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INEQUALITIES OF HERMITE-HADAMARD TYPE FOR COMPOSITE h-CONVEX FUNCTIONS

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Abstract. In this paper we obtain some inequalities of Hermite-Hadamard type for composite convex functions. Applications for AG, AH-h-convex functions, GA, GG, GH-h-convex functions and HA, HG, HH-h-convex function are given.

1. Introduction

We recall here some concepts of convexity that are well known in the literature. Let I be an interval in $\mathbb{R}$.

Definition 1 ([52]). *We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have*

$$f(tx + (1-t)y) \le \frac{1}{t} f(x) + \frac{1}{1-t} f(y). \tag{1.1}$$

Some further properties of this class of functions can be found in [42], [43], [45], [58], [64] and [65]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

Definition 2 ([45]). *We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0, 1]$ we have*

$$f(tx + (1-t)y) \le f(x) + f(y). \tag{1.2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \le \max\{f(x), f(y)\} \tag{1.3}$$

for all $x, y \in I$ and $t \in [0, 1]$.

For some results on P-functions see [45] and [62] while for quasi convex functions, the reader can consult [44].

Definition 3 ([10]). *Let s be a real number, $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

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For some properties of this class of functions see [2], [3], [10], [11], [40], [41], [53], [55] and [67].

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}, (0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 4 ([70])**.** *Let $h : J \to [0, \infty)$ with h not identical to 0. We say that $f : I \to [0, \infty)$ is an h-convex function if for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y) \tag{1.4}$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [70], [9], [56], [68], [66] and [69].

We can introduce now another class of functions.

Definition 5. *We say that the function $f : I \to [0, \infty)$ is of s-Godunova-Levin type, with $s \in [0,1]$, if*

$$f(tx + (1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{1.5}$$

for all $t \in (0,1)$ and $x, y \in I$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of s-Godunova-Levin functions defined on C, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

If $f : I \to [0, \infty)$ is an h-convex function on an interval I of real numbers with $h \in L[0,1]$ and $f \in L[a,b]$ with $a, b \in I, a < b$, then we have the Hermite-Hadamard type inequality obtained by Sarikaya et al. in [66]

$$\begin{aligned} \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b f(u)\,du \\ &= \int_0^1 f((1-\lambda)a + \lambda b)\,d\lambda \leq [f(a) + f(b)] \int_0^1 h(t)\,dt. \end{aligned} \tag{1.6}$$

For an extension of this result to functions defined on convex subsets of linear spaces and refinements, see [31].

In order to extend this result for other classes of functions, we need the following preparations.

Let $g : [a,b] \to [g(a), g(b)]$ be a *continuous strictly increasing function* that is *differentiable* on (a,b).

Definition 6. *A function $f : [a,b] \to \mathbb{R}$ will be called composite-g^{-1} h-convex (concave) on $[a,b]$ if the composite function $f \circ g^{-1} : [g(a), g(b)] \to \mathbb{R}$ is h-convex (concave) in the usual sense on $[g(a), g(b)]$.*

If $f : [a,b] \to \mathbb{R}$ is composite-g^{-1} h-convex on $[a,b]$ then we have the inequality

$$f \circ g^{-1}((1-\lambda)u + \lambda v) \leq h(1-\lambda) f \circ g^{-1}(u) + h(\lambda) f \circ g^{-1}(v) \tag{1.7}$$

for any $u, v \in [g(a), g(b)]$ and $\lambda \in [0,1]$.

This is equivalent to the condition

$$f\circ g^{-1}\left(\left(1-\lambda \right) g\left(t\right) +\lambda g\left(s\right) \right) \leq h\left(1-\lambda \right) f\left(t\right) +h\left(\lambda \right) f\left(s\right) \tag{1.8}$$

for any $t,\ s\in \left[a,b\right]$ and $\lambda \in \left[0,1\right]$.

If we take $g\left(t\right) =\ln t,\ t\in \left[a,b\right] \subset \left(0,\infty \right)$, then the condition (1.8) becomes

$$f\left(t^{1-\lambda }s^{\lambda }\right) \leq h\left(1-\lambda \right) f\left(t\right) +h\left(\lambda \right) f\left(s\right) \tag{1.9}$$

for any $t,\ s\in \left[a,b\right]$ and $\lambda \in \left[0,1\right]$, which is the concept of *GA-h-convexity* as considered in [1].

If we take $g\left(t\right) =-\frac{1}{t},\ t\in \left[a,b\right] \subset \left(0,\infty \right)$, then (1.8) becomes

$$f\left(\frac{ts}{\left(1-\lambda \right) s+\lambda t}\right) \leq h\left(1-\lambda \right) f\left(t\right) +h\left(\lambda \right) f\left(s\right) \tag{1.10}$$

for any $t,\ s\in \left[a,b\right]$ and $\lambda \in \left[0,1\right]$, which is the concept of *HA-h-convexity* as considered in [5].

If $p>0$ and we consider $g\left(t\right) =t^{p},\ t\in \left[a,b\right] \subset \left(0,\infty \right)$, then the condition (1.8) becomes

$$f\left[\left(\left(1-\lambda \right) t^{p}+\lambda s^{p}\right) ^{1/p}\right] \leq h\left(1-\lambda \right) f\left(t\right) +h\left(\lambda \right) f\left(s\right) \tag{1.11}$$

for any $t,\ s\in \left[a,b\right]$ and $\lambda \in \left[0,1\right]$. For $h\left(t\right) =t$ the concept of *p-convexity* was considered in [71].

If we take $g\left(t\right) =\exp t,\ t\in \left[a,b\right]$, then the condition (1.8) becomes

$$f\left[\ln \left(\left(1-\lambda \right) \exp \left(t\right) +\exp g\left(s\right) \right) \right] \leq \left(1-\lambda \right) f\left(t\right) +\lambda f\left(s\right) , \tag{1.12}$$

which is the concept of *LogExp h-convex function* on $\left[a,b\right]$. For $h\left(t\right) =t$, the concept was considered in [28].

Further, assume that $f:\left[a,b\right] \rightarrow \mathcal{J}$, $\mathcal{J}$ an interval of real numbers and $k:\mathcal{J}\rightarrow \mathbb{R}$ a continuous function on $\mathcal{J}$ that is *strictly increasing (decreasing)* on $\mathcal{J}$.

Definition 7. *We say that the function $f:\left[a,b\right] \rightarrow \mathcal{J}$ is k-composite h-convex (concave) on $\left[a,b\right]$, if $k\circ f$ is h-convex (concave) on $\left[a,b\right]$.*

With $g:\left[a,b\right] \rightarrow \left[g\left(a\right) ,g\left(b\right) \right]$ a *continuous strictly increasing function* that is *differentiable* on $\left(a,b\right)$, $f:\left[a,b\right] \rightarrow \mathcal{J}$, $\mathcal{J}$ an interval of real numbers and $k:\mathcal{J}\rightarrow \mathbb{R}$ a continuous function on $\mathcal{J}$ that is *strictly increasing (decreasing)* on $\mathcal{J}$, we can also consider the following concept:

Definition 8. *We say that the function $f:\left[a,b\right] \rightarrow \mathcal{J}$ is k-composite-g^{-1} h-convex (concave) on $\left[a,b\right]$, if $k\circ f\circ g^{-1}$ is h-convex (concave) on $\left[g\left(a\right) ,g\left(b\right) \right]$.*

This definition is equivalent to the condition

$$k\circ f\circ g^{-1}\left(\left(1-\lambda \right) g\left(t\right) +\lambda g\left(s\right) \right) \leq h\left(1-\lambda \right) \left(k\circ f\right) \left(t\right) +h\left(\lambda \right) \left(k\circ f\right) \left(s\right) \tag{1.13}$$

for any $t,\ s\in \left[a,b\right]$ and $\lambda \in \left[0,1\right]$.

If $k:\mathcal{J}\rightarrow \mathbb{R}$ is *strictly increasing (decreasing)* on $\mathcal{J}$, then the condition (1.13) is equivalent to:

$$\begin{aligned} & f\circ g^{-1}\left(\left(1-\lambda \right) g\left(t\right) +\lambda g\left(s\right) \right) \\ & \leq \left(\geq \right) k^{-1}\left[h\left(1-\lambda \right) \left(k\circ f\right) \left(t\right) +h\left(\lambda \right) \left(k\circ f\right) \left(s\right) \right] \end{aligned} \tag{1.14}$$

for any $t,\ s\in \left[a,b\right]$ and $\lambda \in \left[0,1\right]$.

If $k(t)=\ln t$, $t>0$ and $f:[a,b]\to(0,\infty)$, then the fact that f is k-composite h-convex on $[a,b]$ is equivalent to the fact that f is *log-convex* or *multiplicatively convex* or AG-h-convex, namely, for all $x,y\in I$ and $t\in[0,1]$ one has the inequality:

$$f(tx+(1-t)y)\le[f(x)]^{h(t)}[f(y)]^{h(1-t)}. \tag{1.15}$$

A function $f:I\to\mathbb{R}\setminus\{0\}$ is called *AH-h-convex (concave)* on the interval I if the following inequality holds [1]

$$f((1-\lambda)x+\lambda y)\le(\ge)\frac{f(x)f(y)}{h(1-\lambda)f(y)+h(\lambda)f(x)} \tag{1.16}$$

for any $x,y\in I$ and $\lambda\in[0,1]$.

An important case that provides many examples is that one in which the function is assumed to be positive for any $x\in I$. In that situation the inequality (1.16) is equivalent to

$$h(1-\lambda)\frac{1}{f(x)}+h(\lambda)\frac{1}{f(y)}\le(\ge)\frac{1}{f((1-\lambda)x+\lambda y)}$$

for any $x,y\in I$ and $\lambda\in[0,1]$.

Taking into account this fact, we can conclude that the function $f:I\to(0,\infty)$ is AH-h-convex (concave) on I if and only if f is k-composite h-concave (convex) on I with $k:(0,\infty)\to(0,\infty)$, $k(t)=\frac{1}{t}$.

Following [1], we can introduce the concept of *GH-h-convex (concave)* function $f:I\subset(0,\infty)\to\mathbb{R}$ on an interval of positive numbers I as satisfying the condition

$$f\left(x^{1-\lambda}y^{\lambda}\right)\le(\ge)\frac{f(x)f(y)}{h(1-\lambda)f(y)+h(\lambda)f(x)}. \tag{1.17}$$

Since

$$f\left(x^{1-\lambda}y^{\lambda}\right)=f\circ\exp[(1-\lambda)\ln x+\lambda\ln y]$$

and

$$\frac{f(x)f(y)}{h(1-\lambda)f(y)+h(\lambda)f(x)}=\frac{f\circ\exp(\ln x)f\circ\exp(\ln y)}{h(1-\lambda)f\circ\exp(y)+h(\lambda)f\circ\exp(x)}$$

then $f:I\subset(0,\infty)\to\mathbb{R}$ is GH-convex (concave) on I if and only if $f\circ\exp$ is *AH-convex (concave)* on $\ln I:=\{x|\ x=\ln t,\ t\in I\}$. This is equivalent to the fact that f is k-composite-g^{-1} h-concave (convex) on I with $k:(0,\infty)\to(0,\infty)$, $k(t)=\frac{1}{t}$ and $g(t)=\ln t$, $t\in I$.

Following [1], we say that the function $f:I\subset\mathbb{R}\setminus\{0\}\to(0,\infty)$ is *HH-h-convex* if

$$f\left(\frac{xy}{tx+(1-t)y}\right)\le\frac{f(x)f(y)}{h(1-t)f(y)+h(t)f(x)} \tag{1.18}$$

for all $x,y\in I$ and $t\in[0,1]$. If the inequality in (1.18) is reversed, then f is said to be *HH-h-concave.*

We observe that the inequality (1.18) is equivalent to

$$h(1-t)\frac{1}{f(x)}+h(t)\frac{1}{f(y)}\le\frac{1}{f\left(\frac{xy}{tx+(1-t)y}\right)} \tag{1.19}$$

for all $x,y\in I$ and $t\in[0,1]$.

This is equivalent to the fact that f is k-composite-g^{-1} h-concave on $[a,b]$ with $k:(0,\infty)\to(0,\infty)$, $k(t)=\frac{1}{t}$ and $g(t)=-\frac{1}{t}$, $t\in[a,b]$.

The function $f:I\subset(0,\infty)\to(0,\infty)$ is called *GG-h-convex* on the interval I of real umbers $\mathbb{R}$ if [5]

$$f\left(x^{1-\lambda}y^{\lambda}\right)\leq[f(x)]^{h(1-\lambda)}[f(y)]^{h(\lambda)} \tag{1.20}$$

for any $x,\ y\in I$ and $\lambda\in[0,1]$. If the inequality is reversed in (1.20) then the function is called *GG-h-concave.*

For $h(t)=t$, this concept was introduced in 1928 by P. Montel [59], however, the roots of the research in this area can be traced long before him [60]. It is easy to see that [60], the function $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is *GG-h-convex* if and only if the the function $g:[\ln a,\ln b]\to\mathbb{R}$, $g=\ln\circ f\circ\exp$ is h-convex on $[\ln a,\ln b]$. This is equivalent to the fact that f is k-composite-g^{-1} h-convex on $[a,b]$ with $k:(0,\infty)\to\mathbb{R}$, $k(t)=\ln t$ and $g(t)=\ln t$, $t\in[a,b]$.

Following [1] we say that the function $f:I\subset\mathbb{R}\setminus\{0\}\to(0,\infty)$ is *HG-h-convex* if

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leq[f(x)]^{h(1-t)}[f(y)]^{h(t)} \tag{1.21}$$

for all $x,\ y\in I$ and $t\in[0,1]$. If the inequality in (1.8) is reversed, then f is said to be *HG-h-concave.*

Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ and define the associated functions $G_f:\left[\frac{1}{b},\frac{1}{a}\right]\to\mathbb{R}$ defined by $G_f(t)=\ln f\left(\frac{1}{t}\right)$. Then f is *HG-h-convex* on $[a,b]$ iff G_f is h-convex on $\left[\frac{1}{b},\frac{1}{a}\right]$. This is equivalent to the fact that f is k-composite-g^{-1} h-convex on $[a,b]$ with $k:(0,\infty)\to\mathbb{R}$, $k(t)=\ln t$ and $g(t)=-\frac{1}{t}$, $t\in[a,b]$.

We say that the function $f:[a,b]\to(0,\infty)$ is r-h-convex, for $r\neq0$, if

$$f((1-\lambda)x+\lambda y)\leq[h(1-\lambda)f^{r}(y)+h(\lambda)f^{r}(x)]^{1/r} \tag{1.22}$$

for any $x,\ y\in[a,b]$ and $\lambda\in[0,1]$. For $h(t)=t$, the concept was considered in [61],

If $r>0$, then the condition (1.22) is equivalent to

$$f^{r}((1-\lambda)x+\lambda y)\leq h(1-\lambda)f^{r}(y)+h(\lambda)f^{r}(x)$$

namely f is k-composite convex on $[a,b]$ where $k(t)=t^{r}$, $t\geq0$.

If $r<0$, then the condition (1.22) is equivalent to

$$f^{r}((1-\lambda)x+\lambda y)\geq h(1-\lambda)f^{r}(y)+h(\lambda)f^{r}(x)$$

namely f is k-composite h-concave on $[a,b]$ where $k(t)=t^{r}$, $t>0$.

In this paper we obtain some inequalities of Hermite-Hadamard type for composite convex functions. Applications for AG, AH-h-convex functions, GA, GG, GH-h-convex functions and HA, HG, HH-h-convex function are given.

2. Refinements of HH-Inequality

The following representation result holds.

Lemma 1. *Let $f:I\subseteq\mathbb{R}\to\mathbb{C}$ where I is an interval of the real numbers $\mathbb{R}$. Let $y,\ x\in I$ with $y\neq x$ and assume that the mapping $[0,1]\ni t\mapsto f\circ g^{-1}[(1-t)g(x)+tg(y)]$ is Lebesgue integrable on $[0,1]$. Then for any $\lambda\in[0,1]$*

we have the representation

$$\int_0^1 f\circ g^{-1}\left[(1-t)\,g(x)+tg(y)\right]dt \tag{2.1}$$
$$=(1-\lambda)\int_0^1 f\circ g^{-1}\left[(1-t)\left((1-\lambda)\,g(x)+\lambda g(y)\right)+tg(y)\right]dt$$
$$+\lambda\int_0^1 f\circ g^{-1}\left[(1-t)\,g(x)+t\left((1-\lambda)\,g(x)+\lambda g(y)\right)\right]dt.$$

Proof. For $\lambda=0$ and $\lambda=1$ the equality (2.1) is obvious.

Let $\lambda\in(0,1)$. Observe that

$$\int_0^1 f\circ g^{-1}\left[(1-t)\left(\lambda g(y)+(1-\lambda)\,g(x)\right)+tg(y)\right]dt$$
$$=\int_0^1 f\circ g^{-1}\left[\left((1-t)\,\lambda+t\right)g(y)+(1-t)(1-\lambda)\,g(x)\right]dt$$

and

$$\int_0^1 f\circ g^{-1}\left[t\left(\lambda g(y)+(1-\lambda)\,g(x)\right)+(1-t)\,g(x)\right]dt$$
$$=\int_0^1 f\circ g^{-1}\left[t\lambda g(y)+(1-\lambda t)\,g(x)\right]dt.$$

If we make the change of variable $u:=(1-t)\,\lambda+t$ then we have $1-u=(1-t)(1-\lambda)$ and $du=(1-\lambda)\,du$. Then

$$\int_0^1 f\circ g^{-1}\left[\left((1-t)\,\lambda+t\right)g(y)+(1-t)(1-\lambda)\,g(x)\right]dt$$
$$=\frac{1}{1-\lambda}\int_\lambda^1 f\circ g^{-1}\left[ug(y)+(1-u)\,g(x)\right]du.$$

If we make the change of variable $u:=\lambda t$ then we have $du=\lambda dt$ and

$$\int_0^1 f\circ g^{-1}\left[t\lambda g(y)+(1-\lambda t)\,g(x)\right]dt=\frac{1}{\lambda}\int_0^\lambda f\circ g^{-1}\left[ug(y)+(1-u)\,g(x)\right]du.$$

Therefore

$$(1-\lambda)\int_0^1 f\circ g^{-1}\left[(1-t)\left(\lambda g(y)+(1-\lambda)\,g(x)\right)+tg(y)\right]dt$$
$$+\lambda\int_0^1 f\circ g^{-1}\left[t\left(\lambda g(y)+(1-\lambda)\,g(x)\right)+(1-t)\,g(x)\right]dt$$
$$=\int_\lambda^1 f\circ g^{-1}\left[ug(y)+(1-u)\,g(x)\right]du+\int_0^\lambda f\circ g^{-1}\left[ug(y)+(1-u)\,g(x)\right]du$$
$$=\int_0^1 f\circ g^{-1}\left[ug(y)+(1-u)\,g(x)\right]du$$

and the identity (2.1) is proved. □

Theorem 1. *Assume that the function $f : I \subseteq \mathbb{R} \to [0,\infty)$ is a composite-g^{-1} h-convex function with $h \in L[0,1]$. Let y, $x \in I$ with $y \neq x$, then for any $\lambda \in [0,1]$ we have the inequalities*

$$\begin{aligned} &\frac{1}{2h\left(\frac{1}{2}\right)}\left\{(1-\lambda) f\circ g^{-1}\left[\frac{(1-\lambda)g(x)+(\lambda+1)g(y)}{2}\right]\right. \\ &\qquad\left.+\lambda f\circ g^{-1}\left[\frac{(2-\lambda)g(x)+\lambda g(y)}{2}\right]\right\} \\ &\leq \frac{1}{g(y)-g(x)}\int_x^y f(t)\,g'(t)\,dt \\ &\leq \left[f\circ g^{-1}\left((1-\lambda)g(x)+\lambda g(y)\right)+(1-\lambda)f(y)+\lambda f(x)\right]\int_0^1 h(t)\,dt \\ &\leq \left\{[h(1-\lambda)+\lambda] f(x)+[h(\lambda)+1-\lambda] f(y)\right\}\int_0^1 h(t)\,dt. \end{aligned} \tag{2.2}$$

If $f : I \subseteq \mathbb{R} \to [0,\infty)$ is a composite-g^{-1} h-concave function, then the inequalities reverse in (2.2).

Proof. Since $f : I \subseteq \mathbb{R} \to [0,\infty)$ is a composite-g^{-1} h-convex function function, then by Hermite-Hadamard type inequality (1.6) we have

$$\begin{aligned} &\frac{1}{2h\left(\frac{1}{2}\right)} f\circ g^{-1}\left[\frac{(1-\lambda)g(x)+(\lambda+1)g(y)}{2}\right] \\ &\leq \int_0^1 f\circ g^{-1}\left[(1-t)\left((1-\lambda)g(x)+\lambda g(y)\right)+t g(y)\right]dt \\ &\leq \left[f\circ g^{-1}\left((1-\lambda)g(x)+\lambda g(y)\right)+f(y)\right]\int_0^1 h(t)\,dt \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\frac{1}{2h\left(\frac{1}{2}\right)} f\circ g^{-1}\left[\frac{(2-\lambda)g(x)+\lambda g(y)}{2}\right] \\ &\leq \int_0^1 f\circ g^{-1}\left[(1-t)g(x)+t\left((1-\lambda)g(x)+\lambda g(y)\right)\right]dt \\ &\leq \left[f(x)+f\circ g^{-1}\left((1-\lambda)g(x)+\lambda g(y)\right)\right]\int_0^1 h(t)\,dt. \end{aligned} \tag{2.4}$$

Now, if we multiply the inequality (2.3) by $1-\lambda \geq 0$ and (2.4) by $\lambda \geq 0$ and add the obtained inequalities, then we get

$$\begin{aligned} &\frac{1-\lambda}{2h\left(\frac{1}{2}\right)} f\circ g^{-1}\left[\frac{(1-\lambda)g(x)+(\lambda+1)g(y)}{2}\right] \\ &+\frac{\lambda}{2h\left(\frac{1}{2}\right)} f\circ g^{-1}\left[\frac{(2-\lambda)g(x)+\lambda g(y)}{2}\right] \end{aligned} \tag{2.5}$$

$$\leq (1-\lambda)\int_0^1 f\circ g^{-1}\left[(1-t)\left((1-\lambda)g(x)+\lambda g(y)\right)+tg(y)\right]dt$$
$$+\lambda\int_0^1 f\circ g^{-1}\left[(1-t)g(x)+t\left((1-\lambda)g(x)+\lambda g(y)\right)\right]dt$$
$$\leq (1-\lambda)\left[f\circ g^{-1}\left((1-\lambda)g(x)+\lambda g(y)\right)+f(y)\right]\int_0^1 h(t)\,dt$$
$$+\lambda\left[f(x)+f\circ g^{-1}\left((1-\lambda)g(x)+\lambda g(y)\right)\right]\int_0^1 h(t)\,dt$$

and by (2.1) we obtain

$$\begin{aligned}(2.6)\quad &\frac{1}{2h\left(\frac{1}{2}\right)}\left\{(1-\lambda)f\circ g^{-1}\left[\frac{(1-\lambda)g(x)+(\lambda+1)g(y)}{2}\right]\right.\\ &\left.+\lambda f\circ g^{-1}\left[\frac{(2-\lambda)g(x)+\lambda g(y)}{2}\right]\right\}\\ &\leq \int_0^1 f\circ g^{-1}\left[(1-t)g(x)+tg(y)\right]dt\\ &\leq \left[f\circ g^{-1}\left((1-\lambda)g(x)+\lambda g(y)\right)+(1-\lambda)f(y)+\lambda f(x)\right]\int_0^1 h(t)\,dt\\ &\leq \left\{\left[h(1-\lambda)+\lambda\right]f(x)+\left[h(\lambda)+1-\lambda\right]f(y)\right\}\int_0^1 h(t)\,dt,\end{aligned}$$

where the last inequality follows by the definition of composite-g^{-1} h-convexity and performing the required calculation.

By using the change of variable $u=(1-t)g(x)+tg(y)$, we have $du=(g(y)-g(x))\,dt$ and then

$$\int_0^1 f\circ g^{-1}\left[(1-t)g(x)+tg(y)\right]dt=\frac{1}{g(y)-g(x)}\int_{g(x)}^{g(y)} f\circ g^{-1}(u)\,du.$$

If we change the variable $t=g^{-1}(u)$, then $u=g(t)$, which gives that $du=g'(t)\,dt$ and then

$$\int_{g(x)}^{g(y)} f\circ g^{-1}(u)\,du=\int_x^y f(t)\,g'(t)\,dt$$

and the inequality (2.2) is obtained. □

Remark 1. *With the assumptions from Theorem 1, we observe that if we take either $\lambda=0$ or $\lambda=1$ in the first two inequalities in (2.2), then we get (1.6).*

If we take $\lambda = \frac{1}{2}$ *and use the h-convexity of* $f\circ g^{-1}$*, then we get from (2.2) that*

$$\begin{aligned}
&\frac{1}{4h^2\left(\frac{1}{2}\right)} f\circ g^{-1}\left(\frac{g(x)+g(y)}{2}\right) \\
&\le \frac{1}{4h\left(\frac{1}{2}\right)}\left\{f\circ g^{-1}\left(\frac{g(x)+3g(y)}{4}\right)+f\circ g^{-1}\left(\frac{3g(x)+g(y)}{4}\right)\right\} \\
&\le \frac{1}{g(y)-g(x)}\int_x^y f(t)\,g'(t)\,dt \\
&\le \left[f\circ g^{-1}\left(\frac{g(x)+g(y)}{2}\right)+\frac{f(x)+f(y)}{2}\right]\int_0^1 h(t)\,dt \\
&\le \left[h\left(\frac{1}{2}\right)+\frac{1}{2}\right][f(x)+f(y)]\int_0^1 h(t)\,dt,
\end{aligned} \tag{2.7}$$

where $y, x\in I$ *with* $y\neq x$.

Remark 2. *In general, if* $h(\lambda)>0$ *for* $\lambda\in(0,1)$*, then for* $y, x\in I$ *with* $y\neq x$

$$\begin{aligned}
&(1-\lambda) f\circ g^{-1}\left[\frac{(1-\lambda)g(x)+(\lambda+1)g(y)}{2}\right] \\
&\qquad + \lambda f\circ g^{-1}\left[\frac{(2-\lambda)g(x)+\lambda g(y)}{2}\right] \\
&= \frac{1-\lambda}{h(1-\lambda)} h(1-\lambda) f\circ g^{-1}\left[\frac{(1-\lambda)g(x)+(\lambda+1)g(y)}{2}\right] \\
&\qquad + \frac{\lambda}{h(\lambda)} h(\lambda) f\circ g^{-1}\left[\frac{(2-\lambda)g(x)+\lambda g(y)}{2}\right] \\
&\ge \min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\} \\
&\quad\times\left\{h(1-\lambda) f\circ g^{-1}\left[\frac{(1-\lambda)g(x)+(\lambda+1)g(y)}{2}\right]\right. \\
&\qquad\left. + h(\lambda) f\circ g^{-1}\left[\frac{(2-\lambda)g(x)+\lambda g(y)}{2}\right]\right\} \\
&\ge \min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\} \\
&\quad\times f\circ g^{-1}\left[(1-\lambda)\frac{(1-\lambda)g(x)+(\lambda+1)g(y)}{2}+\lambda\frac{(2-\lambda)g(x)+\lambda g(y)}{2}\right] \\
&= \min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\} f\circ g^{-1}\left(\frac{g(x)+g(y)}{2}\right)
\end{aligned}$$

and from (2.2) we get the sequence of inequalities

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)}\min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\} f\circ g^{-1}\left(\frac{g(x)+g(y)}{2}\right) \\
&\le \frac{1}{2h\left(\frac{1}{2}\right)}\left\{(1-\lambda) f\circ g^{-1}\left[\frac{(1-\lambda)g(x)+(\lambda+1)g(y)}{2}\right]\right. \\
&\qquad\left. +\lambda f\circ g^{-1}\left[\frac{(2-\lambda)g(x)+\lambda g(y)}{2}\right]\right\}
\end{aligned} \tag{2.8}$$

$$\leq \frac{1}{g(y)-g(x)}\int_x^y f(t)\,g'(t)\,dt$$
$$\leq \left[f\circ g^{-1}\left((1-\lambda)\,g(x)+\lambda g(y)\right)+(1-\lambda)\,f(y)+\lambda f(x)\right]\int_0^1 h(t)\,dt$$
$$\leq \{[h(1-\lambda)+\lambda]\,f(x)+[h(\lambda)+1-\lambda]\,f(y)\}\int_0^1 h(t)\,dt,$$

for $y,\ x\in I$ with $y\neq x$.

In particular, we have

$$\frac{1}{4h^2\left(\frac{1}{2}\right)}f\circ g^{-1}\left(\frac{g(x)+g(y)}{2}\right) \tag{2.9}$$
$$\leq \frac{1}{4h\left(\frac{1}{2}\right)}\left\{f\circ g^{-1}\left[\frac{(1-\lambda)\,g(x)+(\lambda+1)\,g(y)}{2}\right]\right.$$
$$\left.+f\circ g^{-1}\left[\frac{(2-\lambda)\,g(x)+\lambda g(y)}{2}\right]\right\}$$

$$\leq \frac{1}{g(y)-g(x)}\int_x^y f(t)\,g'(t)\,dt$$
$$\leq \left[f\circ g^{-1}\left(\frac{g(x)+g(y)}{2}\right)+\frac{f(y)+f(x)}{2}\right]\int_0^1 h(t)\,dt$$
$$\leq \left[h\left(\frac{1}{2}\right)+\frac{1}{2}\right][f(x)+f(y)]\int_0^1 h(t)\,dt.$$

In a similar way, if $f:I\subseteq\mathbb{R}\to[0,\infty)$ is a composite-g^{-1} h-concave function, then

$$\frac{1}{2h\left(\frac{1}{2}\right)}\max\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}f\circ g^{-1}\left(\frac{g(x)+g(y)}{2}\right) \tag{2.10}$$
$$\geq \frac{1}{2h\left(\frac{1}{2}\right)}\left\{(1-\lambda)\,f\circ g^{-1}\left[\frac{(1-\lambda)\,g(x)+(\lambda+1)\,g(y)}{2}\right]\right.$$
$$\left.+\lambda f\circ g^{-1}\left[\frac{(2-\lambda)\,g(x)+\lambda g(y)}{2}\right]\right\}$$

$$\geq \frac{1}{g(y)-g(x)}\int_x^y f(t)\,g'(t)\,dt$$
$$\geq \left[f\circ g^{-1}\left((1-\lambda)\,g(x)+\lambda g(y)\right)+(1-\lambda)\,f(y)+\lambda f(x)\right]\int_0^1 h(t)\,dt$$
$$\geq \{[h(1-\lambda)+\lambda]\,f(x)+[h(\lambda)+1-\lambda]\,f(y)\}\int_0^1 h(t)\,dt,$$

for $y,\ x\in I$ with $y\neq x$.

In particular,

$$\begin{aligned}(2.11)\quad & \frac{1}{4h^{2}\left(\frac{1}{2}\right)}f\circ g^{-1}\left(\frac{g(x)+g(y)}{2}\right)\\ &\geq\frac{1}{4h\left(\frac{1}{2}\right)}\left\{f\circ g^{-1}\left[\frac{(1-\lambda)g(x)+(\lambda+1)g(y)}{2}\right]\right.\\ &\left.+f\circ g^{-1}\left[\frac{(2-\lambda)g(x)+\lambda g(y)}{2}\right]\right\}\end{aligned}$$

$$\begin{aligned}&\geq\frac{1}{g(y)-g(x)}\int_{x}^{y}f(t)\,g'(t)\,dt\\ &\geq\left[f\circ g^{-1}\left(\frac{g(x)+g(y)}{2}\right)+\frac{f(y)+f(x)}{2}\right]\int_{0}^{1}h(t)\,dt\\ &\geq\left[h\left(\frac{1}{2}\right)+\frac{1}{2}\right][f(x)+f(y)]\int_{0}^{1}h(t)\,dt.\end{aligned}$$

Corollary 1. *Let $f:I\subseteq\mathbb{R}\rightarrow[0,\infty)$ be a composite-g^{-1} convex function on the interval I in $\mathbb{R}$. Then for any $y,x\in I$ with $y\neq x$ and for any $\lambda\in[0,1]$ we have the inequalities*

$$\begin{aligned}(2.12)\quad & f\circ g^{-1}\left(\frac{g(x)+g(y)}{2}\right)\leq(1-\lambda)f\circ g^{-1}\left[\frac{(1-\lambda)g(x)+(\lambda+1)g(y)}{2}\right]\\ &+\lambda f\circ g^{-1}\left[\frac{(2-\lambda)g(x)+\lambda g(y)}{2}\right]\\ &\leq\frac{1}{g(y)-g(x)}\int_{x}^{y}f(t)\,g'(t)\,dt\\ &\leq\frac{1}{2}\left[f\circ g^{-1}\left((1-\lambda)g(x)+\lambda g(y)\right)+(1-\lambda)f(y)+\lambda f(x)\right]\\ &\leq\frac{f(y)+f(x)}{2}.\end{aligned}$$

We have:

Corollary 2. *Let $f:I\subseteq\mathbb{R}\rightarrow[0,\infty)$ be a composite-g^{-1} Breckner s-convex function on the interval I with $s\in(0,1]$. Then for any $y,x\in I$ with $y\neq x$ and for any $\lambda\in[0,1]$ we have the inequalities*

$$\begin{aligned}(2.13)\quad & \frac{1}{2^{1-s}}\left(\frac{1}{2}-\left|\lambda-\frac{1}{2}\right|\right)^{1-s}f\circ g^{-1}\left(\frac{g(x)+g(y)}{2}\right)\\ &\leq\frac{1}{2^{1-s}}\left\{(1-\lambda)f\circ g^{-1}\left[\frac{(1-\lambda)g(x)+(\lambda+1)g(y)}{2}\right]\right.\\ &\left.+\lambda f\circ g^{-1}\left[\frac{(2-\lambda)g(x)+\lambda g(y)}{2}\right]\right\}\end{aligned}$$

$$(2.14)\quad \leq \frac{1}{g(y)-g(x)}\int_x^y f(t)\,g'(t)\,dt$$
$$\leq \frac{1}{s+1}\left[f\circ g^{-1}\left((1-\lambda)\,g(x)+\lambda g(y)\right)+(1-\lambda)\,f(y)+\lambda f(x)\right]$$
$$\leq \frac{1}{s+1}\left\{\left[(1-\lambda)^s+\lambda\right]f(x)+\left(\lambda^s+1-\lambda\right)f(y)\right\}.$$

We also have:

Corollary 3. *Let $f: I\subseteq\mathbb{R}\to[0,\infty)$ be a composite-g^{-1} of s-Godunova-Levin type on the interval I with $s\in(0,1)$. Then for any $y,\ x\in I$ with $y\neq x$ and for any $\lambda\in(0,1)$ we have the inequalities*

$$(2.15)\quad \frac{1}{2^{1+s}}\left(\frac{1}{2}-\left|\lambda-\frac{1}{2}\right|\right)^{1+s} f\circ g^{-1}\left(\frac{g(x)+g(y)}{2}\right)$$
$$\leq \frac{1}{2^{1+s}}\left\{(1-\lambda)\,f\circ g^{-1}\left[\frac{(1-\lambda)\,g(x)+(\lambda+1)\,g(y)}{2}\right]\right.$$
$$\left.+\lambda f\circ g^{-1}\left[\frac{(2-\lambda)\,g(x)+\lambda g(y)}{2}\right]\right\}$$
$$\leq \frac{1}{g(y)-g(x)}\int_x^y f(t)\,g'(t)\,dt$$
$$\leq \frac{1}{1-s}\left[f\circ g^{-1}\left((1-\lambda)\,g(x)+\lambda g(y)\right)+(1-\lambda)\,f(y)+\lambda f(x)\right]$$
$$\leq \frac{1}{1-s}\left\{\left[(1-\lambda)^{-s}+\lambda\right]f(x)+\left(\lambda^{-s}+1-\lambda\right)f(y)\right\}.$$

More generally, we have:

Corollary 4. *Assume that $g:[a,b]\to[g(a),g(b)]$ is a continuous strictly increasing function that is differentiable on (a,b), $f:[a,b]\to\mathcal{J}$, $\mathcal{J}$ an interval of real numbers and $k:\mathcal{J}\to\mathbb{R}$ is a continuous function on $\mathcal{J}$ that is strictly increasing. If the function $f:[a,b]\to\mathcal{J}$ is k-composite-g^{-1} h-convex on $[a,b]$, then*

$$(2.16)\quad \frac{1}{2h\left(\frac{1}{2}\right)}\min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}k\circ f\circ g^{-1}\left(\frac{g(x)+g(y)}{2}\right)$$
$$\leq \frac{1}{2h\left(\frac{1}{2}\right)}\left\{(1-\lambda)\,k\circ f\circ g^{-1}\left[\frac{(1-\lambda)\,g(x)+(\lambda+1)\,g(y)}{2}\right]\right.$$
$$\left.+\lambda k\circ f\circ g^{-1}\left[\frac{(2-\lambda)\,g(x)+\lambda g(y)}{2}\right]\right\}$$
$$\leq \frac{1}{g(y)-g(x)}\int_x^y (k\circ f)(t)\,g'(t)\,dt$$
$$\leq \left[k\circ f\circ g^{-1}\left((1-\lambda)\,g(x)+\lambda g(y)\right)+(1-\lambda)\,(k\circ f)(y)+\lambda\,(k\circ f)(x)\right]$$
$$\times\int_0^1 h(t)\,dt$$
$$\leq \left\{\left[h(1-\lambda)+\lambda\right](k\circ f)(x)+\left[h(\lambda)+1-\lambda\right](k\circ f)(y)\right\}\int_0^1 h(t)\,dt,$$

for $y,\ x\in[a,b]$ with $y\neq x$.

If the function $f:[a,b]\to\mathcal{J}$ *is* k*-composite-*g^{-1} h*-concave on* $[a,b]$*, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\max\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}k\circ f\circ g^{-1}\left(\frac{g(x)+g(y)}{2}\right) \tag{2.17}$$
$$\geq\frac{1}{2h\left(\frac{1}{2}\right)}\left\{(1-\lambda)k\circ f\circ g^{-1}\left[\frac{(1-\lambda)g(x)+(\lambda+1)g(y)}{2}\right]\right.$$
$$\left.+\lambda k\circ f\circ g^{-1}\left[\frac{(2-\lambda)g(x)+\lambda g(y)}{2}\right]\right\}$$

$$\geq\frac{1}{g(y)-g(x)}\int_x^y(k\circ f)(t)\,g'(t)\,dt$$
$$\geq\left[k\circ f\circ g^{-1}\left((1-\lambda)g(x)+\lambda g(y)\right)+(1-\lambda)(k\circ f)(y)+\lambda(k\circ f)(x)\right]$$
$$\times\int_0^1h(t)\,dt$$
$$\geq\left\{[h(1-\lambda)+\lambda](k\circ f)(x)+[h(\lambda)+1-\lambda](k\circ f)(y)\right\}\int_0^1h(t)\,dt,$$

for $y,x\in[a,b]$ *with* $y\neq x$.

The proof follows by the inequalities (2.8) and (2.10) and we omit the details.

In 1906, Fejér [51], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 2 (Fejér's Inequality)**.** *Consider the integral* $\int_a^b h(x)w(x)\,dx$*, where* h *is a convex function in the interval* (a,b) *and* w *is a positive function in the same interval such that*

$$w(x)=w(a+b-x),\ \text{for any } x\in[a,b]$$

i.e., $y=w(x)$ *is a symmetric curve with respect to the straight line which contains the point* $\left(\frac{1}{2}(a+b),0\right)$ *and is normal to the* x*-axis. Under those conditions the following inequalities are valid:*

$$h\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx\leq\int_a^b h(x)w(x)\,dx\leq\frac{h(a)+h(b)}{2}\int_a^b w(x)\,dx. \tag{2.18}$$

If h *is concave on* (a,b)*, then the inequalities reverse in (2.18).*

If $w:[a,b]\to\mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\to[0,\infty)$, $W(x):=\int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b) and the inverse $W^{-1}:\left[a,\int_a^b w(s)\,ds\right]\to[a,b]$ exists.

Remark 3. *Assume that* $w:[a,b]\to\mathbb{R}$ *is continuous and positive on the interval* $[a,b]$*,* $f:[a,b]\to\mathcal{J}$*,* $\mathcal{J}$ *an interval of real numbers and* $k:\mathcal{J}\to\mathbb{R}$ *is a continuous* *function on $\mathcal{J}$ that is strictly increasing. If the function $f:[a,b]\to\mathcal{J}$ is k-composite-W^{-1} h-convex on $[a,b]$, then we have the weighted inequality*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\min\left\{\frac{1-\lambda}{h\left(1-\lambda\right)},\frac{\lambda}{h\left(\lambda\right)}\right\}k\circ f\circ W^{-1}\left(\frac{\int_a^x w\left(s\right)ds+\int_a^y w\left(s\right)ds}{2}\right) \tag{2.19}$$
$$\leq\frac{1}{2h\left(\frac{1}{2}\right)}\left\{\left(1-\lambda\right)k\circ f\circ W^{-1}\left[\frac{\left(1-\lambda\right)\int_a^x w\left(s\right)ds+\left(\lambda+1\right)\int_a^y w\left(s\right)ds}{2}\right]\right.$$
$$\left.+\lambda k\circ f\circ W^{-1}\left[\frac{\left(2-\lambda\right)\int_a^x w\left(s\right)ds+\lambda\int_a^y w\left(s\right)ds}{2}\right]\right\}$$

$$\leq\frac{1}{\int_x^y w\left(s\right)ds}\int_x^y\left(k\circ f\right)\left(t\right)w\left(t\right)dt$$
$$\leq\left[k\circ f\circ W^{-1}\left(\left(1-\lambda\right)\int_a^x w\left(s\right)ds+\lambda\int_a^y w\left(s\right)ds\right)\right.$$
$$\left.+\left(1-\lambda\right)\left(k\circ f\right)\left(y\right)+\lambda\left(k\circ f\right)\left(x\right)\right]\int_0^1 h\left(t\right)dt$$
$$\leq\left\{\left[h\left(1-\lambda\right)+\lambda\right]\left(k\circ f\right)\left(x\right)+\left[h\left(\lambda\right)+1-\lambda\right]\left(k\circ f\right)\left(y\right)\right\}\int_0^1 h\left(t\right)dt,$$

for any $\lambda\in[0,1]$ and for y, $x\in[a,b]$ with $y\neq x$.

3. Applications for AG and AH-h-Convex Functions

The function $f:[a,b]\to(0,\infty)$ is AG-h-convex means that f is k-composite h-convex on $[a,b]$ with $k\left(t\right)=\ln t$, $t>0$. By making use of Corollary 4 for $g\left(t\right)=t$, we get

$$\left[f\left(\frac{x+y}{2}\right)\right]^{\frac{1}{2h\left(\frac{1}{2}\right)}\min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}} \tag{3.1}$$
$$\leq\left\{f^{(1-\lambda)}\left[\frac{\left(1-\lambda\right)x+\left(\lambda+1\right)y}{2}\right]f^{\lambda}\left[\frac{\left(2-\lambda\right)x+\lambda y}{2}\right]\right\}^{\frac{1}{2h\left(\frac{1}{2}\right)}}$$

$$\leq\exp\left(\frac{1}{y-x}\int_x^y\ln f\left(t\right)dt\right)$$
$$\leq\left[f\left(\left(1-\lambda\right)x+\lambda y\right)f^{(1-\lambda)}\left(y\right)f^{\lambda}\left(x\right)\right]^{\int_0^1 h(t)dt}$$
$$\leq\left\{f^{[h(1-\lambda)+\lambda]}\left(x\right)f^{[h(\lambda)+1-\lambda]}\left(y\right)\right\}^{\int_0^1 h(t)dt},$$

for any $\lambda\in[0,1]$ and x, $y\in[a,b]$ with $y\neq x$.

The function $f:[a,b]\to(0,\infty)$ is AH-h-convex on $[a,b]$ means that f is k-composite h-concave on $[a,b]$ with $k:(0,\infty)\to(0,\infty)$, $k\left(t\right)=\frac{1}{t}$. By making use of Corollary 4 for $g(t)=t$, we get

$$\begin{aligned}(3.2)\quad & \frac{1}{2h\left(\frac{1}{2}\right)}\max\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}f^{-1}\left(\frac{x+y}{2}\right)\\ &\geq\frac{1}{2h\left(\frac{1}{2}\right)}\left\{(1-\lambda)f^{-1}\left[\frac{(1-\lambda)x+(\lambda+1)y}{2}\right]\right.\\ &\left.\qquad+\lambda f^{-1}\left[\frac{(2-\lambda)x+\lambda y}{2}\right]\right\}\\ &\geq\frac{1}{y-x}\int_x^y f^{-1}(t)\,dt\\ &\geq\left[f^{-1}((1-\lambda)x+\lambda y)+(1-\lambda)f^{-1}(y)+\lambda f^{-1}(x)\right]\int_0^1 h(t)\,dt\\ &\geq\left\{[h(1-\lambda)+\lambda]f^{-1}(x)+[h(\lambda)+1-\lambda]f^{-1}(y)\right\}\int_0^1 h(t)\,dt,\end{aligned}$$

for any $\lambda\in[0,1]$ and $x,\,y\in[a,b]$ with $y\neq x$.

4. Applications for GA, GG and GH-h-Convex Functions

If we take $g(t)=\ln t$, $t\in[a,b]\subset(0,\infty)$, then $f:[a,b]\to\mathbb{R}$ is GA-h-convex on $[a,b]$ means that that $f:[a,b]\to\mathbb{R}$ composite-g^{-1} h-convex on $[a,b]$. By making use of Corollary 4 for $k(t)=t$, we get

$$\begin{aligned}(4.1)\quad & \frac{1}{2h\left(\frac{1}{2}\right)}\min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}f(\sqrt{xy})\\ &\leq\frac{1}{2h\left(\frac{1}{2}\right)}\left\{(1-\lambda)f\left(x^{\frac{1-\lambda}{2}}y^{\frac{\lambda+1}{2}}\right)+\lambda f\left(x^{\frac{2-\lambda}{2}}y^{\frac{\lambda}{2}}\right)\right\}\\ &\leq\frac{1}{\ln\left(\frac{y}{x}\right)}\int_x^y\frac{f(t)}{t}dt\\ &\leq\left[f\left(x^{1-\lambda}y^{\lambda}\right)+(1-\lambda)f(y)+\lambda f(x)\right]\int_0^1 h(t)\,dt\\ &\leq\{[h(1-\lambda)+\lambda]f(x)+[h(\lambda)+1-\lambda]f(y)\}\int_0^1 h(t)\,dt,\end{aligned}$$

for any $\lambda\in[0,1]$ and for $y,\,x\in[a,b]$ with $y\neq x$.

The function $f:I\subset(0,\infty)\to(0,\infty)$ is *GG-h-convex* means that f is k-composite-g^{-1} h-convex on $[a,b]$ with $k:(0,\infty)\to\mathbb{R}$, $k(t)=\ln t$ and $g(t)=\ln t$, $t\in[a,b]$. By making use of Corollary 4 we get

$$\begin{aligned}(4.2)\quad & [f(\sqrt{xy})]^{\frac{1}{2h\left(\frac{1}{2}\right)}\min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}}\\ &\leq\left\{f^{(1-\lambda)}\left(x^{\frac{1-\lambda}{2}}y^{\frac{\lambda+1}{2}}\right)f^{\lambda}\left(x^{\frac{2-\lambda}{2}}y^{\frac{\lambda}{2}}\right)\right\}^{\frac{1}{2h\left(\frac{1}{2}\right)}}\end{aligned}$$

$$\leq \exp\left(\frac{1}{\ln\left(\frac{y}{x}\right)}\int_x^y \frac{\ln f(t)}{t}dt\right)$$
$$\leq \left[f\left(x^{1-\lambda}y^{\lambda}\right)f^{\lambda}(x)f^{1-\lambda}(y)\right]^{\int_0^1 h(t)dt}$$
$$\leq \left\{f^{[h(1-\lambda)+\lambda]}(x)f^{[h(\lambda)+1-\lambda]}(y)\right\}^{\int_0^1 h(t)dt},$$

for any $\lambda\in[0,1]$ and for $y,\,x\in[a,b]$ with $y\neq x$.

We also have that $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ is GH-h-convex on $[a,b]$ is equivalent to the fact that f is k-composite-g^{-1} h-concave on $[a,b]$ with $k:(0,\infty)\to(0,\infty)$, $k(t)=\frac{1}{t}$ and $g(t)=\ln t$, $t\in I$. By making use of Corollary 4 we get

$$\text{(4.3)}\quad \frac{1}{2h\left(\frac{1}{2}\right)}\max\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}f^{-1}\left(\sqrt{xy}\right)$$
$$\geq \frac{1}{2h\left(\frac{1}{2}\right)}\left\{\lambda f^{-1}\left(x^{\frac{2-\lambda}{2}}y^{\frac{\lambda}{2}}\right)+(1-\lambda)f^{-1}\left(x^{\frac{1-\lambda}{2}}y^{\frac{\lambda+1}{2}}\right)\right\}$$

$$\geq \frac{1}{\ln\left(\frac{y}{x}\right)}\int_x^y \frac{f^{-1}(t)}{t}dt$$
$$\geq \left[f^{-1}\left(x^{1-\lambda}y^{\lambda}\right)+\lambda f^{-1}(x)+(1-\lambda)f^{-1}(y)\right]\int_0^1 h(t)\,dt$$
$$\geq \left\{[h(1-\lambda)+\lambda]f^{-1}(x)+[h(\lambda)+1-\lambda]f^{-1}(y)\right\}\int_0^1 h(t)\,dt,$$

for any $\lambda\in[0,1]$ and for $y,\,x\in[a,b]$ with $y\neq x$.

5. Applications for HA, HG and HH-h-Convex Functions

Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be an HA-h-convex function on the interval $[a,b]$. This is equivalent to the fact that f is composite-g^{-1} h-convex on $[a,b]$ with the increasing function $g(t)=-\frac{1}{t}$. Then by applying Corollary 4 for $k(t)=t$, we have the inequalities

$$\text{(5.1)}\quad \frac{1}{2h\left(\frac{1}{2}\right)}\min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}f\left(\frac{2xy}{x+y}\right)$$
$$\leq \frac{1}{2h\left(\frac{1}{2}\right)}\left\{(1-\lambda)f\left(\frac{2xy}{(1-\lambda)x+(\lambda+1)y}\right)+\lambda f\left(\frac{2xy}{(2-\lambda)x+\lambda y}\right)\right\}$$

$$\leq \frac{xy}{y-x}\int_x^y \frac{f(t)}{t^2}dt$$
$$\leq \frac{1}{2}\left[f\left(\frac{xy}{(1-\lambda)x+\lambda y}\right)+(1-\lambda)f(x)+\lambda f(y)\right]\int_0^1 h(t)\,dt$$
$$\leq \left\{[h(1-\lambda)+\lambda]f(x)+[h(\lambda)+1-\lambda]f(y)\right\}\int_0^1 h(t)\,dt$$

for any $\lambda\in[0,1]$ and for $y,\,x\in[a,b]$ with $y\neq x$.

Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be an HG-h-convex function on the interval $[a,b]$. This is equivalent to the fact that f is k-composite-g^{-1} h-convex on $[a,b]$

with $k:(0,\infty)\to\mathbb{R}$, $k(t)=\ln t$ and $g(t)=-\frac{1}{t}$, $t\in[a,b]$. Then by applying Corollary 4, we have the inequalities

$$\left[f\left(\frac{2xy}{x+y}\right)\right]^{\frac{1}{2h\left(\frac{1}{2}\right)}\min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}} \tag{5.2}$$
$$\leq\left[f^{1-\lambda}\left(\frac{2xy}{(1-\lambda)x+(\lambda+1)y}\right)f^{\lambda}\left(\frac{2xy}{(2-\lambda)x+\lambda y}\right)\right]^{\frac{1}{2h\left(\frac{1}{2}\right)}}$$
$$\leq\exp\left(\frac{xy}{y-x}\int_x^y\frac{\ln f(t)}{t^2}dt\right)$$
$$\leq\left[f\left(\frac{xy}{(1-\lambda)x+\lambda y}\right)[f(x)]^{1-\lambda}[f(y)]^{\lambda}\right]^{\int_0^1h(t)d}$$
$$\leq\left\{f^{[h(1-\lambda)+\lambda]}(x)\,f^{[h(\lambda)+1-\lambda]}(y)\right\}^{\int_0^1h(t)dt}$$

for any $\lambda\in[0,1]$ and for y, $x\in[a,b]$ with $y\neq x$.

Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be an HH-h-convex function on the interval $[a,b]$. This is equivalent to the fact that f is k-composite-g^{-1} h-concave on $[a,b]$ with $k:(0,\infty)\to(0,\infty)$, $k(t)=\frac{1}{t}$ and $g(t)=-\frac{1}{t}$, $t\in[a,b]$. Then by applying Corollary 4, we have the inequalities

$$\frac{1}{2h\left(\frac{1}{2}\right)}\max\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}f^{-1}\left(\frac{2xy}{x+y}\right) \tag{5.3}$$
$$\geq\frac{1}{2h\left(\frac{1}{2}\right)}\left\{\lambda f^{-1}\left(\frac{2xy}{(2-\lambda)x+\lambda y}\right)+(1-\lambda)f^{-1}\left(\frac{2xy}{(1-\lambda)x+(\lambda+1)y}\right)\right\}$$
$$\geq\frac{xy}{y-x}\int_x^y\frac{f^{-1}(t)}{t^2}dt$$
$$\geq\left[f^{-1}\left(\frac{xy}{(1-\lambda)x+\lambda y}\right)+\lambda f^{-1}(x)+(1-\lambda)f^{-1}(y)\right]\int_0^1h(t)\,dt$$
$$\geq\left\{[h(1-\lambda)+\lambda]f^{-1}(x)+[h(\lambda)+1-\lambda]f^{-1}(y)\right\}\int_0^1h(t)\,dt,$$

for y, $x\in[a,b]$ with $y\neq x$.

Applications for p, r-convex and *LogExp* convex functions can also be provided. However the details are not presented here.

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