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WEIGHTED INEQUALITIES OF OSTROWSKI TYPE FOR FUNCTIONS OF BOUNDED VARIATION AND APPLICATIONS

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ABSTRACT. In this paper we establish some upper bounds for the quantity

$$\left|[g(b)-g(a)]f(x)-\int_a^b f(t)g'(t)\,dt\right|$$

under the assumptions that $g:[a,b]\to[g(a),g(b)]$ is a *continuous strictly increasing function* that is *differentiable* on (a,b) and $f:[a,b]\to\mathbb{C}$ is a function of bounded variation on $[a,b]$. When g is an integral, namely $g(x)=\int_a^x w(s)\,ds$, where $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$, then a refinement and some related versions of an Ostrowski type weighted inequality are provided. Applications for continuous probability density functions supported on finite and infinite intervals and two examples are also given.

1. Introduction

The following inequality of Ostrowski type for functions of bounded variation holds:

Theorem 1 (Dragomir, 1999 [3])**.** *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then for all $x\in[a,b]$, we have the inequality*

$$\left|\int_a^b f(t)\,dt-(b-a)f(x)\right|\le\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f)\,, \tag{1.1}$$

where $\bigvee_a^b(f)$ denotes the total variation of f. The constant $\frac{1}{2}$ is the best possible one.

The best inequality one can get from (1.1) is embodied in:

Corollary 1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation. Then we have the inequality:*

$$\left|\int_a^b f(t)\,dt-(b-a)f\left(\frac{a+b}{2}\right)\right|\le\frac{1}{2}(b-a)\bigvee_a^b(f) \tag{1.2}$$

The constant $\frac{1}{2}$ is best possible.

For related results, see [6], [7] and the survey paper [8].

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In order to extend Ostrowski's type inequality (1.1) to weighted integral, in 2008 Tseng et al. [10] obtained the following result

$$\begin{aligned}
(1.3)\quad & \left| f(x)\int_a^b w(s)\,ds - \int_a^b f(t)\,w(t)\,dt \right| \\
& \leq \frac{1}{2}\left[\int_a^b w(s)\,ds + \left|\int_a^x w(s)\,ds - \int_x^b w(s)\,ds\right|\right]\bigvee_a^b (f),
\end{aligned}$$

for any $x \in [a,b]$, provided that w is continuous and positive on $[a,b]$ and f is of bounded variation on $[a,b]$.

This result was also recaptured from a more general inequality by Liu in [9].

Motivated by the above results, in this paper we establish some upper bounds for the quantity

$$\left|[g(b)-g(a)]\,f(x) - \int_a^b f(t)\,g'(t)\,dt\right|$$

under the assumptions that $g : [a,b] \to [g(a), g(b)]$ is a *continuous strictly increasing function* that is *differentiable* on (a,b) and $f : [a,b] \to \mathbb{C}$ is a function of bounded variation on $[a,b]$. When g is an integral, namely $g(x) = \int_a^x w(s)\,ds$, where $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ and $f : [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$, then a refinement and some related versions of the inequality (1.3) are provided. Applications for continuous probability density functions supported on finite and infinite intervals with two examples are also given.

2. Ostrowski Type Results

We need the following inequality for functions of bounded variation:

Lemma 1. *Let $h : [c,d] \to \mathbb{C}$ be a function of bounded variation on $[c,d]$. Then for all $z \in [c,d]$, we have the inequality*

$$\begin{aligned}
(2.1)\quad & \left|\int_c^d h(t)\,dt - (d-c)\,h(z)\right| \\
& \leq (z-c)\bigvee_c^z (h) + (d-z)\bigvee_z^d (h) \\
& \leq \begin{cases}
\left[\frac{1}{2}(d-c) + \left|z - \frac{c+d}{2}\right|\right]\bigvee_c^d (h), \\[2ex]
\left[(z-c)^p + (d-z)^p\right]^{1/p}\left[\left(\bigvee_c^z (h)\right)^q + \left(\bigvee_z^d (h)\right)^q\right]^{1/q} \\
\text{where } p,\ q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\[2ex]
(d-c)\left[\frac{1}{2}\bigvee_c^d (h) + \frac{1}{2}\left|\bigvee_c^z (h) - \bigvee_z^d (h)\right|\right],
\end{cases}
\end{aligned}$$

where $\bigvee_c^d (h)$ denotes the total variation of h.

Proof. Using the integration by parts formula for Riemann-Stieltjes integrals we have

$$\int_c^z (t-c)\,dh(t) = h(z)(z-c) - \int_c^z h(t)\,dt$$

and

$$\int_z^d (t-d)dh(t) = h(z)(d-z) - \int_z^d h(t)\,dt.$$

If we add the above two equalities, we obtain the following equality of interest, see [5]

$$(2.2) \qquad (d-c)\,h(z) - \int_c^d h(t)\,dt = \int_c^d p(z,t)\,dh(t)\ ,$$

where

$$p(z,t) := \begin{cases} t-c \text{ if } t \in [c,z) \\ t-d \text{ if } z \in [z,d] \end{cases},$$

for all $z, t \in [c,d]$.
It is well known [1, p. 177] that if $q : [\alpha,\beta] \to \mathbb{C}$ is continuous on $[\alpha,\beta]$ and $v : [\alpha,\beta] \to \mathbb{C}$ is of bounded variation on $[\alpha,\beta]$, then

$$(2.3) \qquad \left| \int_\alpha^\beta q(z)\,dv(z) \right| \le \max_{z\in[\alpha,\beta]} |q(z)| \bigvee_\alpha^\beta (v).$$

Applying the property (2.3) we get

$$\begin{aligned}(2.4) \qquad \left|\int_c^d p(z,t)\,dh(t)\right| &= \left|\int_c^z (t-c)\,dh(t) + \int_z^d (t-d)dh(t)\right| \\ &\le \left|\int_c^z (t-c)\,dh(t)\right| + \left|\int_z^d (t-d)dh(t)\right| \\ &\le \max_{t\in[c,z]} |t-c| \bigvee_c^z (h) + \max_{t\in[z,d]} |t-d| \bigvee_z^d (h) \\ &= (z-c)\bigvee_c^z (h) + (d-z)\bigvee_z^d (h)\end{aligned}$$

and then by (2.4), via the identity (2.2), we deduce the first inequality in (2.1).

By utilising Hölder's discrete inequality for two positive numbers, we also have

$$(z-c)\bigvee_c^z (h) + (d-z)\bigvee_z^d (h)$$

$$\leq\begin{cases}\max\{z-c,d-z\}\left[\bigvee_c^z(h)+\bigvee_z^d(h)\right]\\ \\ \left[(z-c)^p+(d-z)^p\right]^{1/p}\left[\left(\bigvee_c^z(h)\right)^q+\left(\bigvee_z^d(h)\right)^q\right]^{1/q}\\ \quad\text{where } p,\ q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1,\\ \\ (z-c+d-z)\max\left\{\bigvee_c^z(h),\bigvee_z^d(h)\right\}\end{cases}$$

$$=\begin{cases}\left[\frac{1}{2}(d-c)+\left|z-\frac{c+d}{2}\right|\right]\bigvee_c^d(h)\\ \\ \left[(z-c)^p+(d-z)^p\right]^{1/p}\left[\left(\bigvee_c^z(h)\right)^q+\left(\bigvee_z^d(h)\right)^q\right]^{1/q}\\ \quad\text{where } p,\ q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1,\\ \\ (d-c)\left[\frac{1}{2}\bigvee_c^d(h)+\frac{1}{2}\left|\bigvee_c^z(h)-\bigvee_z^d(h)\right|\right],\end{cases}$$

which proves the last part of (2.1). □

Corollary 2 (Dragomir, 2014 [7])**.** *Let $h:[c,d]\to\mathbb{C}$ be a function of bounded variation and $p\in(c,d)$ such that $\bigvee_c^p(h)=\bigvee_p^d(h)$. Then we have the inequality*

$$\left|\int_c^d h(t)\,dt-(d-c)\,h(p)\right|\leq\frac{1}{2}(d-c)\bigvee_c^d(h).$$

For other Ostrowski type inequalities, see [2].

Theorem 2. *Let $g:[a,b]\to[g(a),g(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If $f:[a,b]\to\mathbb{C}$ is a function of bounded variation on $[a,b]$, then we have*

$$\left|[g(b)-g(a)]f(x)-\int_a^b f(t)\,g'(t)\,dt\right| \tag{2.5}$$
$$\leq[g(x)-g(a)]\bigvee_a^x(f)+[g(b)-g(x)]\bigvee_x^b(f)$$
$$\leq\begin{cases}\left[\frac{1}{2}[g(b)-g(a)]+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]\bigvee_a^b(f),\\ \\ \left[[g(x)-g(a)]^p+[g(b)-g(x)]^p\right]^{1/p}\left[\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right]^{1/q}\\ \textit{where } p,\ q>1 \textit{ and } \frac{1}{p}+\frac{1}{q}=1,\\ \\ [g(b)-g(a)]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}$$

for all $x\in[a,b]$.

Proof. Assume that $[c,d]\subset[a,b]$. Let $g(c)=z_0<z_1<...<z_{n-1}<z_n=g(d)$, $n\geq1$, a division of the interval $[g(c),g(d)]$. Put $x_i=g^{-1}(z_i)$, $i\in\{0,...,n\}$. Then $c=x_0<x_1<...<z_{n-1}<z_n=c$ is a division of $[c,d]$.

Observe that

$$\sum_{i=0}^{n-1}\left|f\circ g^{-1}\left(z_{i+1}\right)-f\circ g^{-1}\left(z_{i}\right)\right|=\sum_{i=0}^{n-1}\left|f\left(x_{i+1}\right)-f\left(x_{i}\right)\right|,$$

which shows that, if $f:[c,d]\to\mathbb{C}$ is a function of bounded variation on $[c,d]$, then $f\circ g^{-1}:[g(c),g(d)]\to\mathbb{C}$ is of bounded variation on $[g(c),g(d)]$ and the total variation of $f\circ g^{-1}$ on $[g(c),g(d)]$ is the same with the total variation of f on $[c,d]$, namely

$$\bigvee_{g(c)}^{g(d)}\left(f\circ g^{-1}\right)=\bigvee_{c}^{d}(f). \tag{2.6}$$

Now, if we use the inequality (2.1) for the function $h=f\circ g^{-1}$ on the interval $[g(a),g(b)]$ we get for any $z\in[g(a),g(b)]$ that

$$\left|\int_{g(a)}^{g(b)}\left(f\circ g^{-1}\right)(u)\,du-(g(b)-g(a))\left(f\circ g^{-1}\right)(z)\right| \tag{2.7}$$
$$\leq (z-g(a))\bigvee_{g(a)}^{z}\left(f\circ g^{-1}\right)+(g(b)-z)\bigvee_{z}^{g(b)}\left(f\circ g^{-1}\right)$$
$$\leq\begin{cases}\left[\frac{1}{2}(g(b)-g(a))+\left|z-\frac{g(a)+g(b)}{2}\right|\right]\bigvee_{g(a)}^{g(b)}\left(f\circ g^{-1}\right),\\[2ex] \left[(z-g(a))^{p}+(g(b)-z)^{p}\right]^{1/p}\left[\left(\bigvee_{g(a)}^{z}\left(f\circ g^{-1}\right)\right)^{q}+\left(\bigvee_{z}^{g(b)}\left(f\circ g^{-1}\right)\right)^{q}\right]^{1/q}\\ \text{where } p,\ q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1,\\[2ex] (g(b)-g(a))\left[\frac{1}{2}\bigvee_{g(a)}^{g(b)}\left(f\circ g^{-1}\right)+\frac{1}{2}\left|\bigvee_{g(a)}^{z}\left(f\circ g^{-1}\right)-\bigvee_{z}^{g(b)}\left(f\circ g^{-1}\right)\right|\right].\end{cases}$$

Using the property (2.6) and taking $z=g(x)$, $x\in[a,b]$, in (2.7) we then get

$$\left|\int_{g(a)}^{g(b)}\left(f\circ g^{-1}\right)(u)\,du-(g(b)-g(a))f(x)\right| \tag{2.8}$$
$$\leq (g(x)-g(a))\bigvee_{a}^{x}(f)+(g(b)-g(x))\bigvee_{x}^{b}(f)$$
$$\leq\begin{cases}\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]\bigvee_{a}^{b}(f),\\[2ex] \left[(g(x)-g(a))^{p}+(g(b)-g(x))^{p}\right]^{1/p}\left[\left(\bigvee_{a}^{x}(f)\right)^{q}+\left(\bigvee_{x}^{b}(f)\right)^{q}\right]^{1/q}\\ \text{where } p,\ q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1,\\[2ex] (g(b)-g(a))\left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{x}(f)-\bigvee_{x}^{b}(f)\right|\right].\end{cases}$$

Observe also that, by the change of variable $t=g^{-1}(u)$, $u\in[g(a),g(b)]$, we have $u=g(t)$ that gives $du=g'(t)\,dt$ and

$$\int_{g(a)}^{g(b)}\left(f\circ g^{-1}\right)(u)\,du=\int_{a}^{b}f(t)\,g'(t)\,dt. \tag{2.9}$$

By choosing $z = g(x)$ with $x \in [a,b]$ in (2.8) and making use of (2.6) and (2.9) we get the desired result (2.5).

The best constant follows by Lemma 1. $\square$

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a,\, b \in I$ as

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right). \tag{2.10}$$

If $I = \mathbb{R}$ and $g(t) = t$ is the *identity function*, then $M_g(a,b) = A(a,b) := \frac{a+b}{2}$, the *arithmetic mean*. If $I = (0,\infty)$ and $g(t) = \ln t$, then $M_g(a,b) = G(a,b) := \sqrt{ab}$, the *geometric mean*. If $I = (0,\infty)$ and $g(t) = -\frac{1}{t}$, then $M_g(a,b) = H(a,b) := \frac{2ab}{a+b}$, the *harmonic mean*. If $I = (0,\infty)$ and $g(t) = t^p$, $p \neq 0$, then $M_g(a,b) = M_p(a,b) := \left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent p*. Finally, if $I = \mathbb{R}$ and $g(t) = \exp t$, then

$$M_g(a,b) = LME(a,b) := \ln\left(\frac{\exp a + \exp b}{2}\right), \tag{2.11}$$

the *LogMeanExp function*.

Corollary 3. *With the assumptions of Theorem 2 we have*

$$\left|\int_a^b f(t)\,g'(t)\,dt - [g(b)-g(a)]\,f(M_g(a,b))\right| \le \frac{1}{2}[g(b)-g(a)]\bigvee_a^b(f). \tag{2.12}$$

Moreover, if $p \in (a,b)$ is such that $\bigvee_a^p(f) = \bigvee_p^b(f)$, then

$$\left|\int_a^b f(t)\,g'(t)\,dt - [g(b)-g(a)]\,f(p)\right| \le \frac{1}{2}[g(b)-g(a)]\bigvee_a^b(f). \tag{2.13}$$

Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation. We can give the following examples of interest.

a). If we take $g : [a,b] \subset (0,\infty) \to \mathbb{R}$, $g(t) = \ln t$, in (2.5) then we get

$$\begin{aligned} &\left|\ln\left(\frac{b}{a}\right) f(x) - \int_a^b \frac{f(t)}{t}dt\right| \\ &\le \ln\left(\frac{x}{a}\right)\bigvee_a^x(f) + \ln\left(\frac{b}{x}\right)\bigvee_x^b(f) \\ &\le \begin{cases} \left[\frac{1}{2}\ln\left(\frac{b}{a}\right) + \left|\ln\left(\frac{x}{G(a,b)}\right)\right|\right]\bigvee_a^b(f), \\ \\ \left[\left[\ln\left(\frac{x}{a}\right)\right]^p + \left[\ln\left(\frac{b}{x}\right)\right]^p\right]^{1/p}\left[\left(\bigvee_a^x(f)\right)^q + \left(\bigvee_x^b(f)\right)^q\right]^{1/q} \\ \text{where } p,\ q > 1 \text{ and } \frac{1}{p}+\frac{1}{q} = 1, \\ \\ \ln\left(\frac{b}{a}\right)\left[\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^x(f) - \bigvee_x^b(f)\right|\right] \end{cases} \end{aligned} \tag{2.14}$$

for any $x \in [a,b] \subset (0,\infty)$.

In particular, we have

$$\left|\int_a^b \frac{f(t)}{t}dt - \ln\left(\frac{b}{a}\right) f(G(a,b))\right| \leq \frac{1}{2}\ln\left(\frac{b}{a}\right)\bigvee_a^b (f), \tag{2.15}$$

where $G(a,b) := \sqrt{ab}$ is the *geometric mean.*

If $p \in (a,b)$ is such that $\bigvee_a^p (f) = \bigvee_p^b (f)$, then

$$\left|\int_a^b \frac{f(t)}{t}dt - \ln\left(\frac{b}{a}\right) f(p)\right| \leq \frac{1}{2}\ln\left(\frac{b}{a}\right)\bigvee_a^b (f). \tag{2.16}$$

b). If we take $g : [a,b] \subset \mathbb{R} \to (0,\infty)$, $g(t) = \exp t$, in (2.5) then we get

$$\begin{aligned} &\left|(\exp b - \exp a) f(x) - \int_a^b f(t)\exp t dt\right| \\ &\leq (\exp x - \exp a)\bigvee_a^x (f) + (\exp b - \exp x)\bigvee_x^b (f) \\ &\leq \begin{cases} \left[\frac{1}{2}(\exp b - \exp a) + \left|\exp x - \frac{\exp a + \exp b}{2}\right|\right]\bigvee_a^b (f), \\ \\ \left[(\exp x - \exp a)^p + (\exp b - \exp x)^p\right]^{1/p}\left[\left(\bigvee_a^x (f)\right)^q + \left(\bigvee_x^b (f)\right)^q\right]^{1/q} \\ \text{where } p,\ q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ (\exp b - \exp a)\left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right] \end{cases} \end{aligned} \tag{2.17}$$

for any $x \in [a,b]$.

In particular, we have

$$\begin{aligned} &\left|(\exp b - \exp a) f(LME(a,b)) - \int_a^b f(t)\exp t dt\right| \\ &\leq \frac{1}{2}(\exp b - \exp a)\bigvee_a^b (f), \end{aligned} \tag{2.18}$$

where $LME(a,b) := \ln\left(\frac{\exp a + \exp b}{2}\right)$ is the *LogMeanExp function.*

If $p \in (a,b)$ is such that $\bigvee_a^p (f) = \bigvee_p^b (f)$, then

$$\left|(\exp b - \exp a) f(p) - \int_a^b f(t)\exp t dt\right| \leq \frac{1}{2}(\exp b - \exp a)\bigvee_a^b (f). \tag{2.19}$$

c). If we take $g:[a,b]\subset(0,\infty)\to\mathbb{R}$, $g(t)=t^{r}$, $r>0$ in (2.5), then we get

$$\left|(b^{r}-a^{r})f(x)-r\int_{a}^{b}f(t)t^{r-1}dt\right| \tag{2.20}$$

$$\leq (x^{r}-a^{r})\bigvee_{a}^{x}(f)+(b^{r}-x^{r})\bigvee_{x}^{b}(f)$$

$$\leq\begin{cases}\left[\frac{1}{2}(b^{r}-a^{r})+\left|x^{r}-\frac{a^{r}+b^{r}}{2}\right|\right]\bigvee_{a}^{b}(f), \\ \\ \left[(x^{r}-a^{r})^{p}+(b^{r}-x^{r})^{p}\right]^{1/p}\left[\left(\bigvee_{a}^{x}(f)\right)^{q}+\left(\bigvee_{x}^{b}(f)\right)^{q}\right]^{1/q} \\ \text{where } p,\ q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \\ (b^{r}-a^{r})\left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{x}(f)-\bigvee_{x}^{b}(f)\right|\right]\end{cases}$$

for any $x\in[a,b]\subset(0,\infty)$.

In particular, we have

$$\left|(b^{r}-a^{r})f(M_{r}(a,b))-r\int_{a}^{b}f(t)t^{r-1}dt\right|\leq\frac{1}{2}(b^{r}-a^{r})\bigvee_{a}^{b}(f), \tag{2.21}$$

where $M_{r}(a,b):=\left(\frac{a^{r}+b^{r}}{2}\right)^{1/r}$, $r>0$ is the *power mean with exponent* $r>0$.

If $p\in(a,b)$ is such that $\bigvee_{a}^{p}(f)=\bigvee_{p}^{b}(f)$, then

$$\left|(b^{r}-a^{r})f(p)-r\int_{a}^{b}f(t)t^{r-1}dt\right|\leq\frac{1}{2}(b^{r}-a^{r})\bigvee_{a}^{b}(f). \tag{2.22}$$

d). If we take $g:[a,b]\subset(0,\infty)\to\mathbb{R}$, $g(t)=-t^{-r}$, $r>0$ in (2.5), then we get

$$\left|\left(\frac{b^{r}-a^{r}}{b^{r}a^{r}}\right)f(x)-r\int_{a}^{b}f(t)t^{-r-1}dt\right| \tag{2.23}$$

$$\leq\left(\frac{x^{r}-a^{r}}{x^{r}a^{r}}\right)\bigvee_{a}^{x}(f)+\left(\frac{b^{r}-x^{r}}{b^{r}x^{r}}\right)\bigvee_{x}^{b}(f)$$

$$\leq\begin{cases}\left[\frac{1}{2}\left(\frac{b^{r}-a^{r}}{b^{r}a^{r}}\right)+\left|x^{-r}-\frac{a^{-r}+b^{-r}}{2}\right|\right]\bigvee_{a}^{b}(f), \\ \\ \left[\left(\frac{x^{r}-a^{r}}{x^{r}a^{r}}\right)^{p}+\left(\frac{b^{r}-x^{r}}{b^{r}x^{r}}\right)^{p}\right]^{1/p}\left[\left(\bigvee_{a}^{x}(f)\right)^{q}+\left(\bigvee_{x}^{b}(f)\right)^{q}\right]^{1/q} \\ \text{where } p,\ q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left(\frac{b^{r}-a^{r}}{b^{r}a^{r}}\right)\left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{x}(f)-\bigvee_{x}^{b}(f)\right|\right]\end{cases}$$

for any $x\in[a,b]\subset(0,\infty)$.

In particular, we have

$$\left|\left(\frac{b^{r}-a^{r}}{b^{r}a^{r}}\right)f(M_{-r}(a,b))-r\int_{a}^{b}f(t)t^{-r-1}dt\right|\leq\frac{1}{2}\left(\frac{b^{r}-a^{r}}{b^{r}a^{r}}\right)\bigvee_{a}^{b}(f) \tag{2.24}$$

where

$$M_{-r}(a,b) := \left(\frac{a^{-r}+b^{-r}}{2}\right)^{-1/r} = \left(\frac{2a^r b^r}{b^r+a^r}\right)^{1/r}.$$

If $p \in (a,b)$ is such that $\bigvee_a^p (f) = \bigvee_p^b (f)$, then

$$\left|\left(\frac{b^r-a^r}{b^r a^r}\right) f(p) - r\int_a^b f(t)\, t^{-r-1} dt\right| \le \frac{1}{2}\left(\frac{b^r-a^r}{b^r a^r}\right)\bigvee_a^b (f). \tag{2.25}$$

The particular case $r=1$ gives

$$\begin{aligned} &\left|\left(\frac{b-a}{ba}\right) f(x) - \int_a^b \frac{f(t)}{t^2} dt\right| \\ &\le \left(\frac{x-a}{xa}\right)\bigvee_a^x (f) + \left(\frac{b-x}{bx}\right)\bigvee_x^b (f) \\ &\le \begin{cases} \left[\frac{1}{2}\left(\frac{b-a}{ba}\right) + \left|\frac{H(a,b)-x}{H(a,b)x}\right|\right]\bigvee_a^b (f), \\ \\ \left[\left(\frac{x-a}{xa}\right)^p + \left(\frac{b-x}{bx}\right)^p\right]^{1/p}\left[\left(\bigvee_a^x (f)\right)^q + \left(\bigvee_x^b (f)\right)^q\right]^{1/q} \\ \text{where } p,\ q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left(\frac{b-a}{ba}\right)\left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right] \end{cases} \end{aligned} \tag{2.26}$$

for any $x \in [a,b] \subset (0,\infty)$, where $H(a,b) := \frac{2ab}{b+a}$ is the *Harmonic mean.*

In particular, we have

$$\left|\left(\frac{b-a}{ba}\right) f(H(a,b)) - \int_a^b \frac{f(t)}{t^2} dt\right| \le \frac{1}{2}\left(\frac{b-a}{ba}\right)\bigvee_a^b (f). \tag{2.27}$$

If $p \in (a,b)$ is such that $\bigvee_a^p (f) = \bigvee_p^b (f)$, then

$$\left|\left(\frac{b-a}{ba}\right) f(p) - \int_a^b \frac{f(t)}{t^2} dt\right| \le \frac{1}{2}\left(\frac{b-a}{ba}\right)\bigvee_a^b (f). \tag{2.28}$$

3. Weighted Integral Inequalities and Probability Distributions

If $w : [a,b] \to \mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W : [a,b] \to [0,\infty)$, $W(x) := \int_a^x w(s)\, ds$ is strictly increasing and differentiable on (a,b). We have $W'(x) = w(x)$ for any $x \in (a,b)$.

Proposition 1. *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ and $f : [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$, then we have*

$$\left| f(x) \int_a^b w(s)\,ds - \int_a^b f(t)\,w(t)\,dt \right| \tag{3.1}$$
$$\leq \bigvee_a^x (f) \int_a^x w(s)\,ds + \bigvee_x^b (f) \int_x^b w(s)\,ds$$
$$\leq \begin{cases} \frac{1}{2}\left[\int_a^b w(s)\,ds + \left|\int_a^x w(s)\,ds - \int_x^b w(s)\,ds\right|\right] \bigvee_a^b (f), \\ \\ \left[\left(\int_a^x w(s)\,ds\right)^p + \left(\int_x^b w(s)\,ds\right)^p\right]^{1/p} \left[\left(\bigvee_a^x (f)\right)^q + \left(\bigvee_x^b (f)\right)^q\right]^{1/q} \\ \text{where } p,\ q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \frac{1}{2}\left[\bigvee_a^b (f) + \left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right] \int_a^b w(s)\,ds \end{cases}$$

for all $x \in [a,b]$.

In particular, if

$$M_W(a,b) := W^{-1}\left(\frac{1}{2}\int_a^b w(s)\,ds\right),$$

then we have

$$\left| f(M_W(a,b)) \int_a^b w(s)\,ds - \int_a^b f(t)\,w(t)\,dt \right| \leq \frac{1}{2} \bigvee_a^b (f) \int_a^b w(s)\,ds. \tag{3.2}$$

Moreover, if $p \in (a,b)$ is such that $\bigvee_a^p (f) = \bigvee_p^b (f)$, then

$$\left| f(p) \int_a^b w(s)\,ds - \int_a^b f(t)\,w(t)\,dt \right| \leq \frac{1}{2} \bigvee_a^b (f) \int_a^b w(s)\,ds. \tag{3.3}$$

The proof follows by Theorem 2 for $g(x) := \int_a^x w(s)\,ds$, $x \in [a,b]$.

The above result can be extended for infinite intervals I by assuming that the function $f : I \to \mathbb{C}$ is locally of bounded variation on I.

For instance, if $I = [a,\infty)$, $f : [a,\infty) \to \mathbb{C}$ is locally of bounded variation on $[a,\infty)$ with

$$\bigvee_a^\infty (f) := \lim_{b\to\infty} \bigvee_a^b (f) < \infty$$

and $w(s) > 0$ for $s \in [a,\infty)$ with $\int_a^\infty w(s)\,ds = 1$, namely w is a probability density function on $[a,\infty)$, then by (3.1) we get

$$\left|f(x) - \int_a^\infty f(t)\,w(t)\,dt\right| \tag{3.4}$$
$$\leq W(x)\bigvee_a^x (f) + [1 - W(x)]\bigvee_x^\infty (f)$$
$$\leq \begin{cases} \left[\frac{1}{2} + \left|W(x) - \frac{1}{2}\right|\right]\bigvee_a^\infty (f), \\ \\ [W^p(x) + (1 - W(x))^p]^{1/p}\left[\left(\bigvee_a^x (f)\right)^q + \left(\bigvee_x^\infty (f)\right)^q\right]^{1/q} \\ \text{where } p,\ q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \frac{1}{2}\left[\bigvee_a^\infty (f) + \left|\bigvee_a^x (f) - \bigvee_x^\infty (f)\right|\right] \end{cases}$$

for any $x \in [a,\infty)$, where $W(x) := \int_a^x w(s)\,ds$ is the cumulative distribution function.

If $m \in (a,\infty)$ is the *median point* for w, namely $W(m) = \frac{1}{2}$, then by (3.4) we get

$$\left|f(m) - \int_a^\infty f(t)\,w(t)\,dt\right| \leq \frac{1}{2}\bigvee_a^\infty (f). \tag{3.5}$$

Also, if $p \in (a,\infty)$ such that $\bigvee_a^p (f) = \bigvee_p^\infty (f)$, then

$$\left|f(p) - \int_a^\infty f(t)\,w(t)\,dt\right| \leq \frac{1}{2}\bigvee_a^\infty (f). \tag{3.6}$$

In probability theory and statistics, the *beta prime distribution* (also known as *inverted beta distribution* or *beta distribution of the second kind*) is an absolutely continuous probability distribution defined for $x > 0$ with two parameters α and β, having the probability density function:

$$w_{\alpha,\beta}(x) := \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)}$$

where B is *Beta function*

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1},\ \alpha,\ \beta > 0.$$

The cumulative distribution function is

$$W_{\alpha,\beta}(x) = I_{\frac{x}{1+x}}(\alpha,\beta)$$

where I is the *regularized incomplete beta function* defined by

$$I_z(\alpha,\beta) := \frac{B(z;\alpha,\beta)}{B(\alpha,\beta)}.$$

Here $B(\cdot;\alpha,\beta)$ is the *incomplete beta function* defined by

$$B(z;\alpha,\beta) := \int_0^z t^{\alpha-1}(1-t)^{\beta-1},\ \alpha,\ \beta,\ z > 0.$$

Assume that $f:[0,\infty)\to\mathbb{C}$ is locally of bounded variation on $[0,\infty)$ with $\bigvee_0^\infty(f)<\infty$. Using the inequality (3.4) we have for $x>0$ that

$$\left|f(x)-\frac{1}{B(\alpha,\beta)}\int_0^\infty f(t)\,t^{\alpha-1}(1+t)^{-\alpha-\beta}\,dt\right| \tag{3.7}$$
$$\leq I_{\frac{x}{1+x}}(\alpha,\beta)\bigvee_0^x(f)+\left[1-I_{\frac{x}{1+x}}(\alpha,\beta)\right]\bigvee_x^\infty(f)$$
$$\leq\begin{cases}\left[\frac{1}{2}+\left|I_{\frac{x}{1+x}}(\alpha,\beta)-\frac{1}{2}\right|\right]\bigvee_0^\infty(f),\\ \\ \left[\left(I_{\frac{x}{1+x}}(\alpha,\beta)\right)^p+\left(1-I_{\frac{x}{1+x}}(\alpha,\beta)\right)^p\right]^{1/p}\\ \left[\left(\bigvee_0^x(f)\right)^q+\left(\bigvee_x^\infty(f)\right)^q\right]^{1/q}\\ \text{where } p,\ q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1,\\ \\ \frac{1}{2}\left[\bigvee_0^\infty(f)+\left|\bigvee_0^x(f)-\bigvee_x^\infty(f)\right|\right],\end{cases}$$

for $\alpha,\ \beta>0$.

Similar results may be stated for the probability distributions that are supported on the whole axis $\mathbb{R}$. Namely, if $I=\mathbb{R}$, $f:\mathbb{R}\to\mathbb{C}$ is locally of bounded variation on $\mathbb{R}$ with

$$\bigvee_{-\infty}^\infty(f):=\lim_{b\to\infty,a\to-\infty}\bigvee_a^b(f)<\infty$$

and $w(s)>0$ for $s\in\mathbb{R}$ with $\int_{-\infty}^\infty w(s)\,ds=1$, namely w is a probability density function on $\mathbb{R}$, then by (3.1) we get

$$\left|f(x)-\int_{-\infty}^\infty f(t)\,w(t)\,dt\right| \tag{3.8}$$
$$\leq W(x)\bigvee_{-\infty}^x(f)+[1-W(x)]\bigvee_x^\infty(f)$$
$$\leq\begin{cases}\left[\frac{1}{2}+\left|W(x)-\frac{1}{2}\right|\right]\bigvee_{-\infty}^\infty(f),\\ \\ \left[W^p(x)+(1-W(x))^p\right]^{1/p}\left[\left(\bigvee_{-\infty}^x(f)\right)^q+\left(\bigvee_x^\infty(f)\right)^q\right]^{1/q}\\ \text{where } p,\ q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1,\\ \\ \frac{1}{2}\left[\bigvee_{-\infty}^\infty(f)+\left|\bigvee_{-\infty}^x(f)-\bigvee_x^\infty(f)\right|\right]\end{cases}$$

for any $x\in\mathbb{R}$, where $W(x):=\int_{-\infty}^x w(s)\,ds$ is the cumulative distribution function.

If $m\in\mathbb{R}$ is the *median point* for w, namely $W(m)=\frac{1}{2}$, then by (3.4) we get

$$\left|f(m)-\int_{-\infty}^\infty f(t)\,w(t)\,dt\right|\leq\frac{1}{2}\bigvee_{-\infty}^\infty(f). \tag{3.9}$$

Also, if $p\in(a,\infty)$ such that $\bigvee_{-\infty}^p(f)=\bigvee_p^\infty(f)$, then

$$\left|f(p)-\int_{-\infty}^\infty f(t)\,w(t)\,dt\right|\leq\frac{1}{2}\bigvee_{-\infty}^\infty(f). \tag{3.10}$$

In what follows we give an example.

The probability density of the *normal distribution* on $(-\infty,\infty)$ is

$$w_{\mu,\sigma^2}(x) := \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\ x\in\mathbb{R},$$

where μ is the *mean* or *expectation* of the distribution (and also its *median* and *mode*), σ is the *standard deviation*, and σ^2 is the *variance*.

The cumulative distribution function is

$$W_{\mu,\sigma^2}(x) = \frac{1}{2}+\frac{1}{2}\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right),$$

where the *error function* erf is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-t^2\right)dt.$$

If $f:\mathbb{R}\to\mathbb{R}$ is locally of bounded variation with $\bigvee_{-\infty}^{\infty}(f)<\infty$, then from (3.8) we have

$$\left|f(x)-\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}f(t)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dt\right| \tag{3.11}$$
$$\leq\frac{1}{2}\left\{\left[1+\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]\bigvee_{-\infty}^{x}(f)+\left[1-\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]\bigvee_{x}^{\infty}(f)\right\}$$
$$\leq\begin{cases}\frac{1}{2}\left[1+\left|\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right|\right]\bigvee_{-\infty}^{\infty}(f),\\[2ex] \frac{1}{2}\left[\left(1+\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)^p+\left(1-\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)^p\right]^{1/p}\\ \times\left[\left(\bigvee_{-\infty}^{x}(f)\right)^q+\left(\bigvee_{x}^{\infty}(f)\right)^q\right]^{1/q}\\ \text{where } p,\ q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \frac{1}{2}\left[\bigvee_{-\infty}^{\infty}(f)+\left|\bigvee_{-\infty}^{x}(f)-\bigvee_{x}^{\infty}(f)\right|\right]\end{cases}$$

for any $x\in\mathbb{R}$.

In particular, we have

$$\left|f(\mu)-\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}f(t)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dt\right|\leq\frac{1}{2}\bigvee_{-\infty}^{\infty}(f). \tag{3.12}$$

Also, if $p\in\mathbb{R}$ such that $\bigvee_{-\infty}^{p}(f)=\bigvee_{p}^{\infty}(f)$, then

$$\left|f(p)-\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}f(t)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dt\right|\leq\frac{1}{2}\bigvee_{-\infty}^{\infty}(f). \tag{3.13}$$

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