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INEQUALITIES FOR THE FINITE HILBERT TRANSFORM OF FUNCTIONS WITH BOUNDED DIVIDED DIFFERENCES

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Abstract. In this paper we establish some inequalities for the finite Hilbert transform of complex valued functions for which the divided differences in any two points of the interval are bounded. Applications for some particular functions of interest are provided as well.

1. Introduction

Allover this paper, we consider the *finite Hilbert transform* on the open interval (a,b) defined by

$$(Tf)(a,b;t):=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)}{\tau-t}d\tau:=\lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right]\frac{f(\tau)}{\pi(\tau-t)}d\tau$$

for $t\in(a,b)$ and for various classes of functions f for which the above Cauchy Principal Value integral exists, see [13, Section 3.2] or [17, Lemma II.1.1].

For several recent papers devoted to inequalities for the finite Hilbert transform (Tf), see [2]-[10], [14]-[16] and [18]-[19].

The following result holds.

Theorem 1 (Dragomir et al., 2001 [1])**.** *Let $f:[a,b]\to\mathbb{R}$ be a monotonic nondecreasing (nonincreasing) function on $[a,b]$. If the finite Hilbert transform $(Tf)(a,b,\cdot)$ exists in every $t\in(a,b)$, then*

$$(Tf)(a,b;t)\geq(\leq)\frac{1}{\pi}f(t)\ln\left(\frac{b-t}{t-a}\right)\tag{1.1}$$

for all $t\in(a,b)$.

The following result can be useful in practice.

Corollary 1. *Let $f:[a,b]\to\mathbb{R}$ and $e:[a,b]\to\mathbb{R}$, $e(t)=t$ such that $f-me$, $Me-f$ are monotonic nondecreasing, where $m<M$ are given real numbers. If $(Tf)(a,b,\cdot)$ exists in every point $t\in(a,b)$, then we have the inequality*

$$\frac{(b-a)m}{\pi}\leq(Tf)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{b-t}{t-a}\right)\leq\frac{(b-a)M}{\pi}\tag{1.2}$$

for all $t\in(a,b)$.

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Remark 1. *If the function f is differentiable on (a,b) the condition that $f-me$, $Me-f$ are monotonic nondecreasing is equivalent with the following more practical condition*

$$m \le f'(t) \le M \quad \text{for all } t \in (a,b). \tag{1.3}$$

From (1.2) we may deduce the following approximation result

$$\left|(Tf)(a,b;t) - \frac{1}{\pi} f(t) \ln\left(\frac{b-t}{t-a}\right) - \frac{M+m}{2\pi}(b-a)\right| \le \frac{M-m}{2\pi}(b-a) \tag{1.4}$$

for all $t \in (a,b)$.

Motivated by the above results, in this paper we establish some inequalities for the finite Hilbert transform of complex valued functions for which the divided differences in any two points of the interval are bounded. Applications for some particular functions of interest are provided as well.

2. Main Results

For a function $f:(a,b) \to \mathbb{C}$ we define the *divided difference*

$$[f;t,s] := \frac{f(t)-f(s)}{t-s} \text{ for } t,\ s \in (a,b),\ t \neq s.$$

Now, for γ, $\Gamma \in \mathbb{C}$ and (a,b) an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{(a,b),d}(\gamma,\Gamma) := \Big\{ f:(a,b) \to \mathbb{C} \mid \operatorname{Re}\Big[(\Gamma - [f;t,s])\Big(\overline{[f;t,s]} - \overline{\gamma}\Big)\Big] \ge 0, \\ \text{for all } t,\ s \in (a,b),\ t \neq s \Big\} \tag{2.1}$$

and

$$\bar{\Delta}_{(a,b),d}(\gamma,\Gamma) := \Big\{ f:(a,b) \to \mathbb{C} \mid \left|[f;t,s] - \frac{\gamma+\Gamma}{2}\right| \le \frac{1}{2}|\Gamma-\gamma| \\ \text{for all } t,\ s \in (a,b),\ t \neq s \Big\}. \tag{2.2}$$

The following representation result may be stated.

Proposition 1. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that $\bar{U}_{(a,b),d}(\gamma,\Gamma)$ and $\bar{\Delta}_{(a,b),d}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{(a,b),d}(\gamma,\Gamma) = \bar{\Delta}_{(a,b),d}(\gamma,\Gamma). \tag{2.3}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left|z - \frac{\gamma+\Gamma}{2}\right| \le \frac{1}{2}|\Gamma-\gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma})\right] \ge 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma-\gamma|^2 - \left|z - \frac{\gamma+\Gamma}{2}\right|^2 = \operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma})\right]$$

that holds for any $z \in \mathbb{C}$.

The equality (2.3) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 2. *For any* $\gamma,\ \Gamma\in\mathbb{C},\ \gamma\neq\Gamma,$*we have that*

$$\begin{aligned}(2.4)\quad \bar{U}_{(a,b),d}(\gamma,\Gamma) &= \{f:(a,b)\to\mathbb{C}\mid\ (\operatorname{Re}\Gamma-\operatorname{Re}[f;t,s])(\operatorname{Re}[f;t,s]-\operatorname{Re}\gamma)\\ &+(\operatorname{Im}\Gamma-\operatorname{Im}[f;t,s])(\operatorname{Im}[f;t,s]-\operatorname{Im}\gamma)\geq 0 \textit{ for all } t,\ s\in(a,b),\ t\neq s\}.\end{aligned}$$

Now, if we assume that $\operatorname{Re}(\Gamma)\geq\operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma)\geq\operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned}(2.5)\quad \bar{S}_{(a,b),d}(\gamma,\Gamma) &:= \{f:(a,b)\to\mathbb{C}\mid\ \operatorname{Re}(\Gamma)\geq\operatorname{Re}[f;t,s]\geq\operatorname{Re}(\gamma)\\ &\text{and } \operatorname{Im}(\Gamma)\geq\operatorname{Im}[f;t,s]\geq\operatorname{Im}(\gamma)\ \text{ for all } t,\ s\in(a,b),\ t\neq s\}.\end{aligned}$$

One can easily observe that $\bar{S}_{(a,b)}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset\neq\bar{S}_{(a,b),d}(\gamma,\Gamma)\subseteq\bar{U}_{(a,b),d}(\gamma,\Gamma). \tag{2.6}$$

The following result holds:

Theorem 2. *Let* $f:(a,b)\to\mathbb{C}$ *be such that for some* $\gamma,\ \Gamma\in\mathbb{C},\ \gamma\neq\Gamma,$ *we have that* $f\in\bar{\Delta}_{(a,b),d}(\gamma,\Gamma)$. *Then we have the inequality*

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{1}{\pi}\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2\pi}|\Gamma-\gamma| \tag{2.7}$$

for any $t\in(a,b)$.

In particular, for $t=\frac{a+b}{2}$ *we obtain*

$$\left|(Tf)\left(a,b;\frac{a+b}{2}\right)-\frac{1}{\pi}\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2\pi}|\Gamma-\gamma|. \tag{2.8}$$

Proof. Since $f\in\bar{\Delta}_{(a,b),d}(\gamma,\Gamma)$ it follows that

$$\left|f(t)-f(s)-\frac{\gamma+\Gamma}{2}(t-s)\right|\leq\frac{1}{2}|\Gamma-\gamma||t-s|$$

for any $t,\ s\in(a,b)$.

By the continuity of the modulus property, we have

$$|f(t)-f(s)|-\left|\frac{\gamma+\Gamma}{2}\right||t-s|\leq\left|f(t)-f(s)-\frac{\gamma+\Gamma}{2}(t-s)\right|\leq\frac{1}{2}|\Gamma-\gamma||t-s|,$$

for any $t,\ s\in(a,b)$, which implies that

$$|f(t)-f(s)|\leq\frac{1}{2}(|\gamma+\Gamma|+|\Gamma-\gamma|)|t-s|$$

for any $t,\ s\in(a,b)$, showing that f is also Lipschitzian on (a,b). Therefore, we conclude that the finite Hilbert transform $T(f)(a,b;t)$ exists for all $t\in(a,b)$, see [13, Section 3.2] or [17, Lemma II.1.1].

For the mapping , $\mathbf{1}(t)=1$, $t\in(a,b)$, we have

$$\begin{aligned}
(T\mathbf{1})(a,b;t) &= \frac{1}{\pi}PV\int_a^b \frac{1}{\tau-t}d\tau \\
&= \frac{1}{\pi}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}\frac{1}{\tau-t}d\tau+\int_{t+\varepsilon}^b\frac{1}{\tau-t}d\tau\right] \\
&= \frac{1}{\pi}\lim_{\varepsilon\to0+}\left[\ln|\tau-t|\,|_a^{t-\varepsilon}+\ln(\tau-t)|_{t+\varepsilon}^b\right] \\
&= \frac{1}{\pi}\lim_{\varepsilon\to0+}\left[\ln\varepsilon-\ln(t-a)+\ln(b-t)-\ln\varepsilon\right] \\
&= \frac{1}{\pi}\ln\left(\frac{b-t}{t-a}\right),\quad t\in(a,b).
\end{aligned}$$

Then, obviously, for $f:(a,b)\to\mathbb{R}$ we have

$$\begin{aligned}
(Tf)(a,b;t) &= \frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)+f(t)}{\tau-t}d\tau \\
&= \frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau+\frac{f(t)}{\pi}PV\int_a^b\frac{1}{\tau-t}d\tau
\end{aligned}$$

from where we get the equality

$$(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau \tag{2.9}$$

for any $t\in(a,b)$.

Since $f\in\bar{\Delta}_{(a,b),d}(\gamma,\Gamma)$, hence

$$\begin{aligned}
&\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{1}{\pi}\frac{\gamma+\Gamma}{2}\right| \\
&=\left|\frac{1}{\pi}PV\int_a^b\left(\frac{f(\tau)-f(t)}{\tau-t}-\frac{\gamma+\Gamma}{2}\right)d\tau\right| \\
&\le\frac{1}{\pi}PV\int_a^b\left|\frac{f(\tau)-f(t)}{\tau-t}-\frac{\gamma+\Gamma}{2}\right|d\tau\le\frac{1}{2}|\Gamma-\gamma|\frac{1}{\pi}PV\int_a^b d\tau \\
&=\frac{1}{2\pi}|\Gamma-\gamma|
\end{aligned}$$

and the inequality (2.7) is thus obtained. □

Remark 2. *We observe that if $f-me$, $Me-f$ are monotonic nondecreasing, where $m<M$ are given real numbers, then we have that $f\in\bar{\Delta}_{(a,b),d}(m,M)$ and from (2.7) we recapture (1.2).*

We need the following technical lemma:

Lemma 1. *Let $f:(a,b)\to\mathbb{C}$ and $t\in(a,b)$. Provided that all integrals below exists, we have for any $\delta\in\mathbb{C}$ that*

$$\begin{aligned}(2.10)\qquad & \int_a^{t-\varepsilon}\frac{f(\tau)-f(t)}{\tau-t}d\tau+\int_{t+\varepsilon}^{b}\frac{f(\tau)-f(t)}{\tau-t}d\tau\\ &+2\left(\frac{1}{t+\varepsilon-a}\int_a^{t-\varepsilon}f(\tau)\,d\tau-\frac{1}{b-t+\varepsilon}\int_{t+\varepsilon}^{b}f(\tau)\,d\tau\right)\\ &+2\left(\frac{b-t-\varepsilon}{b-t+\varepsilon}-\frac{t-\varepsilon-a}{t+\varepsilon-a}\right)f(t)\\ &=\frac{2}{t+\varepsilon-a}\int_a^{t-\varepsilon}\left(\frac{f(\tau)-f(t)}{\tau-t}-\delta\right)\left(\tau-\frac{a+t-\varepsilon}{2}\right)d\tau\\ &-\frac{2}{b-t+\varepsilon}\int_{t+\varepsilon}^{b}\left(\frac{f(\tau)-f(t)}{\tau-t}-\delta\right)\left(\tau-\frac{b+t+\varepsilon}{2}\right)d\tau,\end{aligned}$$

where $\varepsilon>0$ and such that $\min\{t-a,b-t\}>\varepsilon$.

Proof. We have for any $\delta\in\mathbb{C}$ that

$$\begin{aligned}(2.11)\qquad & \int_a^{t-\varepsilon}\frac{f(\tau)-f(t)}{\tau-t}(\tau-t)\,d\tau\\ &-\int_a^{t-\varepsilon}\frac{f(\tau)-f(t)}{\tau-t}d\tau\frac{1}{t-\varepsilon-a}\int_a^{t-\varepsilon}(\tau-t)\,d\tau\\ &=\int_a^{t-\varepsilon}\left(\frac{f(\tau)-f(t)}{\tau-t}-\delta\right)\left(\tau-t-\frac{1}{t-\varepsilon-a}\int_a^{t-\varepsilon}(s-t)\,ds\right)d\tau\\ &=\int_a^{t-\varepsilon}\left(\frac{f(\tau)-f(t)}{\tau-t}-\delta\right)\left(\tau-\frac{a+t-\varepsilon}{2}\right)d\tau\end{aligned}$$

for $t-a>\varepsilon>0$.

Since

$$\int_a^{t-\varepsilon}\frac{f(\tau)-f(t)}{\tau-t}(\tau-t)\,d\tau=\int_a^{t-\varepsilon}f(\tau)\,d\tau-(t-\varepsilon-a)f(t)$$

and

$$\frac{1}{t-\varepsilon-a}\int_a^{t-\varepsilon}(\tau-t)\,d\tau=-\frac{t+\varepsilon-a}{2},$$

then by (2.11) we get

$$\begin{aligned}&\int_a^{t-\varepsilon}f(\tau)\,d\tau-(t-\varepsilon-a)f(t)+\frac{t+\varepsilon-a}{2}\int_a^{t-\varepsilon}\frac{f(\tau)-f(t)}{\tau-t}d\tau\\ &=\int_a^{t-\varepsilon}\left(\frac{f(\tau)-f(t)}{\tau-t}-\delta\right)\left(\tau-\frac{a+t-\varepsilon}{2}\right)d\tau\end{aligned}$$

from where we obtain

$$\begin{aligned}&\frac{2}{t+\varepsilon-a}\int_a^{t-\varepsilon}f(\tau)\,d\tau-2\left(\frac{t-\varepsilon-a}{t+\varepsilon-a}\right)f(t)+\int_a^{t-\varepsilon}\frac{f(\tau)-f(t)}{\tau-t}d\tau\\ &=\frac{2}{t+\varepsilon-a}\int_a^{t-\varepsilon}\left(\frac{f(\tau)-f(t)}{\tau-t}-\delta\right)\left(\tau-\frac{a+t-\varepsilon}{2}\right)d\tau,\end{aligned}$$

namely

$$\int_{a}^{t-\varepsilon}\frac{f(\tau)-f(t)}{\tau-t}d\tau+\frac{2}{t+\varepsilon-a}\int_{a}^{t-\varepsilon}f(\tau)\,d\tau-2\left(\frac{t-\varepsilon-a}{t+\varepsilon-a}\right)f(t) \tag{2.12}$$
$$=\frac{2}{t+\varepsilon-a}\int_{a}^{t-\varepsilon}\left(\frac{f(\tau)-f(t)}{\tau-t}-\delta\right)\left(\tau-\frac{a+t-\varepsilon}{2}\right)d\tau,$$

for $t-a>\varepsilon>0$.

We have for any $\delta\in\mathbb{C}$ that

$$\int_{t+\varepsilon}^{b}\frac{f(\tau)-f(t)}{\tau-t}(\tau-t)\,d\tau-\int_{t+\varepsilon}^{b}\frac{f(\tau)-f(t)}{\tau-t}d\tau\frac{1}{b-\varepsilon-t}\int_{t+\varepsilon}^{b}(\tau-t)\,d\tau \tag{2.13}$$
$$=\int_{t+\varepsilon}^{b}\left(\frac{f(\tau)-f(t)}{\tau-t}-\delta\right)\left(\tau-t-\frac{1}{b-\varepsilon-t}\int_{t+\varepsilon}^{b}(s-t)\,ds\right)d\tau$$
$$=\int_{a}^{t-\varepsilon}\left(\frac{f(\tau)-f(t)}{\tau-t}-\delta\right)\left(\tau-\frac{b+t+\varepsilon}{2}\right)d\tau$$

for $b-t>\varepsilon>0$.

Since

$$\int_{t+\varepsilon}^{b}\frac{f(\tau)-f(t)}{\tau-t}(\tau-t)\,d\tau=\int_{t+\varepsilon}^{b}f(\tau)\,d\tau-(b-t-\varepsilon)f(t)$$

and

$$\frac{1}{b-\varepsilon-t}\int_{t+\varepsilon}^{b}(\tau-t)\,d\tau=\frac{b-t+\varepsilon}{2},$$

then by (2.13) we get

$$\int_{t+\varepsilon}^{b}f(\tau)\,d\tau-(b-t-\varepsilon)f(t)-\frac{b-t+\varepsilon}{2}\int_{t+\varepsilon}^{b}\frac{f(\tau)-f(t)}{\tau-t}d\tau$$
$$=\int_{t+\varepsilon}^{b}\left(\frac{f(\tau)-f(t)}{\tau-t}-\delta\right)\left(\tau-t-\frac{1}{b-\varepsilon-t}\int_{t+\varepsilon}^{b}(s-t)\,ds\right)d\tau$$
$$=\int_{t+\varepsilon}^{b}\left(\frac{f(\tau)-f(t)}{\tau-t}-\delta\right)\left(\tau-\frac{b+t+\varepsilon}{2}\right)d\tau,$$

namely

$$\frac{2}{b-t+\varepsilon}\int_{t+\varepsilon}^{b}f(\tau)\,d\tau-2\left(\frac{b-t-\varepsilon}{b-t+\varepsilon}\right)f(t)-\int_{t+\varepsilon}^{b}\frac{f(\tau)-f(t)}{\tau-t}d\tau$$
$$=\frac{2}{b-t+\varepsilon}\int_{t+\varepsilon}^{b}\left(\frac{f(\tau)-f(t)}{\tau-t}-\delta\right)\left(\tau-\frac{b+t+\varepsilon}{2}\right)d\tau,$$

which gives

$$\int_{t+\varepsilon}^{b}\frac{f(\tau)-f(t)}{\tau-t}d-\frac{2}{b-t+\varepsilon}\int_{t+\varepsilon}^{b}f(\tau)\,d\tau+2\left(\frac{b-t-\varepsilon}{b-t+\varepsilon}\right)f(t) \tag{2.14}$$
$$=-\frac{2}{b-t+\varepsilon}\int_{t+\varepsilon}^{b}\left(\frac{f(\tau)-f(t)}{\tau-t}-\delta\right)\left(\tau-\frac{b+t+\varepsilon}{2}\right)d\tau,$$

for $b-t>\varepsilon>0$.

If we add (2.12) with (2.14) we deduce the desired equality (2.10). $\square$

Theorem 3. *Let $f:(a,b)\to\mathbb{C}$ be such that for some $\gamma,\ \Gamma\in\mathbb{C},\ \gamma\neq\Gamma,$ we have that $f\in\bar{\Delta}_{(a,b),d}(\gamma,\Gamma)$. Then we have the inequality*

$$\begin{aligned}\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\right.\\ \left.-\frac{2}{\pi}\left(\frac{1}{b-t}\int_t^b f(\tau)\,d\tau-\frac{1}{t-a}\int_a^t f(\tau)\,d\tau\right)\right|\\ \leq\frac{1}{4\pi}\left|\Gamma-\gamma\right|(b-a).\end{aligned}\tag{2.15}$$

In particular, we have

$$\begin{aligned}&\left|(Tf)\left(a,b;\frac{a+b}{2}\right)-\frac{4}{\pi}\left(\frac{1}{b-a}\int_{\frac{a+b}{2}}^b f(\tau)\,d\tau-\frac{1}{b-a}\int_a^{\frac{a+b}{2}} f(\tau)\,d\tau\right)\right|\\ &\leq\frac{1}{4\pi}\left|\Gamma-\gamma\right|(b-a).\end{aligned}\tag{2.16}$$

Proof. By using the equality (2.10) for $\delta=\frac{\gamma+\Gamma}{2}$ and the fact that $f\in\bar{\Delta}_{(a,b),d}(\gamma,\Gamma)$, we have for $\min\{t-a,b-t\}>\varepsilon>0$ that

$$\begin{aligned}&\left|\int_a^{t-\varepsilon}\frac{f(\tau)-f(t)}{\tau-t}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)-f(t)}{\tau-t}d\tau\right.\\ &+2\left(\frac{1}{t+\varepsilon-a}\int_a^{t-\varepsilon}f(\tau)\,d\tau-\frac{1}{b-t+\varepsilon}\int_{t+\varepsilon}^b f(\tau)\,d\tau\right)\\ &\left.+2\left(\frac{b-t-\varepsilon}{b-t+\varepsilon}-\frac{t-\varepsilon-a}{t+\varepsilon-a}\right)f(t)\right|\\ &\leq\frac{2}{t+\varepsilon-a}\left|\int_a^{t-\varepsilon}\left(\frac{f(\tau)-f(t)}{\tau-t}-\frac{\gamma+\Gamma}{2}\right)\left(\tau-\frac{a+t-\varepsilon}{2}\right)d\tau\right|\\ &+\frac{2}{b-t+\varepsilon}\left|\int_{t+\varepsilon}^b\left(\frac{f(\tau)-f(t)}{\tau-t}-\frac{\gamma+\Gamma}{2}\right)\left(\tau-\frac{b+t+\varepsilon}{2}\right)d\tau\right|\\ &\leq\frac{2}{t+\varepsilon-a}\int_a^{t-\varepsilon}\left|\left(\frac{f(\tau)-f(t)}{\tau-t}-\frac{\gamma+\Gamma}{2}\right)\left(\tau-\frac{a+t-\varepsilon}{2}\right)\right|d\tau\\ &+\frac{2}{b-t+\varepsilon}\int_{t+\varepsilon}^b\left|\left(\frac{f(\tau)-f(t)}{\tau-t}-\frac{\gamma+\Gamma}{2}\right)\left(\tau-\frac{b+t+\varepsilon}{2}\right)\right|d\tau\end{aligned}\tag{2.17}$$

$$\begin{aligned}&\leq\frac{1}{2}\left|\Gamma-\gamma\right|\\ &\times\left[\frac{2}{t+\varepsilon-a}\int_a^{t-\varepsilon}\left|\tau-\frac{a+t-\varepsilon}{2}\right|d\tau+\frac{2}{b-t+\varepsilon}\int_{t+\varepsilon}^b\left|\tau-\frac{b+t+\varepsilon}{2}\right|d\tau\right]\\ &=\frac{1}{2}\left|\Gamma-\gamma\right|\left[\frac{2}{t+\varepsilon-a}\frac{(t-\varepsilon-a)^2}{4}+\frac{2}{b-t+\varepsilon}\frac{(b-t-\varepsilon)^2}{4}\right]\\ &=\frac{1}{4}\left|\Gamma-\gamma\right|\left[\frac{(t-\varepsilon-a)^2}{t+\varepsilon-a}+\frac{(b-t-\varepsilon)^2}{b-t+\varepsilon}\right].\end{aligned}$$

By taking the limit over $\varepsilon \to 0+$ in (2.17) we get

$$\left| PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau + 2\left(\frac{1}{t-a} \int_a^t f(\tau)\, d\tau - \frac{1}{b-t} \int_t^b f(\tau)\, d\tau \right) \right| \le \frac{1}{4} |\Gamma - \gamma| \left[\frac{(t-a)^2}{t-a} + \frac{(b-t)^2}{b-t} \right] = \frac{1}{4} |\Gamma - \gamma| (b-a)$$

for $t \in (a,b)$ and by (2.9) we deduce the desired result (2.15). □

Corollary 3. *Let $f : (a,b) \to \mathbb{R}$ and $e : (a,b) \to \mathbb{R}$, $e(t) = t$ such that $f - me$, $Me - f$ are monotonic nondecreasing on (a,b), where $m < M$ are given real numbers. Then*

$$\left| (Tf)(a,b;t) - \frac{f(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right) - \frac{2}{\pi} \left(\frac{1}{b-t} \int_t^b f(\tau)\, d\tau - \frac{1}{t-a} \int_a^t f(\tau)\, d\tau \right) \right| \le \frac{1}{4\pi} (M-m)(b-a) \tag{2.18}$$

for all $t \in (a,b)$.

In particular, we have

$$\left| (Tf)\left(a,b;\frac{a+b}{2}\right) - \frac{4}{\pi} \left(\frac{1}{b-a} \int_{\frac{a+b}{2}}^b f(\tau)\, d\tau - \frac{1}{b-a} \int_a^{\frac{a+b}{2}} f(\tau)\, d\tau \right) \right| \le \frac{1}{4\pi} (M-m)(b-a). \tag{2.19}$$

Remark 3. *If the function f is differentiable on (a,b) and satisfies condition $m \le f'(t) \le M$ for all $t \in (a,b)$, then the inequalities (2.18) and (2.19) are valid.*

3. Some Examples

If we consider the function $f(t) = e^t$, $t \in (a,b)$ a real interval, then

$$(Tf)(a,b;t) = \frac{\exp(t)}{\pi} \left[E_i(b-t) - E_i(a-t) \right], \tag{3.1}$$

where E_i is defined by

$$E_i(x) := PV \int_{-\infty}^x \frac{\exp(s)}{s} ds, \quad x \in \mathbb{R}.$$

Indeed, we have

$$E_i(b-t) - E_i(a-t) = PV \int_{a-t}^{b-t} \frac{\exp(s)}{s} ds = PV \int_a^b \frac{\exp(\tau - t)}{\tau - t} ds = \exp(-t)\, \pi\, (T \exp)(a,b;t)$$

and the equality (3.1) is proved.

We have that $f'(t) = e^t$, $t \in (a,b)$, which shows that $m \le \exp(a) \le f'(t) \le \exp(b) = M$.

By utilising (1.4) we have

$$\left|E_i(b-t)-E_i(a-t)-\ln\left(\frac{b-t}{t-a}\right)-\frac{\exp(a-t)+\exp(b-t)}{2}(b-a)\right| \le \frac{\exp(b-t)-\exp(a-t)}{2}(b-a), \tag{3.2}$$

while from (2.18) we get

$$\left|E_i(b-t)-E_i(a-t)-\ln\left(\frac{b-t}{t-a}\right) -2\left(\frac{\exp(b-t)-1}{b-t}-\frac{1-\exp(a-t)}{t-a}\right)\right| \le \frac{1}{4}\left(\exp(b-t)-\exp(a-t)\right)(b-a) \tag{3.3}$$

for $t\in(a,b)$.

If we take in (3.2) and (3.3) $t=\frac{a+b}{2}$, then we get

$$\left|E_i\left(\frac{b-a}{2}\right)-E_i\left(-\frac{b-a}{2}\right)-\frac{\exp\left(-\frac{b-a}{2}\right)+\exp\left(\frac{b-a}{2}\right)}{2}(b-a)\right| \le \frac{1}{2}\left[\exp\left(\frac{b-a}{2}\right)-\exp\left(-\frac{b-a}{2}\right)\right](b-a),$$

and

$$\left|E_i\left(\frac{b-a}{2}\right)-E_i\left(-\frac{b-a}{2}\right)-4\left(\frac{\exp\left(\frac{b-a}{2}\right)+\exp\left(-\frac{b-a}{2}\right)}{b-a}\right)\right| \le \frac{1}{4}\left[\exp\left(\frac{b-a}{2}\right)-\exp\left(-\frac{b-a}{2}\right)\right](b-a)$$

which, by taking $x=\frac{b-a}{2}>0$, gives

$$\left|E_i(x)-E_i(-x)-\left[\exp(-x)+\exp(x)\right]x\right| \le \left[\exp(x)-\exp(-x)\right]x \tag{3.4}$$

and

$$\left|E_i(x)-E_i(-x)-2\left(\frac{\exp(x)+\exp(-x)}{x}\right)\right| \le \frac{1}{2}\left[\exp(x)-\exp(-x)\right]x \tag{3.5}$$

for $x>0$.

For the function $f(t)=\frac{1}{t}$, with $t\in(a,b)\subset(0,\infty)$ we have

$$(Tf)(a,b;t)=\frac{1}{\pi t}\ln\left(\frac{b-t}{t-a}\right)-\frac{1}{\pi t}\ln\left(\frac{b}{a}\right).$$

Since $f'(t)=-\frac{1}{t^2}$, then $m=-\frac{1}{a^2}\le f'(t)\le-\frac{1}{b^2}=M$, then by (1.4) we have

$$\left|\ln\left(\frac{b}{a}\right)-t\frac{b^2+a^2}{2a^2b^2}(b-a)\right| \le t\frac{b+a}{2a^2b^2}(b-a)^2 \tag{3.6}$$

while from (2.18) we get

$$\left|\ln\left(\frac{b}{a}\right)-2t\left(\frac{\ln t-\ln a}{t-a}-\frac{\ln b-\ln t}{b-t}\right)\right| \le t\frac{b+a}{4a^2b^2}(b-a)^2 \tag{3.7}$$

for $t\in(a,b)\subset(0,\infty)$.

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