Received 11/04/18

GENERALIZED FINITE HILBERT TRANSFORM AND SOME BASIC INEQUALITIES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we consider a generalized finite Hilbert transform of complex valued functions and establish some basic inequalities for several particular classes of interest. Applications for some particular instances of finite Hilbert transforms are given as well.

1. Introduction

Finite Hilbert transform on the open interval (a,b) is defined by

$$(Tf)(a,b;t):=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)}{\tau-t}d\tau:=\lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right]\frac{f(\tau)}{\pi(\tau-t)}d\tau \tag{1.1}$$

for $t\in(a,b)$ and for various classes of functions f for which the above Cauchy Principal Value integral exists, see [13, Section 3.2] or [17, Lemma II.1.1].

We say that the function $f:[a,b]\to\mathbb{R}$ is *α-H-Hölder continuous* on (a,b), if

$$|f(t)-f(s)|\le H|t-s|^{\alpha}\text{ for all }t,\ s\in(a,b),$$

where $\alpha\in(0,1],\ H>0$.

The following theorem holds.

Theorem 1 (Dragomir et al., 2001 [1])**.** *If $f:[a,b]\to\mathbb{R}$ is α-H-Hölder continuous on (a,b), then we have the estimate*

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\right|\le\frac{H}{\alpha\pi}\left[(t-a)^{\alpha}+(b-t)^{\alpha}\right]$$

for all $t\in(a,b)$.

The following two corollaries are natural.

Corollary 1. *Let $f:[a,b]\to\mathbb{R}$ be an L-Lipschitzian mapping on $[a,b]$, i.e. f satisfies the condition*

$$|f(t)-f(s)|\le L|t-s|\quad\textit{for all }t,\ s\in[a,b],\ (L>0).$$

Then we have the inequality

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\right|\le\frac{L(b-a)}{\pi}$$

for all $t\in(a,b)$.

1991 *Mathematics Subject Classification.* 26D15; 26D10.

Key words and phrases. Finite Hilbert transform, Integral inequalities.

RGMIA Res. Rep. Coll. 21 (2018), Art. 36, 20 pp.

Corollary 2. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping on $[a,b]$. If $f'\in L_\infty[a,b]$, then, for all $t\in(a,b)$, we have*

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\right|\le\frac{\|f'\|_\infty(b-a)}{\pi},$$

where $\|f'\|_\infty=\operatorname{essup}_{t\in(a,b)}|f'(t)|<\infty$.

We also have:

Theorem 2 (Dragomir et al., 2001 [1])**.** *Let $f:[a,b]\to\mathbb{R}$ be a monotonic nondecreasing (nonincreasing) function on $[a,b]$. If the finite Hilbert transform $(Tf)(a,b,\cdot)$ exists in every $t\in(a,b)$, then*

$$(Tf)(a,b;t)\ge(\le)\,\frac{1}{\pi}f(t)\ln\left(\frac{b-t}{t-a}\right)$$

for all $t\in(a,b)$.

The following result can be useful in practice.

Corollary 3. *Let $f:[a,b]\to\mathbb{R}$ and $\ell:[a,b]\to\mathbb{R}$, $\ell(t)=t$ such that $f-m\ell$, $M\ell-f$ are monotonic nondecreasing, where m, M are given real numbers. If $(Tf)(a,b,\cdot)$ exists in every point $t\in(a,b)$, then we have the inequality*

$$\frac{(b-a)m}{\pi}\le(Tf)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{b-t}{t-a}\right)\le\frac{(b-a)M}{\pi}\tag{1.2}$$

for all $t\in(a,b)$.

Remark 1. *If the mapping is differentiable on (a,b) the condition that $f-m\ell$, $M\ell-f$ are monotonic nondecreasing is equivalent with the following more practical condition*

$$m\le f'(t)\le M\quad\text{for all } t\in(a,b).$$

From (1.2) we may deduce the following approximation result

$$\left|(Tf)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{b-t}{t-a}\right)-\frac{M+m}{2\pi}(b-a)\right|\le\frac{M-m}{2\pi}(b-a).$$

for all $t\in(a,b)$.

For several recent papers devoted to inequalities for the finite Hilbert transform (Tf), see [2]-[10], [14]-[16] and [18]-[19].

We can naturally generalize the concept of Hilbert transform as follows.

For a *continuous strictly increasing function* $g:[a,b]\to[g(a),g(b)]$ that is *differentiable* on (a,b) we define the following generalization of the finite Hilbert transform of a function $f:(a,b)\to\mathbb{C}$ by

$$\begin{aligned}(T_gf)(a,b;t)&:=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)g'(\tau)}{g(\tau)-g(t)}d\tau\\&:=\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right]\frac{f(\tau)g'(\tau)}{\pi[g(\tau)-g(t)]}d\tau\\&:=\frac{1}{\pi}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)g'(\tau)}{g(\tau)-g(t)}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)g'(\tau)}{g(\tau)-g(t)}d\tau\right]\end{aligned}\tag{1.3}$$

for $t\in(a,b)$, provided the above PV exists.

For $[a,b]\subset(0,\infty)$ and $g(t)=\ln t$, $t\in[a,b]$ we have

$$(T_{\ln}f)(a,b;t):=\frac{1}{\pi}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)}{\tau\ln\left(\frac{\tau}{t}\right)}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)}{\tau\ln\left(\frac{\tau}{t}\right)}d\tau\right] \tag{1.4}$$

where $t\in(a,b)$.

For $g(t)=\exp(\alpha t)$, $t\in[a,b]\subset\mathbb{R}$ with $\alpha>0$ we have

$$\begin{aligned}&\left(T_{\exp(\alpha)}f\right)(a,b;t)\\&:=\frac{1}{\pi}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)\exp(\alpha\tau)}{\exp(\alpha\tau)-\exp(\alpha t)}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)\exp(\alpha\tau)}{\exp(\alpha\tau)-\exp(\alpha t)}d\tau\right]\end{aligned} \tag{1.5}$$

where $t\in(a,b)$.

For $[a,b]\subset(0,\infty)$ and $g(t)=t^r$, $t\in[a,b]$, $r>0$, we have

$$(T_rf)(a,b;t):=\frac{r}{\pi}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)\tau^{r-1}}{\tau^r-t^r}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)\tau^{r-1}}{\tau^r-t^r}d\tau\right], \tag{1.6}$$

where $t\in(a,b)$.

Similarly, we can consider the function $g(t)=-t^{-p}$, $t\in[a,b]\subset(0,\infty)$, $p>0$, and then we have

$$\begin{aligned}(T_{-p}f)(a,b;t)&:=\frac{p}{\pi}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)\tau^{-p-1}}{t^{-p}-\tau^{-p}}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)\tau^{-p-1}}{t^{-p}-\tau^{-p}}d\tau\right]\\&=\frac{pt^p}{\pi}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)}{\tau(\tau^p-t^p)}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)}{\tau(\tau^p-t^p)}d\tau\right],\end{aligned} \tag{1.7}$$

where $t\in(a,b)$.

For $[a,b]\subset\left[-\frac{\pi}{2\rho},\frac{\pi}{2\rho}\right]$ and $g(t)=\sin(\rho t)$, $t\in[a,b]$ where $\rho>0$, we have

$$\begin{aligned}&\left(T_{\sin(\rho)}f\right)(a,b;t)\\&:=\frac{\rho}{\pi}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)\cos(\rho\tau)}{\sin(\rho\tau)-\sin(\rho t)}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)\cos(\rho\tau)}{\sin(\rho\tau)-\sin(\rho t)}d\tau\right]\end{aligned} \tag{1.8}$$

where $t\in(a,b)$.

For $g(t)=\sinh(\sigma t)$, $t\in[a,b]\subset\mathbb{R}$ with $\sigma>0$ we have

$$\begin{aligned}&\left(T_{\sinh(\sigma)}f\right)(a,b;t)\\&:=\frac{\sigma}{\pi}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)\cosh(\sigma\tau)}{\sinh(\sigma\tau)-\sinh(\sigma t)}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)\cosh(\sigma\tau)}{\sinh(\sigma\tau)-\sinh(\sigma t)}d\tau\right]\end{aligned} \tag{1.9}$$

where $t\in(a,b)$.

Similar transforms can be associated to the following functions as well:

$$g(t)=\tan(\rho t),\ t\in[a,b]\subset\left[-\frac{\pi}{2\rho},\frac{\pi}{2\rho}\right]\text{ where }\rho>0,$$

and

$$g(t)=\tanh(\sigma t),\ t\in[a,b]\subset\mathbb{R}\text{ with }\sigma>0.$$

Motivated by the above facts, in this paper we consider the generalized finite Hilbert transform $(T_gf)(a,b;t)$ of complex valued functions f and establish some basic inequalities for several particular classes of interest. Applications for some particular instances of finite Hilbert transforms as the one presented in (1.4)-(1.9) are given as well.

2. Main Results

Consider the function $\mathbf{1}(t)=1,\ t\in(a,b)$. We can state the following basic result:

Lemma 1. *For a continuous strictly increasing function $g:[a,b]\to[g(a),g(b)]$ that is differentiable on (a,b) we have*

$$(T_g\mathbf{1})(a,b;t)=\frac{1}{\pi}\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right),\ t\in(a,b). \tag{2.1}$$

We also have for $f:(a,b)\to\mathbb{C}$ that

$$(T_gf)(a,b;t)=\frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)+\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{g(\tau)-g(t)}g'(\tau)\,d\tau \tag{2.2}$$

for $t\in(a,b)$, provided that the PV from the right hand side of the equality (2.2) exists.

Proof. We have

$$\begin{aligned}(T_g\mathbf{1})(a,b;t)&=\frac{1}{\pi}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}\frac{g'(\tau)}{g(\tau)-g(t)}d\tau+\int_{t+\varepsilon}^b\frac{g'(\tau)}{g(\tau)-g(t)}d\tau\right]\\&=\frac{1}{\pi}\lim_{\varepsilon\to0+}\left[\ln|g(\tau)-g(t)|\big|_a^{t-\varepsilon}+\ln(g(\tau)-g(t))\big|_{t+\varepsilon}^b\right]\\&=\frac{1}{\pi}\lim_{\varepsilon\to0+}[\ln(g(t)-g(t-\varepsilon))-\ln(g(t)-g(a))\\&+\ln(g(b)-g(t))-\ln(g(t+\varepsilon)-g(t))]\\&=\frac{1}{\pi}\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)+\frac{1}{\pi}\lim_{\varepsilon\to0+}\ln\left(\frac{g(t)-g(t-\varepsilon)}{g(t+\varepsilon)-g(t)}\right)\end{aligned} \tag{2.3}$$

for $t\in(a,b)$.

Since g is differentiable, we have

$$\lim_{\varepsilon\to0+}\frac{g(t)-g(t-\varepsilon)}{g(t+\varepsilon)-g(t)}=\lim_{\varepsilon\to0+}\frac{\frac{g(t)-g(t-\varepsilon)}{\varepsilon}}{\frac{g(t+\varepsilon)-g(t)}{\varepsilon}}=\frac{g'(t)}{g'(t)}=1$$

for $t\in(a,b)$, and by (2.3) we get (2.1).

From the definition (1.3) we have

$$\begin{aligned}(T_gf)(a,b;t)&:=\frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t)+f(t))\,g'(\tau)}{g(\tau)-g(t)}d\tau\\&=\frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))\,g'(\tau)\,d\tau}{g(\tau)-g(t)}+\frac{1}{\pi}PV\int_a^b\frac{f(t)\,g'(\tau)\,d\tau}{g(\tau)-g(t)}\\&=\frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))\,g'(\tau)\,d\tau}{g(\tau)-g(t)}+\frac{1}{\pi}f(t)\,PV\int_a^b\frac{g'(\tau)\,d\tau}{g(\tau)-g(t)}\\&=\frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)+\frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))\,g'(\tau)\,d\tau}{g(\tau)-g(t)}\end{aligned}$$

for $t\in(a,b)$, which proves the identity (2.2). $\square$

The following result holds:

Theorem 3. *Assume that g is as in Lemma 1 and $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). If*

$$\left\|\frac{f'}{g'}\right\|_{(a,b),\infty}:=\sup_{s\in(a,b)}\left|\frac{f'(s)}{g'(s)}\right|<\infty,$$

then $(T_g f)(a,b;t)$ exists for all $t\in(a,b)$ and

$$\left|(T_g f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)\right|\le\frac{1}{\pi}\left\|\frac{f'}{g'}\right\|_{(a,b),\infty}[g(b)-g(a)] \tag{2.4}$$

for all $t\in(a,b)$.

Proof. By *Cauchy's mean value theorem*, for any $t,\,\tau\in(a,b)$ with $t\neq\tau$ there exists an s between t and τ such that

$$\frac{f(\tau)-f(t)}{g(\tau)-g(t)}=\frac{f'(s)}{g'(s)},$$

therefore for any $t,\,\tau\in(a,b)$ with $t\neq\tau$ we have

$$\left|\frac{f(\tau)-f(t)}{g(\tau)-g(t)}\right|\le\left\|\frac{f'}{g'}\right\|_{(a,b),\infty}.$$

This implies that

$$\int_a^{t-\varepsilon}\left|\frac{f(\tau)-f(t)}{g(\tau)-g(t)}\right|g'(\tau)\,d\tau\le\left\|\frac{f'}{g'}\right\|_{(a,b),\infty}[g(t-\varepsilon)-g(a)]$$

and

$$\int_{t+\varepsilon}^{b}\left|\frac{f(\tau)-f(t)}{g(\tau)-g(t)}\right|g'(\tau)\,d\tau\le\left\|\frac{f'}{g'}\right\|_{(a,b),\infty}[g(b)-g(t+\varepsilon)]$$

for $t\in(a,b)$ and $\min\{t-a,b-t\}>\varepsilon>0$.

By the triangle inequality for the modulus and the fact that $g'(\tau)>0$ for $t\in(a,b)$, we have

$$\begin{aligned}&\left|\int_a^{t-\varepsilon}\frac{f(\tau)-f(t)}{g(\tau)-g(t)}g'(\tau)\,d\tau+\int_{t+\varepsilon}^{b}\frac{f(\tau)-f(t)}{g(\tau)-g(t)}g'(\tau)\,d\tau\right|\\&\le\int_a^{t-\varepsilon}\left|\frac{f(\tau)-f(t)}{g(\tau)-g(t)}\right|g'(\tau)\,d\tau+\int_{t+\varepsilon}^{b}\left|\frac{f(\tau)-f(t)}{g(\tau)-g(t)}\right|g'(\tau)\,d\tau\\&\le\left\|\frac{f'}{g'}\right\|_{(a,b),\infty}[g(b)-g(t+\varepsilon)+g(t-\varepsilon)-g(a)]\end{aligned} \tag{2.5}$$

for $t\in(a,b)$ and $\min\{t-a,b-t\}>\varepsilon>0$.

By taking the limit over $\varepsilon\to0+$ in (2.5) we get

$$\left|PV\int_a^b\frac{f(\tau)-f(t)}{g(\tau)-g(t)}g'(\tau)\,d\tau\right|\le\left\|\frac{f'}{g'}\right\|_{(a,b),\infty}[g(b)-g(a)] \tag{2.6}$$

for $t\in(a,b)$.

By utilising the equality (2.2) we obtain from (2.6) the desired result (2.4). $\square$

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* a, $b\in I$ as

$$M_g(a,b):=g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent* p. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function*.

Corollary 4. *With the assumptions of Theorem 3, we have*

$$|(T_gf)(a,b;M_g(a,b))|\leq\frac{1}{\pi}\left\|\frac{f'}{g'}\right\|_{(a,b),\infty}[g(b)-g(a)]. \tag{2.7}$$

We also have:

Theorem 4. *Let $g:[a,b]\to[g(a),g(b)]$ be a strictly increasing function that is differentiable on (a,b) and $f:(a,b)\to\mathbb{C}$ such that $f\circ g^{-1}$ is of H-r-Hölder type on $(g(a),g(b))$, where $H>0$, $r\in(0,1]$, then*

$$\begin{aligned}\left|(T_gf)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)\right| \\ \leq\frac{H}{\pi r}\left[(g(b)-g(t))^r+(g(t)-g(a))^r\right]\end{aligned} \tag{2.8}$$

for $t\in(a,b)$.

In particular, in the Lipschitz case, we have for $H=L$ that

$$\left|(T_gf)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)\right|\leq\frac{L}{\pi r}[g(b)-g(a)] \tag{2.9}$$

for $t\in(a,b)$.

Proof. For $t\in(a,b)$ and $\min\{t-a,b-t\}>\varepsilon>0$ we have

$$\begin{aligned}\int_a^{t-\varepsilon}\left|\frac{f(\tau)-f(t)}{g(\tau)-g(t)}\right|g'(\tau)\,d\tau&=\int_a^{t-\varepsilon}\left|\frac{f\circ g^{-1}(g(\tau))-f\circ g^{-1}(g(t))}{g(\tau)-g(t)}\right|g'(\tau)\,d\tau\\&\leq H\int_a^{t-\varepsilon}\frac{|g(\tau)-g(t)|^r}{|g(\tau)-g(t)|}g'(\tau)\,d\tau\\&=H\int_a^{t-\varepsilon}|g(\tau)-g(t)|^{r-1}g'(\tau)\,d\tau\\&=H\int_a^{t-\varepsilon}(g(t)-g(\tau))^{r-1}g'(\tau)\,d\tau\\&=\frac{H}{r}\left[(g(t)-g(a))^r-(g(t)-g(t-\varepsilon))^r\right]\end{aligned}$$

and

$$\begin{aligned}\int_{t+\varepsilon}^{b}\left|\frac{f(\tau)-f(t)}{g(\tau)-g(t)}\right|g'(\tau)\,d\tau &= \int_{t+\varepsilon}^{b}\left|\frac{f\circ g^{-1}(g(\tau))-f\circ g^{-1}(g(t))}{g(\tau)-g(t)}\right|g'(\tau)\,d\tau \\ &\leq H\int_{t+\varepsilon}^{b}\frac{|g(\tau)-g(t)|^{r}}{|g(\tau)-g(t)|}g'(\tau)\,d\tau \\ &= H\int_{t+\varepsilon}^{b}|g(\tau)-g(t)|^{r-1}g'(\tau)\,d\tau \\ &= H\int_{t+\varepsilon}^{b}(g(\tau)-g(t))^{r-1}g'(\tau)\,d\tau \\ &= \frac{H}{r}\left[(g(b)-g(t))^{r}-(g(t+\varepsilon)-g(t))^{r}\right].\end{aligned}$$

By adding these two inequalities, we get

$$\begin{aligned}&\int_{a}^{t-\varepsilon}\left|\frac{f(\tau)-f(t)}{g(\tau)-g(t)}\right|g'(\tau)\,d\tau+\int_{t+\varepsilon}^{b}\left|\frac{f(\tau)-f(t)}{g(\tau)-g(t)}\right|g'(\tau)\,d\tau \\ &\leq \frac{H}{r}\left[(g(b)-g(t))^{r}+(g(t)-g(a))^{r}\right. \\ &\left.-(g(t+\varepsilon)-g(t))^{r}-(g(t)-g(t-\varepsilon))^{r}\right]\end{aligned} \tag{2.10}$$

for $t\in(a,b)$ and $\min\{t-a,b-t\}>\varepsilon>0$.

By using the triangle inequality and taking the limit over $\varepsilon\to 0+$, we get

$$\left|PV\int_{a}^{b}\frac{f(\tau)-f(t)}{g(\tau)-g(t)}g'(\tau)\,d\tau\right|\leq\frac{H}{r}\left[(g(b)-g(t))^{r}+(g(t)-g(a))^{r}\right]$$

for $t\in(a,b)$.

Finally, by making use of the equality (2.2) we deduce the desired result (2.8). □

Corollary 5. *With the assumptions of Theorem 4, we have*

$$|(T_{g}f)(a,b;M_{g}(a,b))|\leq\frac{H}{2^{r-1}\pi r}(g(b)-g(a))^{r}. \tag{2.11}$$

In particular, for $r=1$, we get

$$|(T_{g}f)(a,b;M_{g}(a,b))|\leq\frac{L}{\pi}(g(b)-g(a)). \tag{2.12}$$

For a function $f:(a,b)\to\mathbb{C}$ and an injective function $g:(a,b)\to\mathbb{C}$ we define the *divided difference*

$$[f,g;t,s]:=\frac{f(t)-f(s)}{g(t)-g(s)}\text{ for }t,\ s\in(a,b),\ t\neq s.$$

Now, for $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, an injective function $g:(a,b)\to\mathbb{C}$ and (a,b) a finite interval of real numbers, define the sets of complex-valued functions (see also [11] for a similar definition):

$$\begin{aligned} &(2.13)\quad \bar{U}_{(a,b),g,d}(\gamma,\Gamma) \\ &:= \left\{ f:(a,b)\to\mathbb{C} \mid \operatorname{Re}\left[(\Gamma-[f,g;t,s])\left(\overline{[f,g;t,s]}-\overline{\gamma}\right)\right]\geq 0, \right. \\ &\qquad \left. \text{for all } t,\ s\in(a,b),\ t\neq s \right\} \end{aligned}$$

and

$$(2.14)\quad \bar{\Delta}_{(a,b),g,d}(\gamma,\Gamma) := \left\{ f:(a,b)\to\mathbb{C} \middle| \left|[f,g;t,s]-\frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}|\Gamma-\gamma| \right. \\ \left. \text{for all } t,\ s\in(a,b),\ t\neq s \right\}.$$

The following representation result may be stated.

Proposition 1. *For any* $\gamma,\ \Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, *we have that* $\bar{U}_{(a,b),g,d}(\gamma,\Gamma)$ *and* $\bar{\Delta}_{(a,b),g,d}(\gamma,\Gamma)$ *are nonempty, convex and closed sets and*

$$\bar{U}_{(a,b),d}(\gamma,\Gamma) = \bar{\Delta}_{(a,b),d}(\gamma,\Gamma). \tag{2.15}$$

Proof. We observe that for any $z\in\mathbb{C}$ we have the equivalence

$$\left|z-\frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}|\Gamma-\gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right]\geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma-\gamma|^2 - \left|z-\frac{\gamma+\Gamma}{2}\right|^2 = \operatorname{Re}\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right]$$

that holds for any $z\in\mathbb{C}$.

The equality (2.15) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 6. *For any* $\gamma,\ \Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, *we have that*

$$\bar{U}_{(a,b),g,d}(\gamma,\Gamma) = \{ f:(a,b)\to\mathbb{C} \mid\ (\operatorname{Re}\Gamma-\operatorname{Re}[f,g;t,s])(\operatorname{Re}[f,g;t,s]-\operatorname{Re}\gamma)$$

$$+(\operatorname{Im}\Gamma-\operatorname{Im}[f,g;t,s])(\operatorname{Im}[f,g;t,s]-\operatorname{Im}\gamma)\geq 0 \textit{ for all } t,\ s\in(a,b),\ t\neq s\}.$$

Now, if we assume that $\operatorname{Re}(\Gamma)\geq\operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma)\geq\operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$(2.16)\quad \bar{S}_{(a,b),g,d}(\gamma,\Gamma) := \{ f:(a,b)\to\mathbb{C} \mid\ \operatorname{Re}(\Gamma)\geq\operatorname{Re}[f,g;t,s]\geq\operatorname{Re}(\gamma)$$

$$\text{and } \operatorname{Im}(\Gamma)\geq\operatorname{Im}[f,g;t,s]\geq\operatorname{Im}(\gamma) \text{ for all } t,\ s\in(a,b),\ t\neq s\}.$$

One can easily observe that $\bar{S}_{(a,b)g,d}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset\neq\bar{S}_{(a,b),g,d}(\gamma,\Gamma)\subseteq\bar{U}_{(a,b),g,d}(\gamma,\Gamma). \tag{2.17}$$

We have:

Theorem 5. *Let $g:[a,b]\to[g(a),g(b)]$ be a strictly increasing function that is differentiable on (a,b) and $f:(a,b)\to\mathbb{C}$ such that $f\in\bar{\Delta}_{(a,b),g,d}(\gamma,\Gamma)$ for some γ, $\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$. Then we have*

$$\left|(T_g f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)-\frac{\gamma+\Gamma}{2\pi}(g(b)-g(a))\right| \leq \frac{1}{2\pi}|\Gamma-\gamma|(g(b)-g(a)) \tag{2.18}$$

for all $t\in(a,b)$.

Proof. Since $f\in\bar{\Delta}_{(a,b),d}(\gamma,\Gamma)$ it follows that

$$\left|f(t)-f(s)-\frac{\gamma+\Gamma}{2}(g(t)-g(s))\right|\leq\frac{1}{2}|\Gamma-\gamma||g(t)-g(s)|$$

for any t, $s\in(a,b)$.

By the continuity of the modulus property, we have

$$\begin{aligned}|f(t)-f(s)|-\left|\frac{\gamma+\Gamma}{2}\right||g(t)-g(s)| &\leq\left|f(t)-f(s)-\frac{\gamma+\Gamma}{2}(g(t)-g(s))\right|\\ &\leq\frac{1}{2}|\Gamma-\gamma||g(t)-g(s)|,\end{aligned}$$

for any t, $s\in(a,b)$, which implies that

$$|f(t)-f(s)|\leq\frac{1}{2}(|\gamma+\Gamma|+|\Gamma-\gamma|)|g(t)-g(s)|$$

for any t, $s\in(a,b)$. This can be written as

$$\left|f\circ g^{-1}(g(t))-f\circ g^{-1}(g(s))\right|\leq\frac{1}{2}(|\gamma+\Gamma|+|\Gamma-\gamma|)|g(t)-g(s)|$$

for any t, $s\in(a,b)$, namely $f\circ g^{-1}$ is Lipschitzian with the constant $\frac{1}{2}(|\gamma+\Gamma|+|\Gamma-\gamma|)$ on $(g(a),g(b))$.

Therefore the Cauchy Principal value

$$PV\int_a^b\frac{f(\tau)-f(t)}{g(\tau)-g(t)}g'(\tau)\,d\tau$$

exists (see [13, Section 3.2] or [17, Lemma II.1.1]) and we have

$$\begin{aligned}&\frac{1}{\pi}PV\int_a^b\left(\frac{f(\tau)-f(t)}{g(\tau)-g(t)}-\frac{\gamma+\Gamma}{2}\right)g'(\tau)\,d\tau\\ &=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{g(\tau)-g(t)}g'(\tau)\,d\tau-\frac{\gamma+\Gamma}{2\pi}(g(b)-g(a))\\ &=(T_g f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)-\frac{\gamma+\Gamma}{2\pi}(g(b)-g(a))\end{aligned} \tag{2.19}$$

for any $t\in(a,b)$.

The following property of the Cauchy-Principal Value follows by the properties of integral, modulus and limit,

$$\left|PV\int_a^b A(t,s)\,ds\right|\leq PV\int_a^b|A(t,s)|\,ds, \tag{2.20}$$

assuming that the PVs involved exist for all $t\in(a,b)$.

Using the equality (2.19) and the property (2.20) we get

$$\begin{aligned}&\left|(T_g f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)-\frac{\gamma+\Gamma}{2\pi}(g(b)-g(a))\right|\\&\le\frac{1}{\pi}PV\int_a^b\left|\frac{f(\tau)-f(t)}{g(\tau)-g(t)}-\frac{\gamma+\Gamma}{2}\right|g'(\tau)\,d\tau\le\frac{1}{2\pi}|\Gamma-\gamma|\int_a^b g'(\tau)\,d\tau\\&=\frac{1}{2\pi}|\Gamma-\gamma|(g(b)-g(a))\end{aligned}$$

for all $t\in(a,b)$ and the inequality (2.18) is obtained. □

Corollary 7. *With the assumptions of Theorem 5 we have*

$$\left|(T_g f)(a,b;M_g(a,b))-\frac{\gamma+\Gamma}{2\pi}(g(b)-g(a))\right|\le\frac{1}{2\pi}|\Gamma-\gamma|(g(b)-g(a)).\tag{2.21}$$

The case of monotonic functions $f:(a,b)\to\mathbb{R}$ provides the following simple result:

Proposition 2. *Let $g:[a,b]\to[g(a),g(b)]$ be a strictly increasing function that is differentiable on (a,b) and $f:(a,b)\to\mathbb{R}$ a monotonic nondecreasing (nonincreasing) function so that the generalized finite Hilbert transform $(T_g f)(a,b;t)$ exists, then*

$$(T_g f)(a,b;t)\ge(\le)\frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)\tag{2.22}$$

for any $t\in(a,b)$.

Proof. The proof follows by the representation (2.2) on observing that if $f:(a,b)\to\mathbb{R}$ is a monotonic nondecreasing (nonincreasing) function on (a,b), then for any t, $\tau\in(a,b)$ we have

$$\frac{f(\tau)-f(t)}{g(\tau)-g(t)}\ge(\le)\,0,$$

which implies that

$$PV\int_a^b\frac{f(\tau)-f(t)}{g(\tau)-g(t)}g'(\tau)\,d\tau\ge(\le)\,0$$

for any $t\in(a,b)$. □

Corollary 8. *Let $g:[a,b]\to[g(a),g(b)]$ be a strictly increasing function that is differentiable on (a,b) and $f:(a,b)\to\mathbb{R}$ a function such that for some real numbers $m<M$ we have that $f-mg$ and $Mg-f$ are monotonic nondecreasing on (a,b). If the generalized finite Hilbert transform $(T_g f)(a,b;t)$ exists, then we have*

$$\begin{aligned}\frac{m}{\pi}(g(b)-g(a))&\le(T_g f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)\\&\le\frac{M}{\pi}(g(b)-g(a))\end{aligned}\tag{2.23}$$

for any $t\in(a,b)$.

This can be also written as

$$\left|(T_g f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)-\frac{1}{2\pi}(M+m)(g(b)-g(a))\right| \\ \leq \frac{1}{2\pi}(M-m)(g(b)-g(a)) \tag{2.24}$$

for any $t\in(a,b)$.

Proof. Applying proposition (2.22) for the monotonic nondecreasing function $f-mg$ we have

$$(T_g(f-mg))(a,b;t) \\ \geq \frac{1}{\pi}(f(t)-mg(t))\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right) \\ =\frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)-m\frac{1}{\pi}g(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right) \tag{2.25}$$

for all $t\in(a,b)$.

By the linearity of the generalized Hilbert transform we also have

$$(T_g(f-mg))(a,b;t)=(T_g f)(a,b;t)-m(T_g g)(a,b;t)$$

and by the identity (2.2) for $f=g$ we get

$$(T_g g)(a,b;t)=\frac{1}{\pi}g(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)+\frac{1}{\pi}PV\int_a^b\frac{g(\tau)-g(t)}{g(\tau)-g(t)}g'(\tau)\,d\tau \\ =\frac{1}{\pi}g(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)+\frac{1}{\pi}(g(b)-g(a)),$$

which gives that

$$(T_g(f-mg))(a,b;t) \\ =(T_g f)(a,b;t)-\frac{m}{\pi}g(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)-\frac{m}{\pi}(g(b)-g(a)) \tag{2.26}$$

for all $t\in(a,b)$.

On making use of (2.25) and (2.26) we get

$$(T_g f)(a,b;t)-\frac{m}{\pi}g(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)-\frac{m}{\pi}(g(b)-g(a)) \\ \geq\frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)-\frac{m}{\pi}g(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)$$

for all $t\in(a,b)$, which proves the first inequality in (2.23).

The second part follows in a similar way by considering the monotonic nondecreasing function $Mg-f$. □

Remark 2. *From (2.23) we get for* $t=\frac{a+b}{2}$ *that*

$$\frac{m}{\pi}(g(b)-g(a))\leq(T_g f)\left(a,b;\frac{a+b}{2}\right)\leq\frac{M}{\pi}(g(b)-g(a)), \tag{2.27}$$

where f *and* g *are as in Corollary 8.*

Remark 3. *If f and g are as in Corollary 8, then we observe that*

$$m \leq [f, g; t, s] = \frac{f(t) - f(s)}{g(t) - g(s)} \leq M$$

for all t, $s \in (a, b)$ with $t \neq s$, then by (2.18) for $\Gamma = M$ and $\gamma = m$ we recapture the inequality (2.24) as well.

Remark 4. *We also observe that if $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b) and*

$$mg'(t) \leq f'(t) \leq Mg'(t) \text{ for all } t \in (a, b),$$

then the inequality (2.23) holds.

3. Related Results

The following identity is of interest as well:

Lemma 2. *Let $g : [a, b] \to [g(a), g(b)]$ be a strictly increasing function that is differentiable on (a, b) and $f : (a, b) \to \mathbb{R}$ a locally absolutely continuous function on (a, b), then*

$$(T_g f)(a, b; t) = \frac{1}{\pi} f(t) \ln\left(\frac{g(b) - g(t)}{g(t) - g(a)}\right) + \frac{1}{\pi} PV \int_a^b \left(\int_0^1 \frac{(f' \circ g^{-1})((1-s) g(\tau) + sg(t))}{(g' \circ g^{-1})((1-s) g(\tau) + sg(t))} ds\right) g'(\tau) d\tau \tag{3.1}$$

for any $t \in (a, b)$.

Proof. For an absolutely continuous function $h : [c, d] \to \mathbb{C}$ and for x, $y \in [c, d]$ with $x \neq y$ we have

$$\frac{h(y) - h(x)}{y - x} = \frac{\int_x^y h'(u) du}{y - x}.$$

If we use the change of variable $u = (1 - s) x + sy$, $s \in [0, 1]$ we have $du = (y - x) ds$ and then

$$\frac{\int_x^y h'(u) du}{y - x} = \frac{(y - x) \int_0^1 h'((1 - s) x + sy) ds}{y - x} = \int_0^1 h'((1 - s) x + sy) ds.$$

For t, $\tau \in (a, b)$ with $t \neq \tau$ we then have

$$\begin{aligned}\frac{f(\tau) - f(t)}{g(\tau) - g(t)} &= \frac{f \circ g^{-1}(g(\tau)) - f \circ g^{-1}(g(t))}{g(\tau) - g(t)} \\ &= \int_0^1 \left(f \circ g^{-1}\right)'((1 - s) g(\tau) + sg(t)) ds.\end{aligned}$$

For $z \in (g(a), g(b))$ we have

$$\left(f \circ g^{-1}\right)'(z) = \left(f' \circ g^{-1}\right)(z) \left(g^{-1}\right)'(z) = \frac{\left(f' \circ g^{-1}\right)(z)}{(g' \circ g^{-1})(z)}$$

and therefore

$$\int_0^1 \left(f \circ g^{-1}\right)'((1 - s) g(\tau) + sg(t)) ds = \int_0^1 \frac{\left(f' \circ g^{-1}\right)((1 - s) g(\tau) + sg(t))}{(g' \circ g^{-1})((1 - s) g(\tau) + sg(t))} ds$$

for t, $\tau \in (a, b)$ with $t \neq \tau$.

This implies that

$$PV\int_a^b \frac{f(\tau)-f(t)}{g(\tau)-g(t)}g'(\tau)\,d\tau$$
$$=PV\int_a^b\left(\int_0^1 \frac{\left(f'\circ g^{-1}\right)((1-s)\,g(\tau)+sg(t))}{\left(g'\circ g^{-1}\right)((1-s)\,g(\tau)+sg(t))}ds\right)g'(\tau)\,d\tau$$

for $t\in(a,b)$ and by the equality (2.2) we deduce (3.1). □

Now, for $\varphi,\ \Phi\in\mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\varphi,\Phi):=\left\{g:[a,b]\to\mathbb{C}|\operatorname{Re}\left[(\Phi-g(t))\left(\overline{g(t)}-\overline{\varphi}\right)\right]\geq 0\text{ for a.e. }t\in[a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\varphi,\Phi):=\left\{g:[a,b]\to\mathbb{C}|\ \left|g(t)-\frac{\varphi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\varphi|\ \text{ for a.e. }t\in[a,b]\right\}.$$

The following representation result may be stated.

Proposition 3. *For any $\varphi,\ \Phi\in\mathbb{C}$, $\varphi\neq\Phi$, we have that $\bar{U}_{[a,b]}(\varphi,\Phi)$ and $\bar{\Delta}_{[a,b]}(\varphi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\varphi,\Phi)=\bar{\Delta}_{[a,b]}(\varphi,\Phi).\tag{3.2}$$

The proof is as in Proposition 1.

Corollary 9. *For any $\varphi,\ \Phi\in\mathbb{C}$, $\varphi\neq\Phi$, we have that*

$$\begin{aligned}\bar{U}_{[a,b]}(\varphi,\Phi)=\{g:[a,b]\to\mathbb{C}\ |\ &(\operatorname{Re}\Phi-\operatorname{Re}g(t))(\operatorname{Re}g(t)-\operatorname{Re}\varphi)\\&+(\operatorname{Im}\Phi-\operatorname{Im}g(t))(\operatorname{Im}g(t)-\operatorname{Im}\varphi)\geq 0\ \textit{for a.e. }t\in[a,b]\}.\end{aligned}\tag{3.3}$$

Now, if we assume that $\operatorname{Re}(\Phi)\geq\operatorname{Re}(\varphi)$ and $\operatorname{Im}(\Phi)\geq\operatorname{Im}(\varphi)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_{[a,b]}(\varphi,\Phi):=\{g:[a,b]\to\mathbb{C}\ |\ &\operatorname{Re}(\Phi)\geq\operatorname{Re}g(t)\geq\operatorname{Re}(\varphi)\\&\text{and }\operatorname{Im}(\Phi)\geq\operatorname{Im}g(t)\geq\operatorname{Im}(\varphi)\ \text{ for a.e. }t\in[a,b]\}.\end{aligned}\tag{3.4}$$

One can easily observe that $\bar{S}_{[a,b]}(\varphi,\Phi)$ is closed, convex and

$$\emptyset\neq\bar{S}_{[a,b]}(\varphi,\Phi)\subseteq\bar{U}_{[a,b]}(\varphi,\Phi).\tag{3.5}$$

Theorem 6. *Let $g:[a,b]\to[g(a),g(b)]$ be a strictly increasing function that is differentiable on (a,b) and $f:(a,b)\to\mathbb{R}$ a locally absolutely continuous function on (a,b). Assume that there exists $\varphi,\ \Phi\in\mathbb{C}$, $\varphi\neq\Phi$, such that $\frac{f'}{g'}\in\bar{\Delta}_{[a,b]}(\varphi,\Phi)$, then we have*

$$\begin{aligned}&\left|(T_g f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)-\frac{\varphi+\Phi}{2\pi}(g(b)-g(a))\right|\\&\leq\frac{1}{2\pi}|\Phi-\varphi|(g(b)-g(a))\end{aligned}\tag{3.6}$$

for all $t\in(a,b)$.

Proof. Let t, $\tau \in (a,b)$ with $t \neq \tau$. Since $\frac{f'}{g'} \in \bar{\Delta}_{[a,b]}(\varphi,\Phi)$, hence

$$\left| \frac{\left(f' \circ g^{-1}\right)((1-s)g(\tau)+sg(t))}{\left(g' \circ g^{-1}\right)((1-s)g(\tau)+sg(t))} - \frac{\varphi+\Phi}{2} \right| \leq \frac{1}{2}|\Phi-\varphi|$$

for a.e. $s \in [0,1]$.

Taking the integral over s in this inequality, we get

$$\begin{aligned} &\left| \int_0^1 \frac{\left(f' \circ g^{-1}\right)((1-s)g(\tau)+sg(t))}{\left(g' \circ g^{-1}\right)((1-s)g(\tau)+sg(t))} ds - \frac{\varphi+\Phi}{2} \right| \\ &\leq \int_0^1 \left| \frac{\left(f' \circ g^{-1}\right)((1-s)g(\tau)+sg(t))}{\left(g' \circ g^{-1}\right)((1-s)g(\tau)+sg(t))} - \frac{\varphi+\Phi}{2} \right| ds \leq \frac{1}{2}|\Phi-\varphi| \end{aligned}$$

for t, $\tau \in (a,b)$ with $t \neq \tau$.

Using the property (2.20) we get

$$\begin{aligned} &\left| PV \int_a^b \left(\int_0^1 \frac{\left(f' \circ g^{-1}\right)((1-s)g(\tau)+sg(t))}{\left(g' \circ g^{-1}\right)((1-s)g(\tau)+sg(t))} ds \right) g'(\tau)\, d\tau \right. \\ &\left. -\frac{\varphi+\Phi}{2}(g(b)-g(a)) \right| \\ &\leq PV \int_a^b \left| \int_0^1 \frac{\left(f' \circ g^{-1}\right)((1-s)g(\tau)+sg(t))}{\left(g' \circ g^{-1}\right)((1-s)g(\tau)+sg(t))} ds - \frac{\varphi+\Phi}{2} \right| g'(\tau)\, d\tau \\ &\leq \frac{1}{2}|\Phi-\varphi|(g(b)-g(a)) \end{aligned}$$

for $t \in (a,b)$, and by the equality (3.1) we deduce the desired result (3.6). □

4. Examples

Consider the following *logarithmic finite Hilbert transform*

$$(T_{\ln} f)(a,b;t) := \frac{1}{\pi} \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} \frac{f(\tau)}{\tau \ln\left(\frac{\tau}{t}\right)} d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)}{\tau \ln\left(\frac{\tau}{t}\right)} d\tau \right] \tag{4.1}$$

where $t \in (a,b) \subset (0,\infty)$.

If we assume that if $f:(a,b) \to \mathbb{R}$ is differentiable on (a,b) and

$$\frac{m}{t} \leq f'(t) \leq \frac{M}{t} \text{for all } t \in (a,b), \tag{4.2}$$

then by Remark 4 we have

$$\frac{m}{\pi} \ln\left(\frac{b}{a}\right) \leq (T_{\ln} f)(a,b;t) - \frac{1}{\pi} f(t) \ln\left(\frac{\ln\left(\frac{b}{t}\right)}{\ln\left(\frac{t}{a}\right)}\right) \leq \frac{M}{\pi} \ln\left(\frac{b}{a}\right) \tag{4.3}$$

for all $t \in (a,b)$.

In particular, we have

$$\frac{m}{\pi} \ln\left(\frac{b}{a}\right) \leq (T_{\ln} f)(a,b;G(a,b)) \leq \frac{M}{\pi} \ln\left(\frac{b}{a}\right), \tag{4.4}$$

where $G(a,b) := \sqrt{ab}$ is the *geometric mean* of a, $b > 0$.

This inequality can be extended for complex functions as follows: if $f:(a,b)\to\mathbb{C}$ is locally absolutely continuous on (a,b) and there exists the complex numbers φ, $\Phi\in\mathbb{C}$, $\varphi\neq\Phi$ such that

$$\left|tf'(t)-\frac{\varphi+\Phi}{2}\right|\leq\frac{1}{2}\left|\Phi-\varphi\right| \text{ for a.e. } t\in[a,b], \tag{4.5}$$

then

$$\begin{aligned}&\left|(T_{\ln}f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{\ln\left(\frac{b}{t}\right)}{\ln\left(\frac{t}{a}\right)}\right)-\frac{\varphi+\Phi}{2\pi}\ln\left(\frac{b}{a}\right)\right|\\&\leq\frac{1}{2\pi}\left|\Phi-\varphi\right|\ln\left(\frac{b}{a}\right)\end{aligned} \tag{4.6}$$

for all $t\in(a,b)$.

In particular, we have

$$\left|(T_{\ln}f)(a,b;G(a,b))-\frac{\varphi+\Phi}{2\pi}\ln\left(\frac{b}{a}\right)\right|\leq\frac{1}{2\pi}\left|\Phi-\varphi\right|\ln\left(\frac{b}{a}\right). \tag{4.7}$$

Now, observe that the fact that $f\circ\exp$ is of *H-r-Hölder type* on $(\ln a,\ln b)$, where $H>0,\ r\in(0,1]$ and $(a,b)\subset(0,\infty)$, is equivalent to the inequality

$$\left|f(t)-f(s)\right|\leq H\left|\ln t-\ln s\right|^{r} \text{ for all } t,\ s\in(a,b),$$

then by (2.8) we get

$$\left|(T_{\ln}f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{\ln\left(\frac{b}{t}\right)}{\ln\left(\frac{t}{a}\right)}\right)\right|\leq\frac{H}{\pi r}\left[\left(\ln\left(\frac{b}{t}\right)\right)^{r}+\left(\ln\left(\frac{t}{a}\right)\right)^{r}\right] \tag{4.8}$$

for all $t\in(a,b)$.

In particular, we have

$$\left|(T_{g}f)(a,b;G(a,b))\right|\leq\frac{H}{2^{r-1}\pi r}\left(\ln\left(\frac{b}{a}\right)\right)^{r}. \tag{4.9}$$

Consider the *exponential finite Hilbert transform*

$$\begin{aligned}&\left(T_{\exp(\alpha)}f\right)(a,b;t)\\&:=\frac{1}{\pi}\lim_{\varepsilon\to0+}\left[\int_{a}^{t-\varepsilon}\frac{f(\tau)\exp(\alpha\tau)}{\exp(\alpha\tau)-\exp(\alpha t)}d\tau+\int_{t+\varepsilon}^{b}\frac{f(\tau)\exp(\alpha\tau)}{\exp(\alpha\tau)-\exp(\alpha t)}d\tau\right]\\&=\frac{1}{\pi}\exp(-\alpha t)\\&\times\lim_{\varepsilon\to0+}\left[\int_{a}^{t-\varepsilon}\frac{f(\tau)\exp(\alpha(\tau-t))}{\exp(\alpha(\tau-t))-1}d\tau+\int_{t+\varepsilon}^{b}\frac{f(\tau)\exp(\alpha(\tau-t))}{\exp(\alpha(\tau-t))-1}d\tau\right]\end{aligned} \tag{4.10}$$

where $t\in(a,b)\subset\mathbb{R}$.

If we assume that if $f:(a,b)\to\mathbb{R}$ is differentiable on (a,b) and

$$n\exp(\alpha t)\leq f'(t)\leq N\exp(\alpha t) \text{ for all } t\in(a,b),$$

then by applying Remark 4 for $m=\frac{n}{\alpha}$, $M=\frac{N}{\alpha}$ we have

$$\begin{aligned}\frac{n}{\pi\alpha}\left(\exp(\alpha b)-\exp(\alpha a)\right) &\leq \left(T_{\exp(\alpha)}f\right)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{\exp(\alpha b)-\exp(\alpha t)}{\exp(\alpha t)-\exp(\alpha a)}\right)\\ &\leq \frac{N}{\pi\alpha}\left(\exp(\alpha b)-\exp(\alpha a)\right)\end{aligned} \tag{4.11}$$

for any $t\in(a,b)$.

If we take in (4.11)

$$t=LME_{\alpha}(a,b):=\ln\left(\frac{\exp(\alpha a)+\exp(\alpha b)}{2}\right)^{1/\alpha},$$

then we get

$$\begin{aligned}\frac{n}{\pi\alpha}\left(\exp(\alpha b)-\exp(\alpha a)\right) &\leq \left(T_{\exp(\alpha)}f\right)(a,b;LME_{\alpha}(a,b))\\ &\leq \frac{N}{\pi\alpha}\left(\exp(\alpha b)-\exp(\alpha a)\right).\end{aligned} \tag{4.12}$$

This inequality can be extended for complex functions as follows: if $f:(a,b)\to\mathbb{C}$ is locally absolutely continuous on (a,b) and there exists the complex numbers φ, $\Phi\in\mathbb{C}$, $\varphi\neq\Phi$ such that

$$\left|\frac{f'(t)}{\exp(\alpha t)}-\frac{\varphi+\Phi}{2}\right|\leq\frac{1}{2}\left|\Phi-\varphi\right| \text{ for a.e. } t\in[a,b],$$

then

$$\begin{aligned}&\left|\left(T_{\exp(\alpha)}f\right)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{\exp(\alpha b)-\exp(\alpha t)}{\exp(\alpha t)-\exp(\alpha a)}\right)\right.\\ &\qquad\left.-\frac{\varphi+\Phi}{2\pi\alpha}\left(\exp(\alpha b)-\exp(\alpha a)\right)\right|\\ &\qquad\qquad\leq\frac{1}{2\pi\alpha}\left|\Phi-\varphi\right|\left(\exp(\alpha b)-\exp(\alpha a)\right)\end{aligned} \tag{4.13}$$

for any $t\in(a,b)$.

In particular, we get

$$\begin{aligned}&\left|\left(T_{\exp(\alpha)}f\right)(a,b;LME_{\alpha}(a,b))-\frac{\varphi+\Phi}{2\pi\alpha}\left(\exp(\alpha b)-\exp(\alpha a)\right)\right|\\ &\leq\frac{1}{2\pi\alpha}\left|\Phi-\varphi\right|\left(\exp(\alpha b)-\exp(\alpha a)\right).\end{aligned} \tag{4.14}$$

Now, observe that the fact that $f\circ\left(\frac{1}{\alpha}\ln\right)$ is of *H-r-Hölder type* on $(\exp(\alpha a),\exp(\alpha b))$, where $H>0$, $r\in(0,1]$ and $(a,b)\subset\mathbb{R}$, is equivalent to the inequality

$$\left|f(t)-f(s)\right|\leq H\left|\exp(\alpha t)-\exp(\alpha s)\right|^{r} \text{ for all } t,\ s\in(a,b), \tag{4.15}$$

then by the inequality (2.8) we get

$$\begin{aligned}&\left|\left(T_{\exp(\alpha)}f\right)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{\exp(\alpha b)-\exp(\alpha t)}{\exp(\alpha t)-\exp(\alpha a)}\right)\right|\\ &\leq\frac{H}{\pi r}\left[\left(\exp(\alpha b)-\exp(\alpha t)\right)^{r}+\left(\exp(\alpha t)-\exp(\alpha a)\right)^{r}\right]\end{aligned} \tag{4.16}$$

for any $t\in(a,b)$.

In particular, we have

$$\left|\left(T_{\exp(\alpha)}f\right)(a,b;LME_{\alpha}(a,b))\right|\leq\frac{H}{2^{r-1}\pi r}\left(\exp(\alpha b)-\exp(\alpha a)\right)^{r}. \tag{4.17}$$

For $[a,b]\subset(0,\infty)$ and $g(t)=t^{r}$, $t\in[a,b]$, $r>0$, we consider the *positive r-power Hilbert transform*

$$\left(T_{r}f\right)(a,b;t):=\frac{r}{\pi}\lim_{\varepsilon\to0+}\left[\int_{a}^{t-\varepsilon}\frac{f(\tau)\tau^{r-1}}{\tau^{r}-t^{r}}d\tau+\int_{t+\varepsilon}^{b}\frac{f(\tau)\tau^{r-1}}{\tau^{r}-t^{r}}d\tau\right], \tag{4.18}$$

where $t\in(a,b)$.

If $f:(a,b)\to\mathbb{R}$ is differentiable on (a,b) and

$$mt^{r-1}\leq f'(t)\leq Mt^{r-1}\text{ for all }t\in(a,b), \tag{4.19}$$

then by 2.23

$$\frac{m}{\pi r}\left(b^{r}-a^{r}\right)\leq\left(T_{r}f\right)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{b^{r}-t^{r}}{t^{r}-a^{r}}\right)\leq\frac{M}{\pi r}\left(b^{r}-a^{r}\right) \tag{4.20}$$

for all $t\in(a,b)\subset(0,\infty)$.

In particular, we have

$$\frac{m}{\pi r}\left(b^{r}-a^{r}\right)\leq\left(T_{r}f\right)(a,b;M_{r}(a,b))\leq\frac{M}{\pi r}\left(b^{r}-a^{r}\right) \tag{4.21}$$

where $M_{r}(a,b):=\left(\frac{a^{r}+b^{r}}{2}\right)^{1/r}$

Also, if $f:(a,b)\to\mathbb{C}$ is locally absolutely continuous on (a,b) and there exists the complex numbers φ, $\Phi\in\mathbb{C}$, $\varphi\neq\Phi$ such that

$$\left|\frac{f'(t)}{t^{r-1}}-\frac{\varphi+\Phi}{2}\right|\leq\frac{1}{2}\left|\Phi-\varphi\right|\text{ for a.e. }t\in[a,b], \tag{4.22}$$

then

$$\begin{aligned}&\left|\left(T_{r}f\right)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{b^{r}-t^{r}}{t^{r}-a^{r}}\right)-\frac{\varphi+\Phi}{2\pi r}\left(b^{r}-a^{r}\right)\right|\\&\leq\frac{1}{2\pi r}\left|\Phi-\varphi\right|\left(b^{r}-a^{r}\right)\end{aligned} \tag{4.23}$$

for all $t\in(a,b)\subset(0,\infty)$.

In particular, we have

$$\left|\left(T_{r}f\right)(a,b;M_{r}(a,b))-\frac{\varphi+\Phi}{2\pi r}\left(b^{r}-a^{r}\right)\right|\leq\frac{1}{2\pi r}\left|\Phi-\varphi\right|\left(b^{r}-a^{r}\right). \tag{4.24}$$

The function $f\circ(\cdot)^{1/r}$ is of *H-s-Hölder type on* (a^{r},b^{r}), where $H>0$, $s\in(0,1]$, is equivalent to

$$\left|f(t)-f(u)\right|\leq H\left|t^{r}-u^{r}\right|^{s}\text{ for all }t,\ u\in(a,b),$$

then by (2.8) we have

$$\left|\left(T_{r}f\right)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{b^{r}-t^{r}}{t^{r}-a^{r}}\right)\right|\leq\frac{H}{\pi s}\left[\left(b^{r}-t^{r}\right)^{s}+\left(t^{r}-a^{r}\right)^{s}\right] \tag{4.25}$$

for all $t\in(a,b)\subset(0,\infty)$.

In particular,

$$\left|\left(T_{r}f\right)(a,b;M_{r}(a,b))\right|\leq\frac{H}{2^{s-1}\pi s}\left(b^{r}-a^{r}\right)^{s}. \tag{4.26}$$

The case $r=1$ provides the corresponding results for the regular Hilbert transform, see [1].

Similarly, we can consider the *negative p-power Hilbert transform*

$$(T_{-p}f)(a,b;t) := \frac{pt^p}{\pi} \lim_{\varepsilon\to 0+} \left[\int_a^{t-\varepsilon} \frac{f(\tau)}{\tau(\tau^p - t^p)} d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)}{\tau(\tau^p - t^p)} d\tau \right], \tag{4.27}$$

for $[a,b] \subset (0,\infty)$ and $p>0$.

If $f:(a,b)\to\mathbb{R}$ is differentiable on (a,b) and

$$m \le t^{p+1} f'(t) \le M \text{ for all } t\in(a,b), \tag{4.28}$$

then by (2.23) we have

$$\begin{aligned} \frac{m}{\pi p}\left(\frac{b^p - a^p}{a^p b^p}\right) &\le (T_{-p}f)(a,b;t) - \frac{1}{\pi} f(t) \ln\left(\frac{(b^p - t^p)a^p}{b^p(t^p - a^p)}\right) \\ &\le \frac{M}{\pi p}\left(\frac{b^p - a^p}{a^p b^p}\right) \end{aligned} \tag{4.29}$$

for all $t\in(a,b)$.

In particular,

$$\begin{aligned} \frac{m}{\pi p}\left(\frac{b^p - a^p}{a^p b^p}\right) &\le (T_{-p}f)(a,b;M_{-p}(a,b)) - \frac{1}{\pi} f(t) \ln\left(\frac{(b^p - t^p)a^p}{b^p(t^p - a^p)}\right) \\ &\le \frac{M}{\pi p}\left(\frac{b^p - a^p}{a^p b^p}\right) \end{aligned} \tag{4.30}$$

where $M_{-p}(a,b) := \left(\frac{a^{-p}+b^{-p}}{2}\right)^{-1/p}$.

The case $p=1$ is of interest, since in this case

$$(T_{-1}f)(a,b;t) := \frac{t}{\pi} \lim_{\varepsilon\to 0+} \left[\int_a^{t-\varepsilon} \frac{f(\tau)}{\tau(\tau - t)} d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)}{\tau(\tau - t)} d\tau \right], \tag{4.31}$$

and if

$$m \le t^2 f'(t) \le M \text{ for all } t\in(a,b), \tag{4.32}$$

then

$$\frac{m}{\pi}\left(\frac{b-a}{ab}\right) \le (T_{-1}f)(a,b;t) - \frac{1}{\pi} f(t)\ln\left(\frac{(b-t)a}{b(t-a)}\right) \le \frac{M}{\pi}\left(\frac{b-a}{ab}\right) \tag{4.33}$$

for all $t\in(a,b)$.

In particular, we have

$$\frac{m}{\pi}\left(\frac{b-a}{ab}\right) \le (T_{-1}f)(a,b;H(a,b)) \le \frac{M}{\pi}\left(\frac{b-a}{ab}\right) \tag{4.34}$$

where $H(a,b) := \frac{2ab}{a+b}$ is the *harmonic mean* of a, $b>0$.

Also, if $f:(a,b)\to\mathbb{C}$ is locally absolutely continuous on (a,b) and there exists the complex numbers φ, $\Phi\in\mathbb{C}$, $\varphi\ne\Phi$ such that

$$\left| t^2 f'(t) - \frac{\varphi+\Phi}{2} \right| \le \frac{1}{2}|\Phi - \varphi| \text{ for a.e. } t\in[a,b], \tag{4.35}$$

then

$$\left|(T_{-1}f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{(b-t)a}{b(t-a)}\right)-\frac{\gamma+\Gamma}{2\pi}\left(\frac{b-a}{ab}\right)\right| \le \frac{1}{2\pi}\left|\Gamma-\gamma\right|\left(\frac{b-a}{ab}\right) \tag{4.36}$$

for all $t\in(a,b)$.

In particular, we have

$$\left|(T_{-1}f)(a,b;H(a,b))-\frac{\gamma+\Gamma}{2\pi}\left(\frac{b-a}{ab}\right)\right|\le\frac{1}{2\pi}\left|\Gamma-\gamma\right|\left(\frac{b-a}{ab}\right). \tag{4.37}$$

The fact that $f\circ\left[-(\cdot)^{-1}\right]$ is of *H-s-Hölder type on* $\left(-\frac{1}{a},-\frac{1}{b}\right)$, where $K>0,\ s\in(0,1]$, is equivalent to

$$|f(t)-f(u)|\le K\left|\frac{t-u}{tu}\right|^{s}\text{ for all } t,\ u\in(a,b),$$

then by (2.8) we have

$$\left|(T_{-1}f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{(b-t)a}{b(t-a)}\right)\right|\le\frac{K}{\pi s}\left[\left(\frac{b-t}{bt}\right)^{s}+\left(\frac{t-a}{ta}\right)^{s}\right] \tag{4.38}$$

for all $t\in(a,b)$.

In particular, we have

$$\left|(T_{-1}f)(a,b;H(a,b))\right|\le\frac{K}{2^{s-1}\pi s}\left(\frac{b-a}{ba}\right)^{s}. \tag{4.39}$$

References

[1] N. M. Dragomir, S. S. Dragomir and P. M. Farrell, Some inequalities for the finite Hilbert transform. *Inequality Theory and Applications.* Vol. I, 113–122, Nova Sci. Publ., Huntington, NY, 2001.

[2] N. M. Dragomir, S. S. Dragomir and P. M. Farrell, Approximating the finite Hilbert transform via trapezoid type inequalities. *Comput. Math. Appl.* **43** (2002), no. 10-11, 1359–1369.

[3] N. M. Dragomir, S. S. Dragomir, P. M. Farrell and G. W. Baxter, On some new estimates of the finite Hilbert transform. *Libertas Math.* **22** (2002), 65–75.

[4] N. M. Dragomir, S. S. Dragomir and P. M. Farrell and G. W. Baxter, A quadrature rule for the finite Hilbert transform via trapezoid type inequalities. *J. Appl. Math. Comput.* **13** (2003),no. 1-2, 67–84.

[5] N. M. Dragomir, S. S. Dragomir and P. M. Farrell P. M.; Baxter, G. W. A quadrature rule for the finite Hilbert transform via midpoint type inequalities. *Fixed point theory and applications.* Vol. 5, 11–22, Nova Sci. Publ., Hauppauge, NY, 2004.

[6] S. S. Dragomir, Inequalities for the Hilbert transform of functions whose derivatives are convex. *J. Korean Math. Soc.* **39** (2002), no. 5, 709–729.

[7] S. S. Dragomir, Approximating the finite Hilbert transform via an Ostrowski type inequality for functions of bounded variation. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 4, Article 51, 19 pp

[8] S. S. Dragomir, Approximating the finite Hilbert transform via Ostrowski type inequalities for absolutely continuous functions. *Bull. Korean Math. Soc.* **39** (2002), no. 4, 543–559.

[9] S. S. Dragomir, Some inequalities for the finite Hilbert transform of a product. *Commun. Korean Math. Soc.* **18** (2003), no. 1, 39–57.

[10] S. S. Dragomir, Sharp error bounds of a quadrature rule with one multiple node for the finite Hilbert transform in some classes of continuous differentiable functions. *Taiwanese J. Math.* **9** (2005), no. 1, 95–109.

[11] S. S. Dragomir, The perturbed median principle for integral inequalities with applications. Nonlinear analysis and variational problems, 53–63, Springer Optim. Appl., 35, Springer, New York, 2010.

[12] S. S. Dragomir, Inequalities and approximations for the Finite Hilbert transform: a survey of recent results, Preprint *RGMIA Res. Rep. Coll.* **21** (2018), Art. 30, 90 pp. [Online http://rgmia.org/papers/v21/v21a30.pdf].

[13] F. D. Gakhov, *Boundary Value Problems* (English translation), Pergamon Press, Oxford, 1966.

[14] W. Liu and X. Gao, Approximating the finite Hilbert transform via a companion of Ostrowski's inequality for function of bounded variation and applications. *Appl. Math. Comput.* **247** (2014), 373–385.

[15] W. Liu, X. Gao and Y. Wen, Approximating the finite Hilberttransform via some companions of Ostrowski's inequalities. *Bull. Malays. Math. Sci. Soc.* **39** (2016),no. 4, 1499–1513.

[16] W. Liu and N. Lu, Approximating the finite Hilbert transform via Simpson type inequalities and applications. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **77** (2015),no. 3, 107–122.

[17] S. G. Mikhlin and S. Prössdorf, *Singular Integral Operators* (English translation), Springer Verlag, Berlin, 1986.

[18] S. Wang, X. Gao and N. Lu, A quadrature formula in approximating the finite Hilbert transform via perturbed trapezoid type inequalities. *J. Comput. Anal. Appl.* **22** (2017),no. 2, 239–246.

[19] S. Wang, N. Lu and X. Gao, A quadrature rule for the finite Hilberttransform via Simpson type inequalities and applications. *J. Comput. Anal. Appl.* **22** (2017), no. 2, 229–238.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa