Received 20/03/18

# INEQUALITIES FOR THE FINITE HILBERT TRANSFORM OF A PRODUCT OF TWO FUNCTIONS

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Abstract. In this paper we establish some new inequalities for the finite Hilbert transform of a product of two complex valued functions. Applications for some particular functions of interest are provided as well.

## 1. Introduction

Allover this paper, we consider the *finite Hilbert transform* on the open interval $(a,b)$ defined by

$$(Tf)(a,b;t) := \frac{1}{\pi} PV \int_a^b \frac{f(\tau)}{\tau - t} d\tau := \lim_{\varepsilon \to 0+} \left[ \int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right] \frac{f(\tau)}{\pi(\tau - t)} d\tau$$

for $t \in (a,b)$ and for various classes of functions $f$ for which the above Cauchy Principal Value integral exists, see [15, Section 3.2] or [19, Lemma II.1.1].

For several recent papers devoted to inequalities for the finite Hilbert transform $(Tf)$, see [2]-[11], [16]-[18] and [20]-[21].

We say that the function $g : (a,b) \to \mathbb{C}$ is $L$-$r$-Hölder continuous, or, of *$L$-$r$-Hölder type*, where $L > 0,\ r \in (0,1]$ if

$$|f(t) - f(s)| \le L |t - s|^r \text{ for any } t,\ s \in (a,b).$$

If $r = 1$, we call it *Lipschitzian* on $(a,b)$.

The following result holds.

**Theorem 1** (Dragomir, 2003, [10])**.** *Assume that $f$ is of $L_1$-$r_1$-Hölder type and $g$ is of $L_2$-$r_2$-Hölder type on $(a,b)$, where $L_1,\ L_2 > 0,\ r_1, r_2 \in (0,1]$. Then we have the inequality:*

$$\begin{aligned} &\left| T(fg)(a,b;t) - f(t)T(g)(a,b;t) \right. \\ &\qquad \left. - g(t)T(f)(a,b;t) + \frac{1}{\pi} f(t) g(t) \ln\left(\frac{b-t}{t-a}\right) \right| \\ &\le \frac{L_1 L_2}{\pi(r_1 + r_2)} \left[ (b-t)^{r_1+r_2} + (t-a)^{r_1+r_2} \right] \le \frac{2 L_1 L_2 (b-a)^{r_1+r_2}}{\pi(r_1 + r_2)} \end{aligned} \tag{1.1}$$

*for any $t \in (a,b)$.*

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1991 *Mathematics Subject Classification.* 26D15; 26D10.

*Key words and phrases.* Finite Hilbert transform, Divided difference, Integral inequalities, Grüss integral inequality.



RGMIA Res. Rep. Coll. 21 (2018), Art. 35, 12 pp.

*In particular,*

$$\left|T(fg)\left(a,b;\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)T(g)\left(a,b;\frac{a+b}{2}\right)\right.$$
$$\left.-\,g\left(\frac{a+b}{2}\right)T(f)\left(a,b;\frac{a+b}{2}\right)\right|\leq\frac{L_1L_2(b-a)^{r_1+r_2}}{\pi 2^{r_1+r_2-1}(r_1+r_2)}. \tag{1.2}$$

The following corollary also holds.

**Corollary 1.** *If $f$ and $g$ are Lipschitzian with the constants $K_1$ and $K_2$, then we have the inequality*

$$\left|T(fg)(a,b;t)-f(t)T(g)(a,b;t)\right.$$
$$\left.-g(t)T(f)(a,b;t)+\frac{1}{\pi}f(t)g(t)\ln\left(\frac{b-t}{t-a}\right)\right|$$
$$\leq\frac{K_1K_2}{\pi}\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]\leq\frac{K_1K_2}{2\pi}(b-a)^2 \tag{1.3}$$

*for any $t\in(a,b)$.*

*In particular, for $t=\frac{a+b}{2}$, we have*

$$\left|T(fg)\left(a,b;\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)T(g)\left(a,b;\frac{a+b}{2}\right)\right.$$
$$\left.-g\left(\frac{a+b}{2}\right)T(f)\left(a,b;\frac{a+b}{2}\right)\right|\leq\frac{K_1K_2}{4\pi}(b-a)^2. \tag{1.4}$$

Motivated by the above results, in this paper we establish some new inequalities for the finite Hilbert transform of a product of two complex valued functions. Applications for some particular functions of interest are provided as well.

## 2. Main Results

For a function $f:(a,b)\to\mathbb{C}$ we define the *divided difference*

$$[f;t,s]:=\frac{f(t)-f(s)}{t-s}\text{ for }t,\ s\in(a,b),\ t\neq s.$$

Now, for $\gamma,\ \Gamma\in\mathbb{C}$ and $(a,b)$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{(a,b),d}(\gamma,\Gamma):=\left\{f:(a,b)\to\mathbb{C}\mid\operatorname{Re}\left[(\Gamma-[f;t,s])\left(\overline{[f;t,s]}-\overline{\gamma}\right)\right]\geq 0,\right.$$
$$\left.\text{for all }t,\ s\in(a,b),\ t\neq s\right\} \tag{2.1}$$

and

$$\bar{\Delta}_{(a,b),d}(\gamma,\Gamma):=\left\{f:(a,b)\to\mathbb{C}\,\middle|\,\left|[f;t,s]-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|\right.$$
$$\left.\text{for all }t,\ s\in(a,b),\ t\neq s\right\}. \tag{2.2}$$

The following representation result may be stated.

**Proposition 1.** *For any* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, *we have that* $\bar{U}_{(a,b),d}(\gamma, \Gamma)$ *and* $\bar{\Delta}_{(a,b),d}(\gamma, \Gamma)$ *are nonempty, convex and closed sets and*

$$\bar{U}_{(a,b),d}(\gamma, \Gamma) = \bar{\Delta}_{(a,b),d}(\gamma, \Gamma). \tag{2.3}$$

*Proof.* We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma})\right] \geq 0.$$

This follows by the equality

$$\frac{1}{4} |\Gamma - \gamma|^2 - \left| z - \frac{\gamma + \Gamma}{2} \right|^2 = \operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma})\right]$$

that holds for any $z \in \mathbb{C}$.

The equality (2.3) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

**Corollary 2.** *For any* $\gamma$, $\Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$,*we have that*

$$\begin{aligned} \bar{U}_{(a,b),d}(\gamma, \Gamma) = \{ f : (a,b) \to \mathbb{C} \mid \ & (\operatorname{Re}\Gamma - \operatorname{Re}[f;t,s])(\operatorname{Re}[f;t,s] - \operatorname{Re}\gamma) \\ & + (\operatorname{Im}\Gamma - \operatorname{Im}[f;t,s])(\operatorname{Im}[f;t,s] - \operatorname{Im}\gamma) \geq 0 \textit{ for all } t, \ s \in (a,b), \ t \neq s \}. \end{aligned} \tag{2.4}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{(a,b),d}(\gamma, \Gamma) := \{ f : (a,b) \to \mathbb{C} \mid \ & \operatorname{Re}(\Gamma) \geq \operatorname{Re}[f;t,s] \geq \operatorname{Re}(\gamma) \\ & \text{and } \operatorname{Im}(\Gamma) \geq \operatorname{Im}[f;t,s] \geq \operatorname{Im}(\gamma) \ \text{ for all } t, \ s \in (a,b), \ t \neq s \}. \end{aligned} \tag{2.5}$$

One can easily observe that $\bar{S}_{(a,b)}(\gamma, \Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{(a,b),d}(\gamma, \Gamma) \subseteq \bar{U}_{(a,b),d}(\gamma, \Gamma). \tag{2.6}$$

The following result holds:

**Theorem 2.** *Assume that* $f : (a,b) \to \mathbb{C}$ *belongs to* $\bar{\Delta}_{(a,b),d}(\gamma, \Gamma)$ *for some* $\gamma$, $\Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$ *and* $g : (a,b) \to \mathbb{C}$ *is of* $H$*-*$r$*-Hölder type, where* $H > 0$, $r \in (0,1]$. *Then we have the inequality*

$$\begin{aligned} & \Bigg| T(fg)(a,b;t) - f(t)\, T(g)(a,b;t) - g(t)\, T(f)(a,b;t) \\ & \quad + \frac{f(t)\, g(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right) - \frac{1}{\pi} \frac{\gamma + \Gamma}{2} \left( \int_a^b g(\tau)\, d\tau - g(t)(b-a) \right) \Bigg| \\ & \qquad \leq \frac{1}{2(r+1)\pi} |\Gamma - \gamma| \, H \left[ (b-t)^{r+1} + (t-a)^{r+1} \right] \end{aligned} \tag{2.7}$$

*for all* $t \in (a,b)$.

*In particular, for $r=1$ and Lipschitzian constant $L$, we get*

$$\begin{aligned} &\left| T(fg)(a,b;t) - f(t)T(g)(a,b;t) - g(t)T(f)(a,b;t) \right. \\ &\quad \left. + \frac{f(t)g(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) - \frac{1}{\pi}\frac{\gamma+\Gamma}{2}\left(\int_a^b g(\tau)\,d\tau - g(t)(b-a)\right)\right| \\ &\qquad\qquad \le \frac{1}{2\pi}\left|\Gamma-\gamma\right| L\left[\frac{1}{4}(b-a)^2 + \left(t-\frac{a+b}{2}\right)^2\right] \end{aligned} \tag{2.8}$$

*for all $t\in(a,b)$.*

*Proof.* Since $f\in\bar{\Delta}_{(a,b),d}(\gamma,\Gamma)$ it follows that

$$\left| f(t) - f(s) - \frac{\gamma+\Gamma}{2}(t-s)\right| \le \frac{1}{2}\left|\Gamma-\gamma\right|\left|t-s\right|$$

for any $t,\ s\in(a,b)$.

By the continuity of the modulus property, we have

$$\left|f(t)-f(s)\right| - \left|\frac{\gamma+\Gamma}{2}\right|\left|t-s\right| \le \left|f(t)-f(s)-\frac{\gamma+\Gamma}{2}(t-s)\right| \le \frac{1}{2}\left|\Gamma-\gamma\right|\left|t-s\right|,$$

for any $t,\ s\in(a,b)$, which implies that

$$\left|f(t)-f(s)\right| \le \frac{1}{2}\left(\left|\gamma+\Gamma\right| + \left|\Gamma-\gamma\right|\right)\left|t-s\right|$$

for any $t,\ s\in(a,b)$, showing that $f$ is also Lipschitzian on $(a,b)$.

Since $g$ is also of $H$-$r$-Hölder type, hence it follows that $T(fg)(a,b;t)$, $T(g)(a,b;t)$ and $T(f)(a,b;t)$ exist for all $t\in(a,b)$, see [15, Section 3.2] or [19, Lemma II.1.1].

Now, for any $t,\ \tau\in(a,b)$, we may write that

$$(f(\tau)-f(t))(g(\tau)-g(t)) = f(\tau)g(\tau) + f(t)g(t) - f(t)g(\tau) - f(\tau)g(t)$$

giving

$$\begin{aligned} \frac{f(\tau)g(\tau)}{\tau-t} &= f(t)\cdot\frac{g(\tau)}{\tau-t} + g(t)\cdot\frac{f(\tau)}{\tau-t} - \frac{f(t)g(t)}{\tau-t} \\ &\quad + \frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t} \end{aligned}$$

for any $t,\ \tau\in(a,b)$, $t\neq\tau$.

Consequently,

$$\begin{aligned} & T(fg)(a,b;t) \\ &= \frac{1}{\pi}PV\int_a^b \frac{f(\tau)g(\tau)}{\tau-t}d\tau \\ &= \frac{1}{\pi}f(t)\,PV\int_a^b\frac{g(\tau)}{\tau-t}d\tau + g(t)\frac{1}{\pi}PV\int_a^b\frac{f(\tau)}{\tau-t}d\tau \\ &- \frac{1}{\pi}f(t)g(t)\,PV\int_a^b\frac{d\tau}{\tau-t} + \frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t}d\tau \\ &= f(t)T(g)(a,b;t) + g(t)T(f)(a,b;t) \\ &- \frac{f(t)g(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) + \frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t}d\tau \end{aligned} \tag{2.9}$$

for any $t \in (a,b)$.

For any $\delta \in \mathbb{C}$ we then have

$$\begin{aligned}
&\frac{1}{\pi} PV \int_a^b \left( \frac{f(\tau) - f(t)}{\tau - t} - \delta \right) (g(\tau) - g(t))\, d\tau \\
&= \frac{1}{\pi} PV \int_a^b \left( \frac{f(\tau) - f(t)}{\tau - t} \right) (g(\tau) - g(t))\, d\tau \\
&- \frac{1}{\pi} \delta PV \int_a^b (g(\tau) - g(t))\, d\tau \\
&= \frac{1}{\pi} PV \int_a^b \left( \frac{f(\tau) - f(t)}{\tau - t} \right) (g(\tau) - g(t))\, d\tau \\
&- \frac{1}{\pi} \delta \left( \int_a^b g(\tau)\, d\tau - g(t)(b-a) \right)
\end{aligned} \tag{2.10}$$

for any $t \in (a,b)$.

From (2.9) and (2.10) we get the following equality of interest

$$\begin{aligned}
&T(fg)(a,b;t) - f(t) T(g)(a,b;t) - g(t) T(f)(a,b;t) \\
&+ \frac{f(t) g(t)}{\pi} \ln \left( \frac{b-t}{t-a} \right) - \frac{1}{\pi} \delta \left( \int_a^b g(\tau)\, d\tau - g(t)(b-a) \right) \\
&= \frac{1}{\pi} PV \int_a^b \left( \frac{f(\tau) - f(t)}{\tau - t} - \delta \right) (g(\tau) - g(t))\, d\tau
\end{aligned} \tag{2.11}$$

for any $t \in (a,b)$ and $\delta \in \mathbb{C}$.

The following property of the Cauchy-Principal Value follows by the properties of integral, modulus and limit,

$$\left| PV \int_a^b A(t,s)\, ds \right| \leq PV \int_a^b |A(t,s)|\, ds, \tag{2.12}$$

assuming that the $PVs$ involved exist for all $t \in (a,b)$.

By making use of the identity (2.11) for $\delta = \frac{\gamma+\Gamma}{2}$ and taking into account that $f \in \bar{\Delta}_{(a,b),d}(\gamma, \Gamma)$ we have

$$\begin{aligned}
&\Bigg| T(fg)(a,b;t) - f(t) T(g)(a,b;t) - g(t) T(f)(a,b;t) \\
&+ \frac{f(t) g(t)}{\pi} \ln \left( \frac{b-t}{t-a} \right) - \frac{1}{\pi} \frac{\gamma + \Gamma}{2} \left( \int_a^b g(\tau)\, d\tau - g(t)(b-a) \right) \Bigg| \\
&= \frac{1}{\pi} \left| PV \int_a^b \left( \frac{f(\tau) - f(t)}{\tau - t} - \frac{\gamma + \Gamma}{2} \right) (g(\tau) - g(t))\, d\tau \right| \\
&\leq \frac{1}{\pi} PV \int_a^b \left| \left( \frac{f(\tau) - f(t)}{\tau - t} - \frac{\gamma + \Gamma}{2} \right) (g(\tau) - g(t)) \right| d\tau \\
&\leq \frac{1}{2\pi} |\Gamma - \gamma| \int_a^b |g(\tau) - g(t)|\, d\tau \leq \frac{1}{2\pi} |\Gamma - \gamma| H \int_a^b |\tau - t|^r\, d\tau \\
&= \frac{1}{2(r+1)\pi} |\Gamma - \gamma| H \left[ (b-t)^{r+1} + (t-a)^{r+1} \right]
\end{aligned}$$

for any $t \in (a,b)$, which proves (2.7). □

**Corollary 3.** *Let $f:[a,b]\to\mathbb{R}$ and $m<M$ such that $f-me$, $Me-f$ are monotonic nondecreasing, and $g:(a,b)\to\mathbb{C}$ is of $H$-$r$-Hölder type, where $H>0$, $r\in(0,1]$. Then*

$$\left|T(fg)(a,b;t)-f(t)T(g)(a,b;t)-g(t)T(f)(a,b;t)\right.$$
$$\left.+\frac{f(t)g(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{1}{\pi}\frac{m+M}{2}\left(\int_a^b g(\tau)\,d\tau-g(t)(b-a)\right)\right|$$
$$\leq\frac{1}{2(r+1)\pi}(M-m)L\left[(b-t)^{r+1}+(t-a)^{r+1}\right]\tag{2.13}$$

*for any $t\in(a,b)$ and for $r=1$ and $H=L$*

$$\left|T(fg)(a,b;t)-f(t)T(g)(a,b;t)-g(t)T(f)(a,b;t)\right.$$
$$\left.+\frac{f(t)g(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{1}{\pi}\frac{m+M}{2}\left(\int_a^b g(\tau)\,d\tau-g(t)(b-a)\right)\right|$$
$$\leq\frac{1}{2\pi}(M-m)L\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]\tag{2.14}$$

*for any $t\in(a,b)$.*

**Remark 1.** *The best inequalities we can get from (2.7) and (2.8) are obtained for $t=\frac{a+b}{2}$ and are as follows*

$$\left|T(fg)\left(a,b;\frac{a+b}{2}\right)-f(t)T(g)\left(a,b;\frac{a+b}{2}\right)-g(t)T(f)\left(a,b;\frac{a+b}{2}\right)\right.$$
$$\left.-\frac{1}{\pi}\frac{\gamma+\Gamma}{2}\left(\int_a^b g(\tau)\,d\tau-g\left(\frac{a+b}{2}\right)(b-a)\right)\right|$$
$$\leq\frac{1}{2^{r+1}(r+1)\pi}|\Gamma-\gamma|H(b-a)^{r+1}\tag{2.15}$$

$$\left|T(fg)\left(a,b;\frac{a+b}{2}\right)-f(t)T(g)\left(a,b;\frac{a+b}{2}\right)-g(t)T(f)\left(a,b;\frac{a+b}{2}\right)\right.$$
$$\left.-\frac{1}{\pi}\frac{\gamma+\Gamma}{2}\left(\int_a^b g(\tau)\,d\tau-g\left(\frac{a+b}{2}\right)(b-a)\right)\right|\leq\frac{1}{8\pi}|\Gamma-\gamma|L(b-a)^2.\tag{2.16}$$

**Remark 2.** *If the function $f$ is differentiable on $(a,b)$ the condition that $f-me$, $Me-f$ are monotonic nondecreasing is equivalent with the following more practical condition*

$$m\leq f'(t)\leq M\quad\textit{for all } t\in(a,b).\tag{2.17}$$

Now, for $\varphi,\Phi\in\mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\varphi,\Phi):=\left\{g:[a,b]\to\mathbb{C}\,|\,\operatorname{Re}\left[(\Phi-g(t))\left(\overline{g(t)}-\overline{\varphi}\right)\right]\geq 0\ \text{ for each }\ t\in[a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\varphi,\Phi) := \left\{ g : [a,b] \to \mathbb{C} | \ \left| g(t) - \frac{\varphi+\Phi}{2} \right| \le \frac{1}{2} |\Phi - \varphi| \text{ for each } \ t \in [a,b] \right\}.$$

The following representation result may be stated.

**Proposition 2.** *For any $\varphi, \Phi \in \mathbb{C}$, $\varphi \ne \Phi$, we have that $\bar{U}_{[a,b]}(\varphi,\Phi)$ and $\bar{\Delta}_{[a,b]}(\varphi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\varphi,\Phi) = \bar{\Delta}_{[a,b]}(\varphi,\Phi). \tag{2.18}$$

The proof is as in Proposition 1.

**Corollary 4.** *For any $\varphi, \Phi \in \mathbb{C}$, $\varphi \ne \Phi$, we have that*

$$\bar{U}_{[a,b]}(\varphi,\Phi) = \{ g : [a,b] \to \mathbb{C} \mid (\operatorname{Re}\Phi - \operatorname{Re} g(t))(\operatorname{Re} g(t) - \operatorname{Re}\varphi) + (\operatorname{Im}\Phi - \operatorname{Im} g(t))(\operatorname{Im} g(t) - \operatorname{Im}\varphi) \ge 0 \textit{ for each } t \in [a,b] \}. \tag{2.19}$$

Now, if we assume that $\operatorname{Re}(\Phi) \ge \operatorname{Re}(\varphi)$ and $\operatorname{Im}(\Phi) \ge \operatorname{Im}(\varphi)$, then we can define the following set of functions as well:

$$\bar{S}_{[a,b]}(\varphi,\Phi) := \{ g : [a,b] \to \mathbb{C} \mid \operatorname{Re}(\Phi) \ge \operatorname{Re} g(t) \ge \operatorname{Re}(\varphi) \text{ and } \operatorname{Im}(\Phi) \ge \operatorname{Im} g(t) \ge \operatorname{Im}(\varphi) \text{ for each } \ t \in [a,b] \}. \tag{2.20}$$

One can easily observe that $\bar{S}_{[a,b]}(\varphi,\Phi)$ is closed, convex and

$$\emptyset \ne \bar{S}_{[a,b]}(\varphi,\Phi) \subseteq \bar{U}_{[a,b]}(\varphi,\Phi). \tag{2.21}$$

We use the following Grüss type inequality for complex valued functions:

**Lemma 1** (Dragomir, 1999 [6])**.** *Assume that $f$, $g : [a,b] \to \mathbb{C}$ are integrable and there exist $\varphi$, $\Phi$, $\psi$, $\Psi \in \mathbb{C}$, $\varphi \ne \Phi$, $\psi \ne \Psi$ such that $f \in \bar{\Delta}_{[a,b]}(\varphi,\Phi)$ and $g \in \bar{\Delta}_{[a,b]}(\psi,\Psi)$, then*

$$\left| \frac{1}{b-a}\int_a^b f(x) g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \frac{1}{b-a}\int_a^b g(x)\,dx \right| \le \frac{1}{4} |\Phi - \varphi| |\Psi - \psi|. \tag{2.22}$$

*The constant $\frac{1}{4}$ is best possible in (2.22).*

**Theorem 3.** *Assume that the functions $f$, $g : (a,b) \to \mathbb{C}$ are differentiable on $(a,b)$ and the derivatives $f'$, $g'$ satisfy the conditions that $f' \in \bar{\Delta}_{(a,b)}(\varphi,\Phi)$ and $g' \in \bar{\Delta}_{(a,b)}(\psi,\Psi)$ for some complex constants $\varphi, \Phi, \psi, \Psi$ with $\varphi \ne \Phi$, $\psi \ne \Psi$. Then*

$$\begin{aligned} &\Bigg| T(fg)(a,b;t) - f(t)T(g)(a,b;t) - g(t)T(f)(a,b;t) \\ &+ \frac{f(t)g(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right) \\ &- \frac{1}{\pi}\left[ \int_t^b (b-\tau) f'(\tau) g'(\tau)\,d\tau - \int_a^t (\tau - a) f'(\tau) g'(\tau)\,d\tau \right] \Bigg| \\ &\le \frac{1}{4\pi} |\Phi - \varphi| |\Psi - \psi| \left[ \frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2}\right)^2 \right] \end{aligned} \tag{2.23}$$

*for all $t \in (a,b)$.*

*In particular, we have*

$$
\begin{aligned}
&\left| T(fg)\left(a,b;\frac{a+b}{2}\right) - f\left(\frac{a+b}{2}\right) T(g)\left(a,b;\frac{a+b}{2}\right)\right. \\
&- g\left(\frac{a+b}{2}\right) T(f)\left(a,b;\frac{a+b}{2}\right) \\
&\left.-\frac{1}{\pi}\left[\int_{\frac{a+b}{2}}^{b}(b-\tau) f'(\tau) g'(\tau)\,d\tau - \int_{a}^{\frac{a+b}{2}}(\tau-a) f'(\tau) g'(\tau)\,d\tau\right]\right| \\
&\le \frac{1}{16\pi}\left|\Phi-\varphi\right|\left|\Psi-\psi\right|(b-a)^2 .
\end{aligned}
\tag{2.24}
$$

*Proof.* Observe that

$$
\begin{aligned}
&\frac{1}{\pi} PV\int_a^b \left(\frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t} - \int_t^{\tau} f'(s) g'(s)\,ds\right) d\tau \\
&\qquad = \frac{1}{\pi} PV\int_a^b \frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t}\,d\tau \\
&\qquad\qquad - \frac{1}{\pi} PV\int_a^b \left(\int_t^{\tau} f'(s) g'(s)\,ds\right) d\tau
\end{aligned}
\tag{2.25}
$$

for all $t \in (a,b)$.

From (2.9) we have

$$
\begin{aligned}
&\frac{1}{\pi} PV\int_a^b \frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t}\,d\tau \\
&\qquad = T(fg)(a,b;t) - f(t) T(g)(a,b;t) - g(t) T(f)(a,b;t) \\
&\qquad\qquad + \frac{f(t) g(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)
\end{aligned}
$$

and by using the integration by parts, we get

$$
\begin{aligned}
&\frac{1}{\pi} PV\int_a^b \left(\int_t^{\tau} f'(s) g'(s)\,ds\right) d\tau \\
&= \frac{1}{\pi} \int_a^b \left(\int_t^{\tau} f'(s) g'(s)\,ds\right) d\tau \\
&= \frac{1}{\pi}\left[\left(\int_t^{\tau} f'(s) g'(s)\,ds\right)\tau\Big|_a^b - \int_a^b \tau f'(\tau) g'(\tau)\,d\tau\right] \\
&= \frac{1}{\pi}\left[\left(\int_t^{b} f'(s) g'(s)\,ds\right) b - \left(\int_t^{a} f'(s) g'(s)\,ds\right) a - \int_a^b \tau f'(\tau) g'(\tau)\,d\tau\right] \\
&= \frac{1}{\pi}\left[\int_t^{b} f'(s) g'(s)\,ds + a\int_a^{t} f'(s) g'(s)\,ds\right. \\
&\left. - \int_a^t \tau f'(\tau) g'(\tau)\,d\tau - \int_t^b \tau f'(\tau) g'(\tau)\,d\tau\right] \\
&= \frac{1}{\pi}\left[\int_t^{b} (b-\tau) f'(\tau) g'(\tau)\,d\tau - \int_a^{t} (\tau-a) f'(\tau) g'(\tau)\,d\tau\right]
\end{aligned}
$$

for all $t \in (a,b)$.

By (2.25) we then have the following identity of interest

$$\begin{aligned}
(2.26)\qquad & \frac{1}{\pi} PV \int_a^b \left( \frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau - t} - \int_t^\tau f'(s)\, g'(s)\, ds \right) d\tau \\
&= T(fg)(a,b;t) - f(t)\, T(g)(a,b;t) - g(t)\, T(f)(a,b;t) \\
&+ \frac{f(t)\, g(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right) \\
&- \frac{1}{\pi}\left[\int_t^b (b-\tau) f'(\tau)\, g'(\tau)\, d\tau - \int_a^t (\tau - a) f'(\tau)\, g'(\tau)\, d\tau\right]
\end{aligned}$$

for all $t \in (a,b)$.

By taking the modulus in (2.26) and using the property (2.12), we have

$$\begin{aligned}
(2.27)\qquad & \Bigg| T(fg)(a,b;t) - f(t)\, T(g)(a,b;t) - g(t)\, T(f)(a,b;t) \\
&+ \frac{f(t)\, g(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right) \\
&- \frac{1}{\pi}\left[\int_t^b (b-\tau) f'(\tau)\, g'(\tau)\, d\tau - \int_a^t (\tau - a) f'(\tau)\, g'(\tau)\, d\tau\right] \Bigg| \\
&\le \frac{1}{\pi} PV \int_a^b \left| \frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau - t} - \int_t^\tau f'(s)\, g'(s)\, ds \right| d\tau
\end{aligned}$$

for all $t \in (a,b)$.

Using the inequality (2.22) for the derivatives $f'$ and $g'$ we get

$$\begin{aligned}
& \left| \frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau - t} - \int_t^\tau f'(s)\, g'(s)\, ds \right| \\
&= \left| \frac{\int_t^\tau f'(s)\, ds \int_t^\tau g'(s)\, ds}{\tau - t} - \int_t^\tau f'(s)\, g'(s)\, ds \right| \le \frac{1}{4} |\Phi - \varphi|\, |\Psi - \psi|\, |\tau - t|
\end{aligned}$$

for all $t, \tau \in (a,b)$ with $t \neq \tau$.

Therefore

$$\begin{aligned}
(2.28)\qquad & \frac{1}{\pi} PV \int_a^b \left| \frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau - t} - \int_t^\tau f'(s)\, g'(s)\, ds \right| d\tau \\
&\le \frac{1}{\pi} PV \int_a^b \frac{1}{4} |\Phi - \varphi|\, |\Psi - \psi|\, |\tau - t|\, d\tau \\
&= \frac{1}{4\pi} |\Phi - \varphi|\, |\Psi - \psi|\, PV \int_a^b |\tau - t|\, d\tau \\
&= \frac{1}{4\pi} |\Phi - \varphi|\, |\Psi - \psi| \int_a^b |\tau - t|\, d\tau \\
&= \frac{1}{4\pi} |\Phi - \varphi|\, |\Psi - \psi|\, \frac{(b-t)^2 + (t-a)^2}{2} \\
&= \frac{1}{4\pi} |\Phi - \varphi|\, |\Psi - \psi| \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2}\right)^2\right]
\end{aligned}$$

for all $t \in (a,b)$.

On making use of (2.27) and (2.28) we get the desired result (2.23). $\square$

## 3. An Example

Consider the identity function $\ell(t) = t$, $t \in \mathbb{R}$ and $(a,b)$ a finite interval. Then

$$T(\ell)(a,b;t) = \frac{1}{\pi} PV \int_a^b \frac{\tau}{\tau - t} d\tau = \frac{1}{\pi} PV \int_a^b \frac{\tau - t + t}{\tau - t} d\tau = \frac{1}{\pi}\int_a^b d\tau + \frac{t}{\pi} PV \int_a^b \frac{d\tau}{\tau - t} = \frac{1}{\pi}(b-a) + \frac{t}{\pi}\ln\left(\frac{b-t}{t-a}\right) \tag{3.1}$$

for $t \in (a,b)$.

We observe that $\ell$ is Lipschitzian with the constant $L = 1$.

Assume that $f : (a,b) \to \mathbb{C}$ belongs to $\bar{\Delta}_{(a,b),d}(\gamma,\Gamma)$ for some $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, then by the inequality (2.8) for $g = \ell$ we have

$$\left| T(\ell f)(a,b;t) - f(t) T(\ell)(a,b;t) - tT(f)(a,b;t) + \frac{tf(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) - \frac{1}{\pi}\frac{\gamma+\Gamma}{2}\left(\int_a^b \tau d\tau - t(b-a)\right)\right| \leq \frac{1}{2\pi}|\Gamma - \gamma|\left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2}\right)^2\right],$$

namely

$$\left| T(\ell f)(a,b;t) - f(t)\left[\frac{1}{\pi}(b-a) + \frac{t}{\pi}\ln\left(\frac{b-t}{t-a}\right)\right] - tT(f)(a,b;t) + \frac{tf(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) - \frac{1}{\pi}\frac{\gamma+\Gamma}{2}\left(\frac{b^2-a^2}{2} - t(b-a)\right)\right| \leq \frac{1}{2\pi}|\Gamma - \gamma|\left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2}\right)^2\right],$$

which is equivalent to

$$\left| T(\ell f)(a,b;t) - \frac{1}{\pi} f(t)(b-a) - tT(f)(a,b;t) - \frac{1}{\pi}\frac{\gamma+\Gamma}{2}(b-a)\left(\frac{a+b}{2} - t\right)\right| \leq \frac{1}{2\pi}|\Gamma - \gamma|\left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2}\right)^2\right], \tag{3.2}$$

for $t \in (a,b)$.

In particular, we have

$$\left|T(\ell f)\left(a,b;\frac{a+b}{2}\right)-\frac{1}{\pi}f\left(\frac{a+b}{2}\right)(b-a)\right.$$
$$\left.-\frac{a+b}{2}T(f)\left(a,b;\frac{a+b}{2}\right)\right|\leq\frac{1}{8\pi}\left|\Gamma-\gamma\right|(b-a)^{2}. \tag{3.3}$$

If we consider the function $f(t)=e^{t}$, $t\in(a,b)$ a real interval, then

$$(Tf)(a,b;t)=\frac{\exp(t)}{\pi}\left[E_{i}(b-t)-E_{i}(a-t)\right], \tag{3.4}$$

where $E_i$ is defined by

$$E_{i}(x):=PV\int_{-\infty}^{x}\frac{\exp(s)}{s}ds,\quad x\in\mathbb{R}.$$

Indeed, we have

$$E_{i}(b-t)-E_{i}(a-t)=PV\int_{a-t}^{b-t}\frac{\exp(s)}{s}ds=PV\int_{a}^{b}\frac{\exp(\tau-t)}{\tau-t}ds$$
$$=\exp(-t)\,\pi\,(T\exp)(a,b;t)$$

and the equality (3.4) is proved.

We have that $f'(t)=e^{t}$, $t\in(a,b)$, which shows that $m\leq\exp(a)\leq f'(t)\leq\exp(b)=M$ for $t\in(a,b)$. Then by (3.2) for $\Gamma=M=\exp(b)$ and $\gamma=m=\exp(a)$ we get

$$\left|T(\ell\exp)(a,b;t)-\frac{1}{\pi}e^{t}(b-a)-\frac{t\exp(t)}{\pi}\left[E_{i}(b-t)-E_{i}(a-t)\right]\right.$$
$$\left.-\frac{1}{\pi}\frac{\exp(a)+\exp(b)}{2}(b-a)\left(\frac{a+b}{2}-t\right)\right|$$
$$\leq\frac{1}{2\pi}\left[\exp(b)-\exp(a)\right]\left[\frac{1}{4}(b-a)^{2}+\left(t-\frac{a+b}{2}\right)^{2}\right], \tag{3.5}$$

for $t\in(a,b)$.

In particular,

$$\left|T(\ell\exp)\left(a,b;\frac{a+b}{2}\right)-\frac{1}{\pi}e^{\frac{a+b}{2}}(b-a)\right.$$
$$\left.-\frac{\frac{a+b}{2}\exp\left(\frac{a+b}{2}\right)}{\pi}\left[E_{i}\left(\frac{b-a}{2}\right)-E_{i}\left(-\frac{b-a}{2}\right)\right]\right|$$
$$\leq\frac{1}{8\pi}\left[\exp(b)-\exp(a)\right](b-a)^{2}. \tag{3.6}$$

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