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INEQUALITIES FOR A GENERALIZED FINITE HILBERT TRANSFORM OF DIFFERENTIABLE FUNCTIONS WITH CONVEX DERIVATIVES

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Abstract. In this paper we consider a generalized finite Hilbert transform and establish some inequalities for differentiable functions whose derivative are either convex or, in the complex case, has the modulus convex. Applications for some particular instances of finite Hilbert transforms are given as well.

1. Introduction

Finite Hilbert transform on the open interval (a,b) is defined by

$$(Tf)(a,b;t) := \frac{1}{\pi} PV \int_a^b \frac{f(\tau)}{\tau - t} d\tau := \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right] \frac{f(\tau)}{\pi(\tau - t)} d\tau \tag{1.1}$$

for $t \in (a,b)$ and for various classes of functions f for which the above Cauchy Principal Value integral exists, see [13, Section 3.2] or [17, Lemma II.1.1].

We say that the function $f : [a,b] \to \mathbb{R}$ is *α-H-Hölder continuous* on (a,b), if

$$|f(t) - f(s)| \le H |t-s|^\alpha \text{ for all } t,\ s \in (a,b),$$

where $\alpha \in (0,1]$, $H > 0$.

The following theorem holds.

Theorem 1 (Dragomir et al., 2001 [1])**.** *If $f : [a,b] \to \mathbb{R}$ is α-H-Hölder continuous on (a,b), then we have the estimate*

$$\left| (Tf)(a,b;t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) \right| \le \frac{H}{\alpha \pi} \left[(t-a)^\alpha + (b-t)^\alpha \right]$$

for all $t \in (a,b)$.

The following two corollaries are natural.

Corollary 1. *Let $f : [a,b] \to \mathbb{R}$ be an L-Lipschitzian mapping on $[a,b]$, i.e. f satisfies the condition*

$$|f(t) - f(s)| \le L |t-s| \quad \textit{for all } t,\ s \in [a,b],\ (L > 0).$$

Then we have the inequality

$$\left| (Tf)(a,b;t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) \right| \le \frac{L(b-a)}{\pi}$$

for all $t \in (a,b)$.

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Corollary 2. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a,b]$. If $f' \in L_\infty [a,b]$, then, for all $t \in (a,b)$, we have*

$$\left| (Tf)(a,b;t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) \right| \leq \frac{\|f'\|_\infty (b-a)}{\pi},$$

where $\|f'\|_\infty = \operatorname{essup}_{t\in(a,b)} |f'(t)| < \infty$.

We also have:

Theorem 2 (Dragomir et al., 2001 [1])**.** *Let $f : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing (nonincreasing) function on $[a,b]$. If the finite Hilbert transform $(Tf)(a,b,\cdot)$ exists in every $t \in (a,b)$, then*

$$(Tf)(a,b;t) \geq (\leq) \frac{1}{\pi} f(t) \ln \left(\frac{b-t}{t-a} \right)$$

for all $t \in (a,b)$.

The following result can be useful in practice.

Corollary 3. *Let $f : [a,b] \to \mathbb{R}$ and $\ell : [a,b] \to \mathbb{R}$, $\ell(t) = t$ such that $f - m\ell$, $M\ell - f$ are monotonic nondecreasing, where m, M are given real numbers. If $(Tf)(a,b,\cdot)$ exists in every point $t \in (a,b)$, then we have the inequality*

$$\frac{(b-a)m}{\pi} \leq (Tf)(a,b;t) - \frac{1}{\pi} f(t) \ln \left(\frac{b-t}{t-a} \right) \leq \frac{(b-a)M}{\pi} \tag{1.2}$$

for all $t \in (a,b)$.

Remark 1. *If the mapping is differentiable on (a,b) the condition that $f - m\ell$, $M\ell - f$ are monotonic nondecreasing is equivalent with the following more practical condition*

$$m \leq f'(t) \leq M \quad \text{for all } t \in (a,b).$$

From (1.2) we may deduce the following approximation result

$$\left| (Tf)(a,b;t) - \frac{1}{\pi} f(t) \ln \left(\frac{b-t}{t-a} \right) - \frac{M+m}{2\pi}(b-a) \right| \leq \frac{M-m}{2\pi}(b-a).$$

for all $t \in (a,b)$.

The following result also holds.

Theorem 3 (Dragomir, 2002, [6])**.** *Assume that the differentiable function $f : (a,b) \to \mathbb{R}$ is such that f' is convex on (a,b). Then the Hilbert transform $(Tf)(a,b;\cdot)$ exists in every point $t \in (a,b)$ and:*

$$\begin{aligned} &\frac{2}{\pi}\left[f\left(\frac{t+b}{2}\right) - f\left(\frac{t+a}{2}\right) \right] + \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) \\ &\leq (Tf)(a,b;t) \\ &\leq \frac{1}{2\pi}\left[f(b) - f(a) + (b-a) f'(t) \right] + \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) \end{aligned} \tag{1.3}$$

for any $t \in (a,b)$.

In particular, for $t=\frac{a+b}{2}$ we get

$$\frac{2}{\pi}\left[f\left(\frac{a+3b}{4}\right)-f\left(\frac{3a+b}{2}\right)\right]\leq(Tf)\left(a,b;\frac{a+b}{2}\right)\leq\frac{1}{2\pi}\left[f(b)-f(a)+(b-a)f'\left(\frac{a+b}{2}\right)\right] \tag{1.4}$$

For several recent papers devoted to inequalities for the finite Hilbert transform (Tf), see [2]-[10], [14]-[16] and [18]-[19].

We can naturally generalize the concept of Hilbert transform as follows.

For a *continuous strictly increasing function* $g:[a,b]\to[g(a),g(b)]$ that is *differentiable* on (a,b) we define the following generalization of the finite Hilbert transform of a function $f:(a,b)\to\mathbb{C}$ by

$$\begin{aligned}(T_gf)(a,b;t)&:=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)g'(\tau)}{g(\tau)-g(t)}d\tau\\&:=\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right]\frac{f(\tau)g'(\tau)}{\pi[g(\tau)-g(t)]}d\tau\\&:=\frac{1}{\pi}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)g'(\tau)}{g(\tau)-g(t)}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)g'(\tau)}{g(\tau)-g(t)}d\tau\right]\end{aligned} \tag{1.5}$$

for $t\in(a,b)$, provided the above PV exists.

For $[a,b]\subset(0,\infty)$ and $g(t)=\ln t$, $t\in[a,b]$ we have

$$(T_{\ln}f)(a,b;t):=\frac{1}{\pi}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)}{\tau\ln\left(\frac{\tau}{t}\right)}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)}{\tau\ln\left(\frac{\tau}{t}\right)}d\tau\right] \tag{1.6}$$

where $t\in(a,b)$.

For $g(t)=\exp(\alpha t)$, $t\in[a,b]\subset\mathbb{R}$ with $\alpha>0$ we have

$$\begin{aligned}&\left(T_{\exp(\alpha)}f\right)(a,b;t)\\&:=\frac{1}{\pi}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)\exp(\alpha\tau)}{\exp(\alpha\tau)-\exp(\alpha t)}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)\exp(\alpha\tau)}{\exp(\alpha\tau)-\exp(\alpha t)}d\tau\right]\end{aligned} \tag{1.7}$$

where $t\in(a,b)$.

For $[a,b]\subset(0,\infty)$ and $g(t)=t^r$, $t\in[a,b]$, $r>0$, we have

$$(T_rf)(a,b;t):=\frac{r}{\pi}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)\tau^{r-1}}{\tau^r-t^r}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)\tau^{r-1}}{\tau^r-t^r}d\tau\right], \tag{1.8}$$

where $t\in(a,b)$.

Similarly, we can consider the function $g(t)=-t^{-p}$, $t\in[a,b]\subset(0,\infty)$, $p>0$, and then we have

$$\begin{aligned}(T_{-p}f)(a,b;t)&:=\frac{p}{\pi}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)\tau^{-p-1}}{t^{-p}-\tau^{-p}}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)\tau^{-p-1}}{t^{-p}-\tau^{-p}}d\tau\right]\\&=\frac{pt^p}{\pi}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)}{\tau(\tau^p-t^p)}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)}{\tau(\tau^p-t^p)}d\tau\right],\end{aligned} \tag{1.9}$$

where $t\in(a,b)$.

For $[a,b] \subset \left[-\frac{\pi}{2\rho}, \frac{\pi}{2\rho}\right]$ and $g(t) = \sin(\rho t)$, $t \in [a,b]$ where $\rho > 0$, we have

$$\begin{aligned} &\left(T_{\sin(\rho)} f\right)(a,b;t) \\ &:= \frac{\rho}{\pi} \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} \frac{f(\tau)\cos(\rho\tau)}{\sin(\rho\tau) - \sin(\rho t)} d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)\cos(\rho\tau)}{\sin(\rho\tau) - \sin(\rho t)} d\tau \right] \end{aligned} \tag{1.10}$$

where $t \in (a,b)$.

For $g(t) = \sinh(\sigma t)$, $t \in [a,b] \subset \mathbb{R}$ with $\sigma > 0$ we have

$$\begin{aligned} &\left(T_{\sinh(\sigma)} f\right)(a,b;t) \\ &:= \frac{\sigma}{\pi} \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} \frac{f(\tau)\cosh(\sigma\tau)}{\sinh(\sigma\tau) - \sinh(\sigma t)} d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)\cosh(\sigma\tau)}{\sinh(\sigma\tau) - \sinh(\sigma t)} d\tau \right] \end{aligned} \tag{1.11}$$

where $t \in (a,b)$.

Similar transforms can be associated to the following functions as well:

$$g(t) = \tan(\rho t), \ t \in [a,b] \subset \left[-\frac{\pi}{2\rho}, \frac{\pi}{2\rho}\right] \text{ where } \rho > 0,$$

and

$$g(t) = \tanh(\sigma t), \ t \in [a,b] \subset \mathbb{R} \text{ with } \sigma > 0.$$

Motivated by the above results, we establish in this paper some inequalities for differentiable functions whose derivative are either convex or, in the complex case, has the modulus convex. Applications for some particular instances of finite Hilbert transforms are given as well.

2. Main Results

Consider the function $\mathbf{1}(t) = 1$, $t \in (a,b)$. We need the following preliminary results:

Lemma 1. *For a continuous strictly increasing function $g : [a,b] \to [g(a), g(b)]$ that is differentiable on (a,b) we have*

$$(T_g \mathbf{1})(a,b;t) = \frac{1}{\pi} \ln\left(\frac{g(b) - g(t)}{g(t) - g(a)}\right), \ t \in (a,b). \tag{2.1}$$

We also have for $f : (a,b) \to \mathbb{C}$ that

$$(T_g f)(a,b;t) = \frac{1}{\pi} f(t) \ln\left(\frac{g(b) - g(t)}{g(t) - g(a)}\right) + \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{g(\tau) - g(t)} g'(\tau)\, d\tau \tag{2.2}$$

for $t \in (a,b)$, provided that the PV from the right hand side of the equality (2.2) exists.

Proof. We have

$$
\begin{aligned}
(2.3)\qquad (T_g\mathbf{1})(a,b;t) &= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}\frac{g'(\tau)}{g(\tau)-g(t)}d\tau+\int_{t+\varepsilon}^b\frac{g'(\tau)}{g(\tau)-g(t)}d\tau\right]\\
&= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\ln|g(\tau)-g(t)||_a^{t-\varepsilon}+\ln(g(\tau)-g(t))|_{t+\varepsilon}^b\right]\\
&= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\ln(g(t)-g(t-\varepsilon))-\ln(g(t)-g(a))\right.\\
&\quad\left.+\ln(g(b)-g(t))-\ln(g(t+\varepsilon)-g(t))\right]\\
&= \frac{1}{\pi}\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)+\frac{1}{\pi}\lim_{\varepsilon\to 0+}\ln\left(\frac{g(t)-g(t-\varepsilon)}{g(t+\varepsilon)-g(t)}\right)
\end{aligned}
$$

for $t\in(a,b)$.

Since g is differentiable, we have

$$\lim_{\varepsilon\to 0+}\frac{g(t)-g(t-\varepsilon)}{g(t+\varepsilon)-g(t)}=\lim_{\varepsilon\to 0+}\frac{\frac{g(t)-g(t-\varepsilon)}{\varepsilon}}{\frac{g(t+\varepsilon)-g(t)}{\varepsilon}}=\frac{g'(t)}{g'(t)}=1$$

for $t\in(a,b)$, and by (2.3) we get (2.1).

From the definition (1.5) we have

$$
\begin{aligned}
(T_gf)(a,b;t) &:= \frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t)+f(t))g'(\tau)}{g(\tau)-g(t)}d\tau\\
&= \frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))g'(\tau)d\tau}{g(\tau)-g(t)}+\frac{1}{\pi}PV\int_a^b\frac{f(t)g'(\tau)d\tau}{g(\tau)-g(t)}\\
&= \frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))g'(\tau)d\tau}{g(\tau)-g(t)}+\frac{1}{\pi}f(t)PV\int_a^b\frac{g'(\tau)d\tau}{g(\tau)-g(t)}\\
&= \frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)+\frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))g'(\tau)d\tau}{g(\tau)-g(t)}
\end{aligned}
$$

for $t\in(a,b)$, which proves the identity (2.2). □

The following identity is of interest as well:

Lemma 2. *Let $g:[a,b]\to[g(a),g(b)]$ be a strictly increasing function that is differentiable on (a,b) and $f:(a,b)\to\mathbb{R}$ a locally absolutely continuous function on (a,b), then*

$$
\begin{aligned}
(2.4)\qquad (T_gf)(a,b;t) &= \frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)\\
&\quad+\frac{1}{\pi}PV\int_a^b\left(\int_0^1\frac{(f'\circ g^{-1})((1-s)g(\tau)+sg(t))}{(g'\circ g^{-1})((1-s)g(\tau)+sg(t))}ds\right)g'(\tau)d\tau
\end{aligned}
$$

for any $t\in(a,b)$.

Proof. For an absolutely continuous function $h:[c,d]\to\mathbb{C}$ and for $x,\ y\in[c,d]$ with $x\neq y$ we have

$$\frac{h(y)-h(x)}{y-x}=\frac{\int_x^y h'(u)du}{y-x}.$$

If we use the change of variable $u=(1-s)x+sy$, $s\in[0,1]$ we have $du=(y-x)\,ds$ and then

$$\frac{\int_x^y h'(u)\,du}{y-x}=\frac{(y-x)\int_0^1 h'((1-s)x+sy)\,ds}{y-x}=\int_0^1 h'((1-s)x+sy)\,ds.$$

For $t,\,\tau\in(a,b)$ with $t\neq\tau$ we then have

$$\begin{aligned}\frac{f(\tau)-f(t)}{g(\tau)-g(t)}&=\frac{f\circ g^{-1}(g(\tau))-f\circ g^{-1}(g(t))}{g(\tau)-g(t)}\\&=\int_0^1\left(f\circ g^{-1}\right)'((1-s)g(\tau)+sg(t))\,ds.\end{aligned}$$

For $z\in(g(a),g(b))$ we have

$$\left(f\circ g^{-1}\right)'(z)=\left(f'\circ g^{-1}\right)(z)\left(g^{-1}\right)'(z)=\frac{\left(f'\circ g^{-1}\right)(z)}{\left(g'\circ g^{-1}\right)(z)} \tag{2.5}$$

and therefore

$$\int_0^1\left(f\circ g^{-1}\right)'((1-s)g(\tau)+sg(t))\,ds=\int_0^1\frac{\left(f'\circ g^{-1}\right)((1-s)g(\tau)+sg(t))}{\left(g'\circ g^{-1}\right)((1-s)g(\tau)+sg(t))}ds$$

for $t,\,\tau\in(a,b)$ with $t\neq\tau$.

This implies that

$$\begin{aligned}&PV\int_a^b\frac{f(\tau)-f(t)}{g(\tau)-g(t)}g'(\tau)\,d\tau\\&=PV\int_a^b\left(\int_0^1\frac{\left(f'\circ g^{-1}\right)((1-s)g(\tau)+sg(t))}{\left(g'\circ g^{-1}\right)((1-s)g(\tau)+sg(t))}ds\right)g'(\tau)\,d\tau\end{aligned}$$

for $t\in(a,b)$ and by the equality (2.2) we deduce (2.4). □

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a,\,b\in I$ as

$$M_g(a,b):=g^{-1}\left(\frac{g(a)+g(b)}{2}\right). \tag{2.6}$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent p*. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function*.

We have the following result:

Theorem 4. *Let $g:[a,b]\to[g(a),g(b)]$ be a strictly increasing function that is differentiable on (a,b) and $f:(a,b)\to\mathbb{R}$ a differentiable function on (a,b). Assume that $\frac{f'\circ g^{-1}}{g'\circ g^{-1}}$ is convex on $(g(a),g(b))$, then*

$$\begin{aligned}(2.7)\qquad &\frac{2}{\pi}\left[f\left(M_g(b,t)\right)-f\left(M_g(t,a)\right)\right]\\ &\leq (T_g f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{g(b)-g(t)}{g(t)-g(a)}\right)\\ &\leq \frac{1}{2\pi}\left[f(b)-f(a)+\frac{f'(t)}{g'(t)}\left[g(b)-g(a)\right]\right]\end{aligned}$$

for any $t\in(a,b)$.

In particular, for $t=M_g(a,b)$ we get

$$\begin{aligned}(2.8)\qquad &\frac{2}{\pi}\left[f\circ g^{-1}\left(\frac{3g(b)+g(a)}{4}\right)-f\circ g^{-1}\left(\frac{g(b)+3g(a)}{4}\right)\right]\\ &\leq (T_g f)(a,b;M_g(a,b))\\ &\leq \frac{1}{2\pi}\left[f(b)-f(a)+\frac{f'\circ g^{-1}\left(\frac{g(a)+g(b)}{2}\right)}{g'\circ g^{-1}\left(\frac{g(a)+g(b)}{2}\right)}\left[g(b)-g(a)\right]\right].\end{aligned}$$

Proof. If a function $\varphi:[0,1]\to\mathbb{R}$ is convex on $[0,1]$, then by Hermite-Hadamard inequality we have

$$(2.9)\qquad \varphi\left(\frac{1}{2}\right)\leq\int_0^1\varphi(s)\,ds\leq\frac{1}{2}\left[\varphi(0)+\varphi(1)\right].$$

Let $t,\ \tau\in(a,b)$ with $t\neq\tau$. Since $\frac{f'\circ g^{-1}}{g'\circ g^{-1}}$ is convex on $(g(a),g(b))$, then the function

$$\varphi(s):=\frac{\left(f'\circ g^{-1}\right)\left((1-s)g(\tau)+sg(t)\right)}{\left(g'\circ g^{-1}\right)\left((1-s)g(\tau)+sg(t)\right)}$$

is convex on $[0,1]$ and by (2.9) we get

$$\begin{aligned}(2.10)\qquad \frac{\left(f'\circ g^{-1}\right)\left(\frac{g(\tau)+g(t)}{2}\right)}{\left(g'\circ g^{-1}\right)\left(\frac{g(\tau)+g(t)}{2}\right)} &\leq\int_0^1\frac{\left(f'\circ g^{-1}\right)\left((1-s)g(\tau)+sg(t)\right)}{\left(g'\circ g^{-1}\right)\left((1-s)g(\tau)+sg(t)\right)}ds\\ &\leq\frac{1}{2}\left[\frac{\left(f'\circ g^{-1}\right)(g(\tau))}{\left(g'\circ g^{-1}\right)(g(\tau))}+\frac{\left(f'\circ g^{-1}\right)(g(t))}{\left(g'\circ g^{-1}\right)(g(t))}\right]\\ &=\frac{1}{2}\left[\frac{f'(\tau)}{g'(\tau)}+\frac{f'(t)}{g'(t)}\right]\end{aligned}$$

for $t,\ \tau\in(a,b)$ with $t\neq\tau$.

If we multiply the inequality (2.10) by $g'(\tau) \geq 0$ and take the PV, then we get

$$PV\int_a^b \frac{\left(f'\circ g^{-1}\right)\left(\frac{g(\tau)+g(t)}{2}\right)}{\left(g'\circ g^{-1}\right)\left(\frac{g(\tau)+g(t)}{2}\right)} g'(\tau)\,d\tau \tag{2.11}$$
$$\leq PV\int_a^b \left(\int_0^1 \frac{\left(f'\circ g^{-1}\right)\left((1-s)\,g(\tau)+sg(t)\right)}{\left(g'\circ g^{-1}\right)\left((1-s)\,g(\tau)+sg(t)\right)} ds\right) g'(\tau)\,d\tau$$
$$\leq \frac{1}{2} PV\int_a^b \left[\frac{f'(\tau)}{g'(\tau)} + \frac{f'(t)}{g'(t)}\right] g'(\tau)\,d\tau$$

for $t \in (a,b)$.

Observe that

$$PV\int_a^b \left[\frac{f'(\tau)}{g'(\tau)} + \frac{f'(t)}{g'(t)}\right] g'(\tau)\,d\tau = \int_a^b \frac{f'(\tau)}{g'(\tau)} g'(\tau)\,d\tau + \frac{f'(t)}{g'(t)} \int_a^b g'(\tau)\,d\tau \tag{2.12}$$
$$= f(b) - f(a) + \frac{f'(t)}{g'(t)}\left[g(b) - g(a)\right]$$

for $t \in (a,b)$.

Also, by the identity (2.5) we have

$$\frac{\left(f'\circ g^{-1}\right)\left(\frac{g(\tau)+g(t)}{2}\right)}{\left(g'\circ g^{-1}\right)\left(\frac{g(\tau)+g(t)}{2}\right)} = \left(f\circ g^{-1}\right)'\left(\frac{g(\tau)+g(t)}{2}\right)$$

for $t,\ \tau \in (a,b)$, which gives

$$PV\int_a^b \frac{\left(f'\circ g^{-1}\right)\left(\frac{g(\tau)+g(t)}{2}\right)}{\left(g'\circ g^{-1}\right)\left(\frac{g(\tau)+g(t)}{2}\right)} g'(\tau)\,d\tau \tag{2.13}$$
$$= \int_a^b \frac{\left(f'\circ g^{-1}\right)\left(\frac{g(\tau)+g(t)}{2}\right)}{\left(g'\circ g^{-1}\right)\left(\frac{g(\tau)+g(t)}{2}\right)} g'(\tau)\,d\tau$$
$$= \int_a^b \left(f\circ g^{-1}\right)'\left(\frac{g(\tau)+g(t)}{2}\right) g'(\tau)\,d\tau$$
$$= 2\int_a^b \left(f\circ g^{-1}\right)'\left(\frac{g(\tau)+g(t)}{2}\right) d\left(\frac{g(\tau)+g(t)}{2}\right)$$
$$= 2\left[\left(f\circ g^{-1}\right)\left(\frac{g(b)+g(t)}{2}\right) - \left(f\circ g^{-1}\right)\left(\frac{g(a)+g(t)}{2}\right)\right]$$
$$= 2\left[f\left(M_g(b,t)\right) - f\left(M_g(t,a)\right)\right]$$

for $t \in (a,b)$.

From (2.11) we get

$$\begin{aligned}
&2\left[f\left(M_g\left(b,t\right)\right)-f\left(M_g\left(t,a\right)\right)\right]\\
&\leq PV\int_a^b\left(\int_0^1\frac{\left(f'\circ g^{-1}\right)\left(\left(1-s\right)g\left(\tau\right)+sg\left(t\right)\right)}{\left(g'\circ g^{-1}\right)\left(\left(1-s\right)g\left(\tau\right)+sg\left(t\right)\right)}ds\right)g'\left(\tau\right)d\tau\\
&\leq\frac{1}{2}\left[f\left(b\right)-f\left(a\right)+\frac{f'\left(t\right)}{g'\left(t\right)}\left[g\left(b\right)-g\left(a\right)\right]\right]
\end{aligned}$$

for $t\in(a,b)$ and by using the identity (2.4) we obtain the desired result (2.7). □

Remark 2. *If we take $g(t)=t$, $t\in(a,b)$ in Theorem 4, then we recapture Theorem 3.*

We have:

Theorem 5. *Let $g:[a,b]\to[g(a),g(b)]$ be a strictly increasing function that is differentiable on (a,b) and $f:(a,b)\to\mathbb{C}$ a differentiable function on (a,b). Assume that $\left|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right|$ is convex on $(g(a),g(b))$, then*

$$\begin{aligned}
&\left|\left(T_gf\right)\left(a,b;t\right)-\frac{1}{\pi}f\left(t\right)\ln\left(\frac{g\left(b\right)-g\left(t\right)}{g\left(t\right)-g\left(a\right)}\right)\right|\\
&\leq\frac{1}{2\pi}\left[\int_a^b\left|f'\left(\tau\right)\right|d\tau+\frac{\left|f'\left(t\right)\right|}{g'\left(t\right)}\left(g\left(b\right)-g\left(a\right)\right)\right]
\end{aligned}\tag{2.14}$$

for any $t\in(a,b)$.

In particular, for $t=M_g(a,b)$ we get

$$\begin{aligned}
&\left|\left(T_gf\right)\left(a,b;M_g\left(a,b\right)\right)\right|\\
&\leq\frac{1}{2\pi}\left[\int_a^b\left|f'\left(\tau\right)\right|d\tau+\frac{\left|f'\left(M_g\left(a,b\right)\right)\right|}{g'\left(M_g\left(a,b\right)\right)}\left[g\left(b\right)-g\left(a\right)\right]\right].
\end{aligned}\tag{2.15}$$

Proof. By Hermite-Hadamard inequality for the convex function $\left|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right|$ we get

$$\begin{aligned}
&\int_0^1\left|\frac{\left(f'\circ g^{-1}\right)\left(\left(1-s\right)g\left(\tau\right)+sg\left(t\right)\right)}{\left(g'\circ g^{-1}\right)\left(\left(1-s\right)g\left(\tau\right)+sg\left(t\right)\right)}\right|ds\\
&\leq\frac{1}{2}\left[\left|\frac{\left(f'\circ g^{-1}\right)\left(g\left(\tau\right)\right)}{\left(g'\circ g^{-1}\right)\left(g\left(\tau\right)\right)}\right|+\left|\frac{\left(f'\circ g^{-1}\right)\left(g\left(t\right)\right)}{\left(g'\circ g^{-1}\right)\left(g\left(t\right)\right)}\right|\right]\\
&=\frac{1}{2}\left[\left|\frac{f'\left(\tau\right)}{g'\left(\tau\right)}\right|+\left|\frac{f'\left(t\right)}{g'\left(t\right)}\right|\right]
\end{aligned}$$

for $t,\ \tau \in (a,b)$, which gives

$$\begin{aligned} PV\int_a^b \left(\int_0^1 \left| \frac{(f' \circ g^{-1})((1-s)g(\tau)+sg(t))}{(g' \circ g^{-1})((1-s)g(\tau)+sg(t))} \right| ds \right) g'(\tau)\,d\tau & \\ \leq \frac{1}{2} PV \int_a^b \left[\left| \frac{f'(\tau)}{g'(\tau)} \right| + \left| \frac{f'(t)}{g'(t)} \right| \right] g'(\tau)\,d\tau & \\ = \frac{1}{2} \int_a^b \left[\left| \frac{f'(\tau)}{g'(\tau)} \right| + \left| \frac{f'(t)}{g'(t)} \right| \right] g'(\tau)\,d\tau & \\ = \frac{1}{2} \left[\int_a^b |f'(\tau)|\,d\tau + \left| \frac{f'(t)}{g'(t)} \right| (g(b)-g(a)) \right] & \end{aligned} \tag{2.16}$$

for $t \in (a,b)$.

By using the identity (2.4) we get the desired result (2.14). □

Remark 3. *If we take $g(t)=t$, $t \in (a,b)$ in Theorem 5, then, by assuming that $|f'|$ is convex on (a,b), we get*

$$\left| (Tf)(a,b;t) - \frac{1}{\pi} f(t) \ln\left(\frac{b-t}{t-a} \right) \right| \leq \frac{1}{2\pi} \left[\int_a^b |f'(\tau)|\,d\tau + |f'(t)|(b-a) \right] \tag{2.17}$$

for any $t \in (a,b)$.

In particular, for $t = \frac{a+b}{2}$ we get

$$\left| (Tf)\left(a,b;\frac{a+b}{2}\right) \right| \leq \frac{1}{2\pi} \left[\int_a^b |f'(\tau)|\,d\tau + \left| f'\left(\frac{a+b}{2}\right) \right| (b-a) \right]. \tag{2.18}$$

3. Examples

For $[a,b] \subset (0,\infty)$ and $g(t) = \ln t$, $t \in [a,b]$, consider the following *Logarithmic Finite Hilbert transform*

$$(T_{\ln} f)(a,b;t) := \frac{1}{\pi} \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} \frac{f(\tau)}{\tau \ln\left(\frac{\tau}{t}\right)} d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)}{\tau \ln\left(\frac{\tau}{t}\right)} d\tau \right] \tag{3.1}$$

where $t \in (a,b) \subset (0,\infty)$.

Let $f : (a,b) \to \mathbb{R}$ be a differentiable function on (a,b). Assume that $\exp \cdot (f' \circ \exp)$ is convex on $(\ln a, \ln b)$, then by Theorem 4

$$\begin{aligned} \frac{2}{\pi} [f(G(t,b)) - f(G(a,t))] & \\ \leq (T_{\ln} f)(a,b;t) - \frac{1}{\pi} f(t) \ln\left(\frac{\ln\left(\frac{b}{t}\right)}{\ln\left(\frac{t}{a}\right)} \right) & \\ \leq \frac{1}{2\pi} \left[f(b) - f(a) + t f'(t) \ln\left(\frac{b}{a}\right) \right] & \end{aligned} \tag{3.2}$$

for any $t \in (a,b)$.

In particular, for $t=G(a,b)$ we get

$$\begin{aligned}
&\frac{2}{\pi}\left[f\left(\sqrt[4]{b^3a}\right)-f\left(\sqrt[4]{a^3b}\right)\right] \\
&\leq (T_{\ln}f)(a,b;G(a,b)) \\
&\leq \frac{1}{2\pi}\left[f(b)-f(a)+G(a,b)f'(G(a,b))\ln\left(\frac{b}{a}\right)\right].
\end{aligned} \tag{3.3}$$

Let $f:(a,b)\to\mathbb{C}$ be a differentiable function on (a,b). Assume that $\exp\cdot(|f'|\circ\exp)$ is convex on $(\ln a,\ln b)$, then by Theorem 5

$$\begin{aligned}
&\left|(T_{\ln}f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{\ln\left(\frac{b}{t}\right)}{\ln\left(\frac{t}{a}\right)}\right)\right| \\
&\leq \frac{1}{2\pi}\left[\int_a^b |f'(\tau)|\,d\tau + t\,|f'(t)|\ln\left(\frac{b}{a}\right)\right]
\end{aligned} \tag{3.4}$$

for any $t\in(a,b)$.

In particular, for $t=G(a,b)$ we get

$$\begin{aligned}
&|(T_{\ln}f)(a,b;G(a,b))| \\
&\leq \frac{1}{2\pi}\left[\int_a^b |f'(\tau)|\,d\tau + G(a,b)\,|f'(G(a,b))|\ln\left(\frac{b}{a}\right)\right].
\end{aligned} \tag{3.5}$$

For $g(t)=\exp(\alpha t)$, $t\in[a,b]\subset\mathbb{R}$ with $\alpha>0$, consider the *Exponential Finite Hilbert transform*

$$\begin{aligned}
&\left(T_{\exp(\alpha)}f\right)(a,b;t) \\
&:= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)\exp(\alpha\tau)}{\exp(\alpha\tau)-\exp(\alpha t)}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)\exp(\alpha\tau)}{\exp(\alpha\tau)-\exp(\alpha t)}d\tau\right] \\
&= \frac{1}{\pi}\exp(-\alpha t) \\
&\times \lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)\exp(\alpha(\tau-t))}{\exp(\alpha(\tau-t))-1}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)\exp(\alpha(\tau-t))}{\exp(\alpha(\tau-t))-1}d\tau\right]
\end{aligned} \tag{3.6}$$

where $t\in(a,b)\subset\mathbb{R}$.

Let $f:(a,b)\to\mathbb{R}$ be a differentiable function on (a,b). Assume that $\frac{f'\circ\left(\frac{1}{\alpha}\ln\right)}{\alpha\ell}$ is convex on $(\exp(\alpha a),\exp(\alpha b))$, where $\ell(t)=t$, $t\in(a,b)$, then by Theorem 4 we have

$$\begin{aligned}
&\frac{2}{\pi}\left[f(LME_\alpha(t,b))-f(LME_\alpha(a,t))\right] \\
&\leq \left(T_{\exp(\alpha)}f\right)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{\exp(\alpha b)-\exp(\alpha t)}{\exp(\alpha t)-\exp(\alpha a)}\right) \\
&\leq \frac{1}{2\pi}\left[f(b)-f(a)+\frac{f'(t)}{\alpha\exp(\alpha t)}\left[\exp(\alpha b)-\exp(\alpha a)\right]\right]
\end{aligned} \tag{3.7}$$

for any $t\in(a,b)$.

For $t = LME_\alpha(a,b)$ we get

$$\frac{2}{\pi}\left\{ f\left[\ln\left(\frac{3\exp(\alpha b)+\exp(\alpha a)}{4}\right)^{1/\alpha}\right] - f\left[\ln\left(\frac{3\exp(\alpha a)+\exp(\alpha b)}{4}\right)^{1/\alpha}\right]\right\} \le \left(T_{\exp(\alpha)}f\right)(a,b;LME_\alpha(a,b)) \le \frac{1}{2\pi}\left[f(b)-f(a)+\frac{f'(LME_\alpha(a,b))}{\alpha\left(\frac{\exp(\alpha a)+\exp(\alpha b)}{2}\right)}[\exp(\alpha b)-\exp(\alpha a)]\right]. \tag{3.8}$$

Let $f:(a,b)\to\mathbb{C}$ be a differentiable function on (a,b). Assume that $\frac{|f'|\circ\left(\frac{1}{\alpha}\ln\right)}{\alpha\ell}$ is convex on $(\exp(\alpha a),\exp(\alpha b))$, then by Theorem 5 we have

$$\left|\left(T_{\exp(\alpha)}f\right)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{\exp(\alpha b)-\exp(\alpha t)}{\exp(\alpha t)-\exp(\alpha a)}\right)\right| \le \frac{1}{2\pi}\left[\int_a^b|f'(\tau)|\,d\tau+\frac{|f'(t)|}{\alpha\exp(\alpha t)}[\exp(\alpha b)-\exp(\alpha a)]\right] \tag{3.9}$$

for any $t\in(a,b)$.

For $t = LME_\alpha(a,b)$ we get

$$\left|\left(T_{\exp(\alpha)}f\right)(a,b;LME_\alpha(a,b))\right| \le \frac{1}{2\pi}\left[\int_a^b|f'(\tau)|\,d\tau+\frac{|f'(LME_\alpha(a,b))|}{\alpha\left(\frac{\exp(\alpha a)+\exp(\alpha b)}{2}\right)}[\exp(\alpha b)-\exp(\alpha a)]\right]. \tag{3.10}$$

Let $g:[a,b]\subset(0,\infty)\to\mathbb{R}$, $g(t)=-\frac{1}{t}$ and the *Harmonic Finite Hilbert transform*

$$(T_{-1}f)(a,b;t) := \frac{t}{\pi}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)}{\tau(\tau-t)}d\tau+\int_{t+\varepsilon}^b\frac{f(\tau)}{\tau(\tau-t)}d\tau\right], \tag{3.11}$$

where $t\in(a,b)$.

Let $f:(a,b)\to\mathbb{C}$ be a differentiable function on (a,b). Assume that $(\cdot)^{-2}\cdot f'\circ(\cdot)^{-2}$ is convex on $\left(-\frac{1}{a},-\frac{1}{b}\right)$, then by Theorem 4 we have

$$\frac{2}{\pi}[f(H(b,t))-f(H(t,a))] \le (T_{-1}f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{(b-t)a}{b(t-a)}\right) \le \frac{1}{2\pi}\left[f(b)-f(a)+\frac{t^2}{ab}f'(t)(b-a)\right] \tag{3.12}$$

for any $t\in(a,b)$.

In particular, for $t=H(a,b)$ we get

$$\frac{2}{\pi}\left[f\left(\frac{4ab}{3a+b}\right)-f\left(\frac{4ab}{a+3b}\right)\right] \le (T_{-1}f)(a,b;H(a,b)) \le \frac{1}{2\pi}\left[f(b)-f(a)+\frac{H^{2}(a,b)}{ab}f'(H(a,b))(b-a)\right]. \tag{3.13}$$

Let $f:(a,b)\to\mathbb{C}$ be a differentiable function on (a,b). Assume that $(\cdot)^{-2}\cdot|f'|\circ(\cdot)^{-2}$ is convex on $\left(-\frac{1}{a},-\frac{1}{b}\right)$, then by Theorem 5 we have

$$\left|(T_{-1}f)(a,b;t)-\frac{1}{\pi}f(t)\ln\left(\frac{(b-t)a}{b(t-a)}\right)\right| \le \frac{1}{2\pi}\left[\int_a^b |f'(\tau)|\,d\tau+\frac{t^{2}}{ab}|f'(t)|(b-a)\right] \tag{3.14}$$

for any $t\in(a,b)$.

In particular,

$$|(T_{-1}f)(a,b;H(a,b))| \le \frac{1}{2\pi}\left[\int_a^b |f'(\tau)|\,d\tau+\frac{H^{2}(a,b)}{ab}|f'(H(a,b))|(b-a)\right]. \tag{3.15}$$

References

[1] N. M. Dragomir, S. S. Dragomir and P. M. Farrell, Some inequalities for the finite Hilbert transform. *Inequality Theory and Applications.* Vol. I, 113–122, Nova Sci. Publ., Huntington, NY, 2001.

[2] N. M. Dragomir, S. S. Dragomir and P. M. Farrell, Approximating the finite Hilbert transform via trapezoid type inequalities. *Comput. Math. Appl.* **43** (2002), no. 10-11, 1359–1369.

[3] N. M. Dragomir, S. S. Dragomir, P. M. Farrell and G. W. Baxter, On some new estimates of the finite Hilbert transform. *Libertas Math.* **22** (2002), 65–75.

[4] N. M. Dragomir, S. S. Dragomir and P. M. Farrell and G. W. Baxter, A quadrature rule for the finite Hilbert transform via trapezoid type inequalities. *J. Appl. Math. Comput.* **13** (2003),no. 1-2, 67–84.

[5] N. M. Dragomir, S. S. Dragomir and P. M. Farrell and G. W. Baxter, A quadrature rule for the finite Hilbert transform via midpoint type inequalities. *Fixed point theory and applications.* Vol. 5, 11–22, Nova Sci. Publ., Hauppauge, NY, 2004.

[6] S. S. Dragomir, Inequalities for the Hilbert transform of functions whose derivatives are convex. *J. Korean Math. Soc.* **39** (2002), no. 5, 709–729.

[7] S. S. Dragomir, Approximating the finite Hilbert transform via an Ostrowski type inequality for functions of bounded variation. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 4, Article 51, 19 pp

[8] S. S. Dragomir, Approximating the finite Hilbert transform via Ostrowski type inequalities for absolutely continuous functions. *Bull. Korean Math. Soc.* **39** (2002), no. 4, 543–559.

[9] S. S. Dragomir, Some inequalities for the finite Hilbert transform of a product. *Commun. Korean Math. Soc.* **18** (2003), no. 1, 39–57.

[10] S. S. Dragomir, Sharp error bounds of a quadrature rule with one multiple node for the finite Hilbert transform in some classes of continuous differentiable functions. *Taiwanese J. Math.* **9** (2005), no. 1, 95–109.

[11] S. S. Dragomir, The perturbed median principle for integral inequalities with applications. Nonlinear analysis and variational problems, 53–63, Springer Optim. Appl., 35, Springer, New York, 2010.

[12] S. S. Dragomir, Inequalities and approximations for the Finite Hilbert transform: a survey of recent results, Preprint *RGMIA Res. Rep. Coll.* **21** (2018), Art. 30, 90 pp. [Online http://rgmia.org/papers/v21/v21a30.pdf].

[13] F. D. Gakhov, *Boundary Value Problems* (English translation), Pergamon Press, Oxford, 1966.

[14] W. Liu and X. Gao, Approximating the finite Hilbert transform via a companion of Ostrowski's inequality for function of bounded variation and applications. *Appl. Math. Comput.* **247** (2014), 373–385.

[15] W. Liu, X. Gao and Y. Wen, Approximating the finite Hilberttransform via some companions of Ostrowski's inequalities. *Bull. Malays. Math. Sci. Soc.* **39** (2016),no. 4, 1499–1513.

[16] W. Liu and N. Lu, Approximating the finite Hilbert transform via Simpson type inequalities and applications. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **77** (2015),no. 3, 107–122.

[17] S. G. Mikhlin and S. Prössdorf, *Singular Integral Operators* (English translation), Springer Verlag, Berlin, 1986.

[18] S. Wang, X. Gao and N. Lu, A quadrature formula in approximating the finite Hilbert transform via perturbed trapezoid type inequalities. *J. Comput. Anal. Appl.* **22** (2017),no. 2, 239–246.

[19] S. Wang, N. Lu and X. Gao, A quadrature rule for the finite Hilberttransform via Simpson type inequalities and applications. *J. Comput. Anal. Appl.* **22** (2017), no. 2, 229–238.

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