t-a)\exp t\right]
\end{aligned} \tag{3.2}$$

for any $t\in(a,b)$.

This is equivalent to

$$\begin{aligned}
\ln\left(\frac{b-t}{t-a}\right)+b-t+1-\exp(a-t) &\leq \left[E_i(b-t)-E_i(a-t)\right] \\
&\leq \ln\left(\frac{b-t}{t-a}\right)+t-a-1+\exp(b-t)
\end{aligned} \tag{3.3}$$

for any $t\in(a,b)$.

Further, if we take $t=\frac{a+b}{2}$ in (3.3), then we get

$$\begin{aligned}
\frac{b-a}{2}+1-\exp\left(-\frac{b-a}{2}\right) &\leq \left[E_i\left(\frac{b-a}{2}\right)-E_i\left(-\frac{b-a}{2}\right)\right] \\
&\leq \frac{b-a}{2}-1+\exp\left(\frac{b-a}{2}\right).
\end{aligned} \tag{3.4}$$

If we take in this inequality $\frac{b-a}{2}=x>0$, then we have

$$-\exp(-x)+x+1\leq E_i(x)-E_i(-x)\leq \exp(x)+x-1 \tag{3.5}$$

for any $x>0$.

From the inequality (3.5) written for the function exp we have

$$\begin{aligned}
\frac{1}{\pi}(b-a)\exp(a) &\leq \frac{1}{\pi}(b-a)\frac{\exp t-\exp a}{t-a} \\
&\leq \frac{\exp t}{\pi}\left[E_i(b-t)-E_i(a-t)\right]-\frac{\exp t}{\pi}\ln\left(\frac{b-t}{t-a}\right) \\
&\leq \frac{1}{\pi}(b-a)\frac{\exp b-\exp t}{b-t}\leq \frac{1}{\pi}(b-a)\exp b,
\end{aligned} \tag{3.6}$$

for any $t\in(a,b)$, which is equivalent to

$$\begin{aligned}
(b-a)\exp(a-t) &\leq (b-a)\frac{1-\exp(a-t)}{t-a} \\
&\leq E_i(b-t)-E_i(a-t)-\ln\left(\frac{b-t}{t-a}\right) \\
&\leq (b-a)\frac{\exp(b-t)-1}{b-t}\leq (b-a)\exp(b-t),
\end{aligned} \tag{3.7}$$

for any $t\in(a,b)$.

If we take $t=\frac{a+b}{2}$ in (3.7), then we get

$$(b-a)\exp\left(-\frac{b-a}{2}\right)\leq(b-a)\frac{1-\exp\left(-\frac{b-a}{2}\right)}{\frac{b-a}{2}}\leq E_i\left(\frac{b-a}{2}\right)-E_i\left(-\frac{b-a}{2}\right)\leq(b-a)\frac{\exp\left(\frac{b-a}{2}\right)-1}{\frac{b-a}{2}}\leq(b-a)\exp\left(\frac{b-a}{2}\right). \tag{3.8}$$

If we take in this inequality $\frac{b-a}{2}=x>0$, then we have

$$2x\exp(-x)\leq 2[1-\exp(-x)]\leq E_i(x)-E_i(-x)\leq 2[\exp(x)-1]\leq 2x\exp(x). \tag{3.9}$$

Using the inequality (2.17) for the convex function exp we get

$$\left|\frac{\exp(t)}{\pi}[E_i(b-t)-E_i(a-t)]-\frac{\exp(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{2}{\pi}\left(\frac{\exp b-\exp t}{b-t}-\frac{\exp t-\exp a}{t-a}\right)\right|\leq\frac{1}{2\pi}(t-a)\left[\exp t-\frac{\exp t-\exp a}{t-a}\right]+\frac{1}{2\pi}(b-t)\left[\frac{\exp b-\exp t}{b-t}-\exp t\right] \tag{3.10}$$

for $t\in(a,b)$.

This can be written in an equivalent form as

$$\left|[E_i(b-t)-E_i(a-t)]-\ln\left(\frac{b-t}{t-a}\right)-2\left(\frac{\exp(b-t)-1}{b-t}-\frac{1-\exp(a-t)}{t-a}\right)\right|\leq\frac{1}{2}(t-a)\left[1-\frac{1-\exp(a-t)}{t-a}\right]+\frac{1}{2}(b-t)\left[\frac{\exp(b-t)-1}{b-t}-1\right] \tag{3.11}$$

for $t\in(a,b)$.

Now, if in (3.11) we take $t=\frac{a+b}{2}$, then we get

$$\left|\left[E_i\left(\frac{b-a}{2}\right)-E_i\left(-\frac{b-a}{2}\right)\right]-2\left(\frac{\exp\left(\frac{b-a}{2}\right)-1}{\frac{b-a}{2}}-\frac{1-\exp\left(-\frac{b-a}{2}\right)}{\frac{b-a}{2}}\right)\right|\leq\frac{1}{2}\left(\frac{b-a}{2}\right)\left[1-\frac{1-\exp\left(-\frac{b-a}{2}\right)}{\frac{b-a}{2}}\right]+\frac{1}{2}\left(\frac{b-a}{2}\right)\left[\frac{\exp\left(\frac{b-a}{2}\right)-1}{\frac{b-a}{2}}-1\right]$$

namely

$$\left|\left[E_i\left(\frac{b-a}{2}\right)-E_i\left(-\frac{b-a}{2}\right)\right]\right.$$
$$\left.-\frac{4}{b-a}\left(\exp\left(\frac{b-a}{2}\right)+\exp\left(-\frac{b-a}{2}\right)-2\right)\right|$$
$$\leq\frac{1}{2}\left[\exp\left(\frac{b-a}{2}\right)+\exp\left(-\frac{b-a}{2}\right)-2\right]. \tag{3.12}$$

If we take $\frac{b-a}{2}=x>0$ in (3.12), then we get

$$\left|\left[E_i(x)-E_i(-x)\right]-\frac{4}{x}\left(\frac{\exp(x)+\exp(-x)}{2}-1\right)\right|$$
$$\leq\frac{\exp(x)+\exp(-x)}{2}-1 \tag{3.13}$$

for any $x>0$.

We denote by $\ell(t)=t$, the identity function.

For the function $\ell^{-1}(t)=\frac{1}{t}$, with $t\in(a,b)\subset(0,\infty)$ we have

$$\begin{aligned}&\left(T\ell^{-1}\right)(a,b;t)\\&=\frac{1}{\pi}PV\int_a^b\frac{\tau^{-1}}{\tau-t}d\tau=\frac{1}{\pi}PV\int_a^b\frac{\tau^{-1}-t^{-1}}{\tau-t}d\tau+\frac{1}{\pi t}PV\int_a^b\frac{1}{\tau-t}d\tau\\&=-\frac{1}{\pi t}\int_a^b\frac{d\tau}{\tau}+\frac{1}{\pi t}\ln\left(\frac{b-t}{t-a}\right)=\frac{1}{\pi t}\ln\left(\frac{b-t}{t-a}\right)-\frac{1}{\pi t}\ln\left(\frac{b}{a}\right).\end{aligned}$$

If we use the inequality (1.3) for the function ℓ^{-1} we get

$$\frac{1}{t}-\frac{1}{a}-\frac{1}{t^2}(b-t)\leq-\frac{1}{t}\ln\left(\frac{b}{a}\right)\leq\frac{1}{b}-\frac{1}{t}-\frac{1}{t^2}(t-a),$$

which gives us

$$2-\frac{t^2+ab}{bt}\leq\ln\left(\frac{b}{a}\right)\leq\frac{t^2+ab}{at}-2 \tag{3.14}$$

for any $t\in(a,b)$.

If we take in (3.14) $t=\sqrt{ab}$, then we get the inequality

$$2\left(\frac{b-\sqrt{ab}}{b}\right)\leq\ln\left(\frac{b}{a}\right)\leq2\left(\frac{\sqrt{ab}-a}{a}\right) \tag{3.15}$$

for $b>a>0$.

From the inequality (2.17) written for ℓ^{-1} we have

$$\left|\frac{1}{t}\ln\left(\frac{b}{a}\right)+2\left(\frac{\ln b-\ln t}{b-t}-\frac{\ln t-\ln a}{t-a}\right)\right|$$
$$\leq\frac{1}{2}(t-a)\left(-\frac{1}{t^2}-\frac{\frac{1}{t}-\frac{1}{a}}{t-a}\right)+\frac{1}{2}(b-t)\left(\frac{\frac{1}{b}-\frac{1}{t}}{b-t}+\frac{1}{t^2}\right), \tag{3.16}$$

which gives us

$$\left|\ln\left(\frac{b}{a}\right)-2t\left(\frac{\ln t-\ln a}{t-a}-\frac{\ln b-\ln t}{b-t}\right)\right|\leq\frac{1}{2t}\left[\frac{(t-a)^2}{a}+\frac{(b-t)^2}{b}\right], \tag{3.17}$$

for any $t \in (a, b)$.

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