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WEIGHTED INEQUALITIES OF TRAPEZOID TYPE FOR ABSOLUTELY CONTINUOUS FUNCTIONS AND APPLICATIONS

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Abstract. In this paper we establish some upper bounds for the quantity

$$\left|(g(x)-g(a))f(a)+(g(b)-g(x))f(b)-\int_a^b f(t)g'(t)\,dt\right|$$

under the assumptions that $g:[a,b]\to[g(a),g(b)]$ is a *continuous strictly increasing function* that is *differentiable* on (a,b) and $f:[a,b]\to\mathbb{C}$ is an absolutely continuous function on $[a,b]$. When g is an integral, namely $g(x)=\int_a^x w(s)\,ds$, where $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$, then some weighted inequalities that generalize the Trapezoid inequality are provided. Applications for continuous probability density functions supported on finite and infinite intervals with two examples are also given.

1. Introduction

The following trapezoid type inequality holds [4].

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be an L-Lipschitzian mapping on $[a,b]$. Then we have the inequality*

$$\left|\int_a^b f(t)\,dt-[(x-a)f(a)+(b-x)f(b)]\right| \le\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]L \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best in (1.1).

If we choose $x=\frac{a+b}{2}$, then we have [9]

$$\left|\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)\right|\le\frac{1}{4}(b-a)^2L, \tag{1.2}$$

which is the "trapezoid inequality". Note that the trapezoid inequality (1.2) is, in a sense, the best possible inequality we can obtain from (1.1). In addition, the constant $\frac{1}{4}$ is the best possible one, providing the sharpest bound in the class.

Corollary 1. *Assume that $f:[a,b]\to\mathbb{C}$ is absolutely continuous with $f'\in L_\infty[a,b]$, namely f' is essentially bounded, and put $\|f'\|_\infty=\operatorname{essup}_{t\in(a,b)}|f'(t)|<$*

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∞. *Then we have the inequality*

$$\left|\int_a^b f(t)\,dt - [(x-a)f(a) + (b-x)f(b)]\right| \leq \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \|f'\|_\infty \tag{1.3}$$

for all $x \in [a,b]$. *The constant* $\frac{1}{4}$ *is best in (1.3).*

In particular, we have

$$\left|\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a)\right| \leq \frac{1}{4}(b-a)^2 \|f'\|_\infty . \tag{1.4}$$

The following trapezoid type integral inequality for mappings of bounded variation holds [8], [12] and [3]:

Theorem 2. *Let* $f : [a,b] \to \mathbb{R}$ *be a mapping of bounded variation.*

We then have the inequality:

$$\left|\int_a^b f(t)\,dt - [(x-a)f(a) + (b-x)f(b)]\right| \leq \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f) \tag{1.5}$$

holding for all $x \in [a,b]$, *where* $\bigvee_a^b (f)$ *denotes the total variation of* f *on the interval* $[a,b]$.

The constant $\frac{1}{2}$ *is the best possible one.*

If we choose $x = \frac{a+b}{2}$, then we get [11]:

$$\left|\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a)\right| \leq \frac{1}{2}(b-a) \bigvee_a^b (f), \tag{1.6}$$

which is the *"trapezoid" inequality*. Note that the trapezoid inequality (1.6) is in a sense the best possible inequality we can get from (1.5). Also, the constant $\frac{1}{2}$ is the best possible.

If $w : [a,b] \to \mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W : [a,b] \to [0,\infty)$, $W(x) := \int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x) = w(x)$ for any $x \in (a,b)$.

In 2004 Tseng et al. [24] proved a weighted trapezoid inequality, which essentially can be written as

$$\left|\frac{f(a)\int_a^x w(s)\,ds + f(b)\int_x^b w(s)\,ds}{\int_a^b w(s)\,ds} - \frac{1}{\int_a^b w(s)\,ds}\int_a^b f(t)w(t)\,dt\right| \leq \frac{1}{2}\left[1 + \left|\frac{\int_x^b w(s)\,ds - \int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\right|\right] \bigvee_a^b (f) \tag{1.7}$$

for any $x \in [a,b]$.

For related result concerning the Trapezoid inequality, see [1]-[2], [5]-[7] and [9]-[23].

Motivated by the above results, in this paper we establish some upper bounds for the quantity

$$\left|(g(x)-g(a))f(a)+(g(b)-g(x))f(b)-\int_a^b f(t)g'(t)\,dt\right|$$

under the assumptions that $g:[a,b]\to[g(a),g(b)]$ is a *continuous strictly increasing function* that is *differentiable* on (a,b) and $f:[a,b]\to\mathbb{C}$ is an absolutely continuous function on $[a,b]$. When g is an integral, namely $g(x)=\int_a^x w(s)\,ds$, where $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$, then some weighted inequalities that generalize the Trapezoid inequality are provided. Applications for continuous probability density functions supported on finite and infinite intervals with two examples are also given.

2. The Main Results

We need the following result:

Lemma 1. *Let $h:[c,d]\to\mathbb{C}$ be an absolutely continuous function on $[c,d]$ whose derivative $h'\in L_\infty[c,d]$. Then*

$$\left|\frac{(z-c)h(c)+(d-z)h(d)}{d-c}-\frac{1}{d-c}\int_c^d h(t)\,dt\right| \tag{2.1}$$

$$\leq\frac{1}{2}\left[\|h'\|_{[c,z],\infty}\left(\frac{z-c}{d-c}\right)^2+\|h'\|_{[z,d],\infty}\left(\frac{d-z}{d-c}\right)^2\right](d-c)$$

$$\leq\begin{cases}\|h'\|_{[c,d],\infty}\left[\frac{1}{4}+\left(\frac{z-\frac{c+d}{2}}{d-c}\right)^2\right](d-c);\\[2ex] \frac{1}{2}\left[\|h'\|^{\alpha}_{[c,z],\infty}+\|h'\|^{\alpha}_{[z,d],\infty}\right]^{\frac{1}{\alpha}}\left[\left(\frac{z-c}{d-c}\right)^{2\beta}+\left(\frac{d-z}{d-c}\right)^{2\beta}\right]^{\frac{1}{\beta}}(d-c),\\ \text{where }\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\[2ex] \frac{1}{2}\left[\|h'\|_{[c,z],\infty}+\|h'\|_{[z,d],\infty}\right]\left[\frac{1}{2}+\left|\frac{z-\frac{c+d}{2}}{d-c}\right|\right]^2(d-c)\end{cases}$$

for all $z\in[c,d]$, where $\|\cdot\|_{[m,n],\infty}$ denotes the usual sup-norm on $L_\infty[m,n]$, i.e., we recall that

$$\|g\|_{[m,n],\infty}=\operatorname*{essup}_{t\in[m,n]}|g(t)|<\infty.$$

Proof. Let $z\in(c,d)$. Using the integration by parts formula we have,

$$\int_c^z(t-z)h'(t)\,dt+\int_z^d(t-z)h'(t)\,dt \tag{2.2}$$

$$=(z-c)h(c)-\int_c^z h(t)\,dt+(d-z)h(d)-\int_z^d h(t)\,dt$$

$$=(z-c)h(c)+(d-z)h(d)-\int_c^d h(t)\,dt.$$

Taking the modulus and using the triangle inequality we have

$$\begin{aligned}&\left|(z-c)\,h(c)+(d-z)\,h(d)-\int_c^d h(t)\,dt\right|\\&\leq\left|\int_c^z (t-z)\,h'(t)\,dt\right|+\left|\int_z^d (t-z)\,h'(t)\,dt\right|\\&\leq\int_c^z (z-t)\,|h'(t)|\,dt+\int_z^d (t-z)\,|h'(t)|\,dt\\&\leq\frac{1}{2}\left[\|h'\|_{[c,z],\infty}(z-c)^2+\|h'\|_{[z,d],\infty}(d-z)^2\right]\end{aligned}$$

for all $z\in[c,d]$, and the first inequality in (2.1) is proved.

Now, let us observe that

$$\begin{aligned}&\|h'\|_{[c,z],\infty}(z-c)^2+\|h'\|_{[z,d],\infty}(d-z)^2\\&\leq\max\left\{\|h'\|_{[c,z],\infty},\|h'\|_{[z,d],\infty}\right\}\left[(z-c)^2+(d-z)^2\right]\\&=\max\left\{\|h'\|_{[c,z],\infty},\|h'\|_{[z,d],\infty}\right\}\left[\frac{1}{2}(d-c)^2+2\left(z-\frac{c+d}{2}\right)^2\right]\\&=(d-c)^2\max\left\{\|h'\|_{[c,z],\infty},\|h'\|_{[z,d],\infty}\right\}\left[\frac{1}{2}+2\cdot\frac{\left(z-\frac{c+d}{2}\right)^2}{(d-c)^2}\right]\\&=(d-c)^2\|h'\|_{[c,d],\infty}\left[\frac{1}{2}+2\cdot\frac{\left(z-\frac{c+d}{2}\right)^2}{(d-c)^2}\right],\end{aligned}$$

and the first part of the second inequality in (2.1) is proved.

For the second inequality, we employ the elementary inequality for real numbers which can be derived from *Hölder's discrete inequality*

$$0\leq ms+nt\leq(m^\alpha+n^\alpha)^{\frac{1}{\alpha}}\times\left(s^\beta+t^\beta\right)^{\frac{1}{\beta}}, \tag{2.3}$$

provided that m, s, n, $t\geq0$, $\alpha>1$ and $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Using (2.3), we obtain

$$\begin{aligned}&\|h'\|_{[c,z],\infty}(z-c)^2+\|h'\|_{[z,d],\infty}(d-z)^2\\&\leq\left(\|h'\|^\alpha_{[c,z],\infty}+\|h'\|^\alpha_{[z,d],\infty}\right)^{\frac{1}{\alpha}}\left[(z-c)^{2\beta}+(d-z)^{2\beta}\right]^{\frac{1}{\beta}}\end{aligned}$$

and the second part of the second inequality in (2.1) is also obtained.

Finally, we observe that

$$\begin{aligned}&\|h'\|_{[c,z],\infty}(z-c)^2+\|h'\|_{[z,d],\infty}(d-z)^2\\&\leq\max\left\{(z-c)^2,(d-z)^2\right\}\left[\|h'\|_{[c,z],\infty}+\|h'\|_{[z,d],\infty}\right]\\&=\left[\frac{d-c}{2}+\left|z-\frac{c+d}{2}\right|\right]^2\left[\|h'\|_{[c,z],\infty}+\|h'\|_{[z,d],\infty}\right]\end{aligned}$$

and the last part of the second inequality in (2.1) is proved. □

The following corollary is also natural.

Corollary 2. *Under the above assumptions, we have the trapezoid inequality*

$$\begin{aligned}(2.4)\qquad &\left|\frac{h(c)+h(d)}{2}-\frac{1}{d-c}\int_c^d h(t)\,dt\right| \\ &\leq \frac{1}{8}(d-c)\left[\|h'\|_{\left[c,\frac{c+d}{2}\right],\infty}+\|h'\|_{\left[\frac{c+d}{2},d\right],\infty}\right] \\ &\leq \begin{cases}\frac{1}{4}(d-c)\|h'\|_{[c,d],\infty}; \\ \\ \frac{1}{2^{\frac{3\beta-1}{\beta}}}(d-c)\left[\|h'\|^{\alpha}_{\left[c,\frac{c+d}{2}\right],\infty}+\|h'\|^{\alpha}_{\left[\frac{c+d}{2},d\right],\infty}\right]^{\frac{1}{\alpha}}, \\ \text{where } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1.\end{cases}\end{aligned}$$

We have:

Theorem 3. *Let $g:[a,b]\to[g(a),g(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If $f:[a,b]\to\mathbb{C}$ is absolutely continuous on $[a,b]$ and $\frac{f'}{g'}$ is essentially bounded, namely $\frac{f'}{g'}\in L_\infty[a,b]$, then we have*

$$(2.5)\qquad \left|\frac{(g(x)-g(a))f(a)+(g(b)-g(x))f(b)}{g(b)-g(a)}-\frac{1}{g(b)-g(a)}\int_a^b f(t)\,g'(t)\,dt\right|$$

$$\leq \frac{1}{2}\left[\left\|\frac{f'}{g'}\right\|_{[a,x],\infty}\left(\frac{g(x)-g(a)}{g(b)-g(a)}\right)^2+\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}\left(\frac{g(b)-g(x)}{g(b)-g(a)}\right)^2\right](g(b)-g(a))$$

$$\leq \begin{cases}\left\|\frac{f'}{g'}\right\|_{[a,b],\infty}\left[\frac{1}{4}+\left(\frac{g(x)-\frac{g(a)+g(b)}{2}}{g(b)-g(a)}\right)^2\right](g(b)-g(a)); \\ \\ \frac{1}{2}\left[\left\|\frac{f'}{g'}\right\|^{\alpha}_{[a,x],\infty}+\left\|\frac{f'}{g'}\right\|^{\alpha}_{[x,b],\infty}\right]^{\frac{1}{\alpha}} \\ \times\left[\left(\frac{g(x)-g(a)}{g(b)-g(a)}\right)^{2\beta}+\left(\frac{g(b)-g(x)}{g(b)-g(a)}\right)^{2\beta}\right]^{\frac{1}{\beta}}(g(b)-g(a)), \\ \text{where } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \frac{1}{2}\left[\left\|\frac{f'}{g'}\right\|_{[a,x],\infty}+\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}\right] \\ \times\left[\frac{1}{2}+\left|\frac{g(x)-\frac{g(a)+g(b)}{2}}{g(b)-g(a)}\right|\right]^2(g(b)-g(a))\end{cases}$$

for all $x\in[a,b]$.

Proof. Assume that $[c,d]\subset[a,b]$. If $f:[c,d]\to\mathbb{C}$ is absolutely continuous on $[c,d]$, then $f\circ g^{-1}:[g(c),g(d)]\to\mathbb{C}$ is absolutely continuous on $[g(c),g(d)]$ and using the chain rule and the derivative of inverse functions we have

$$(2.6)\qquad \left(f\circ g^{-1}\right)'(z)=\left(f'\circ g^{-1}\right)(z)\left(g^{-1}\right)'(z)=\frac{\left(f'\circ g^{-1}\right)(z)}{\left(g'\circ g^{-1}\right)(z)}$$

for almost every (a.e.) $z\in[g(c),g(d)]$.

If $x\in[c,d]$, then by taking $z=g(x)$, we get

$$\left(f\circ g^{-1}\right)'(z)=\frac{\left(f'\circ g^{-1}\right)(g(x))}{\left(g'\circ g^{-1}\right)(g(x))}=\frac{f'(x)}{g'(x)}.$$

Therefore, since $\frac{f'}{g'}\in L_\infty[c,d]$, hence $\left(f\circ g^{-1}\right)'\in L_\infty[g(c),g(d)]$.

Now, if we use the inequality (2.1) for the function $h=f\circ g^{-1}$ on the interval $[g(a),g(b)]$, then we get for any $z\in[g(a),g(b)]$ that

$$\left|\frac{(z-g(a))f(a)+(g(b)-z)f(b)}{g(b)-g(a)}-\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)}f\circ g^{-1}(t)\,dt\right| \tag{2.7}$$
$$\leq\frac{1}{2}\left[\left\|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right\|_{[g(a),z],\infty}\left(\frac{z-g(a)}{g(b)-g(a)}\right)^2+\left\|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right\|_{[z,g(b)],\infty}\left(\frac{g(b)-z}{g(b)-g(a)}\right)^2\right]\times(g(b)-g(a))$$

$$\leq\begin{cases}\left\|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right\|_{[g(a),g(b)],\infty}\left[\frac{1}{4}+\left(\frac{z-\frac{g(a)+g(b)}{2}}{g(b)-g(a)}\right)^2\right](g(b)-g(a)); \\[2ex] \frac{1}{2}\left[\left\|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right\|_{[g(a),z],\infty}^{\alpha}+\left\|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right\|_{[z,g(b)],\infty}^{\alpha}\right]^{\frac{1}{\alpha}} \\ \times\left[\left(\frac{z-g(a)}{g(b)-g(a)}\right)^{2\beta}+\left(\frac{g(b)-z}{g(b)-g(a)}\right)^{2\beta}\right]^{\frac{1}{\beta}}(g(b)-g(a)), \\ \text{where }\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[2ex] \frac{1}{2}\left[\left\|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right\|_{[g(a),z],\infty}+\left\|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right\|_{[z,g(b)],\infty}\right] \\ \times\left[\frac{1}{2}+\left|\frac{z-\frac{g(a)+g(b)}{2}}{g(b)-g(a)}\right|\right]^2(g(b)-g(a)).\end{cases}$$

Taking $z=g(x)$, $x\in[a,b]$, in (2.7) we then get

$$\left|\frac{(g(x)-g(a))f(a)+(g(b)-g(x))f(b)}{g(b)-g(a)}-\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)}f\circ g^{-1}(t)\,dt\right| \tag{2.8}$$
$$\leq\frac{1}{2}\left[\left\|\frac{f'}{g'}\right\|_{[a,x],\infty}\left(\frac{g(x)-g(a)}{g(b)-g(a)}\right)^2+\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}\left(\frac{g(b)-g(x)}{g(b)-g(a)}\right)^2\right](g(b)-g(a))$$

$$\leq \begin{cases} \left\| \frac{f'}{g'} \right\|_{[a,b],\infty} \left[\frac{1}{4} + \left(\frac{g(x) - \frac{g(a)+g(b)}{2}}{g(b)-g(a)} \right)^2 \right] (g(b) - g(a)); \\ \\ \frac{1}{2} \left[\left\| \frac{f'}{g'} \right\|_{[a,x],\infty}^{\alpha} + \left\| \frac{f'}{g'} \right\|_{[x,b],\infty}^{\alpha} \right]^{\frac{1}{\alpha}} \\ \times \left[\left(\frac{g(x)-g(a)}{g(b)-g(a)} \right)^{2\beta} + \left(\frac{g(b)-g(x)}{g(b)-g(a)} \right)^{2\beta} \right]^{\frac{1}{\beta}} (g(b) - g(a)), \\ \text{where } \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \\ \frac{1}{2} \left[\left\| \frac{f'}{g'} \right\|_{[a,x],\infty} + \left\| \frac{f'}{g'} \right\|_{[x,b],\infty} \right] \\ \times \left[\frac{1}{2} + \left| \frac{g(x) - \frac{g(a)+g(b)}{2}}{g(b)-g(a)} \right| \right]^2 (g(b) - g(a)) \end{cases}$$

since

$$\left\| \frac{f' \circ g^{-1}}{g' \circ g^{-1}} \right\|_{[g(a),g(x)],\infty} = \left\| \frac{f'}{g'} \right\|_{[a,x],\infty}$$

and

$$\left\| \frac{f' \circ g^{-1}}{g' \circ g^{-1}} \right\|_{[g(x),g(b)],\infty} = \left\| \frac{f'}{g'} \right\|_{[x,b],\infty}.$$

Observe also that, by the change of variable $t = g^{-1}(u)$, $u \in [g(a), g(b)]$, we have $u = g(t)$ that gives $du = g'(t)\,dt$ and

$$\int_{g(a)}^{g(b)} \left(f \circ g^{-1} \right)(u)\,du = \int_a^b f(t)\,g'(t)\,dt. \tag{2.9}$$

Finally, by making use of (2.8) we deduce the desired result (2.5). □

Remark 1. *If we take $x = \frac{a+b}{2}$ in 2.5, then we get*

$$\left| \frac{\left(g\left(\frac{a+b}{2}\right) - g(a) \right) f(a) + \left(g(b) - g\left(\frac{a+b}{2}\right) \right) f(b)}{g(b) - g(a)} - \frac{1}{g(b) - g(a)} \int_a^b f(t)\,g'(t)\,dt \right| \tag{2.10}$$
$$\leq \frac{1}{2} \left[\left\| \frac{f'}{g'} \right\|_{\left[a, \frac{a+b}{2}\right],\infty} \left(\frac{g\left(\frac{a+b}{2}\right) - g(a)}{g(b) - g(a)} \right)^2 + \left\| \frac{f'}{g'} \right\|_{\left[\frac{a+b}{2}, b\right],\infty} \left(\frac{g(b) - g\left(\frac{a+b}{2}\right)}{g(b) - g(a)} \right)^2 \right] \times (g(b) - g(a))$$

$$\leq \begin{cases} \left\Vert \frac{f'}{g'}\right\Vert_{[a,b],\infty} \left[\frac{1}{4} + \left(\frac{g\left(\frac{a+b}{2}\right) - \frac{g(a)+g(b)}{2}}{g(b)-g(a)}\right)^2\right] (g(b) - g(a)); \\ \\ \frac{1}{2}\left[\left\Vert \frac{f'}{g'}\right\Vert^{\alpha}_{\left[a,\frac{a+b}{2}\right],\infty} + \left\Vert \frac{f'}{g'}\right\Vert^{\alpha}_{\left[\frac{a+b}{2},b\right],\infty}\right]^{\frac{1}{\alpha}} \\ \times \left[\left(\frac{g\left(\frac{a+b}{2}\right)-g(a)}{g(b)-g(a)}\right)^{2\beta} + \left(\frac{g(b)-g\left(\frac{a+b}{2}\right)}{g(b)-g(a)}\right)^{2\beta}\right]^{\frac{1}{\beta}} (g(b) - g(a)), \\ \textit{where } \ \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \\ \frac{1}{2}\left[\left\Vert \frac{f'}{g'}\right\Vert_{\left[a,\frac{a+b}{2}\right],\infty} + \left\Vert \frac{f'}{g'}\right\Vert_{\left[\frac{a+b}{2},b\right],\infty}\right] \\ \times \left[\frac{1}{2} + \left|\frac{g\left(\frac{a+b}{2}\right) - \frac{g(a)+g(b)}{2}}{g(b)-g(a)}\right|\right]^2 (g(b) - g(a)) \end{cases}$$

for all $x \in [a, b]$.

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a, b \in I$ as

$$M_g(a,b) := g^{-1}\left(\frac{g(a) + g(b)}{2}\right). \tag{2.11}$$

If $I = \mathbb{R}$ and $g(t) = t$ is the *identity function*, then $M_g(a,b) = A(a,b) := \frac{a+b}{2}$, the *arithmetic mean*. If $I = (0,\infty)$ and $g(t) = \ln t$, then $M_g(a,b) = G(a,b) := \sqrt{ab}$, the *geometric mean*. If $I = (0,\infty)$ and $g(t) = -\frac{1}{t}$, then $M_g(a,b) = H(a,b) := \frac{2ab}{a+b}$, the *harmonic mean*. If $I = (0,\infty)$ and $g(t) = t^p$, $p \neq 0$, then $M_g(a,b) = M_p(a,b) := \left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent* p. Finally, if $I = \mathbb{R}$ and $g(t) = \exp t$, then

$$M_g(a,b) = LME(a,b) := \ln\left(\frac{\exp a + \exp b}{2}\right), \tag{2.12}$$

the *LogMeanExp function*.

Corollary 3. *With the assumptions of Theorem 3 we have*

$$\begin{aligned} &\left|\frac{f(a) + f(b)}{2} - \frac{1}{g(b) - g(a)}\int_a^b f(t)\, g'(t)\, dt\right| \\ &\leq \frac{1}{8}\left[\left\Vert \frac{f'}{g'}\right\Vert_{[a,M_g(a,b)],\infty} + \left\Vert \frac{f'}{g'}\right\Vert_{[M_g(a,b),b],\infty}\right] (g(b) - g(a)) \\ &\leq \begin{cases} \frac{1}{4}\left\Vert \frac{f'}{g'}\right\Vert_{[a,b],\infty} (g(b) - g(a)); \\ \\ \frac{1}{2^{\frac{3\beta-1}{\beta}}}\left[\left\Vert \frac{f'}{g'}\right\Vert^{\alpha}_{[a,M_g(a,b)],\infty} + \left\Vert \frac{f'}{g'}\right\Vert^{\alpha}_{[M_g(a,b),b],\infty}\right]^{\frac{1}{\alpha}} (g(b) - g(a)), \\ \textit{where } \ \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1. \end{cases} \end{aligned} \tag{2.13}$$

Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. We can give the following examples of interest.

a). If we take $g:[a,b]\subset(0,\infty)\to\mathbb{R}$, $g(t)=\ln t$, in (2.5) and assume that $\ell f'\in L_{\infty}[a,b]$ where $\ell(t):=t$, then we get

$$\left|\frac{f(a)\ln\left(\frac{x}{a}\right)+f(b)\ln\left(\frac{b}{x}\right)}{\ln\left(\frac{b}{a}\right)}-\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b\frac{f(t)}{t}dt\right| \tag{2.14}$$
$$\leq\frac{1}{2}\left[\|\ell f'\|_{[a,x],\infty}\left[\frac{\ln\left(\frac{x}{a}\right)}{\ln\left(\frac{b}{a}\right)}\right]^2+\|\ell f'\|_{[x,b],\infty}\left[\frac{\ln\left(\frac{b}{x}\right)}{\ln\left(\frac{b}{a}\right)}\right]^2\right]\ln\left(\frac{b}{a}\right)$$
$$\leq\begin{cases}\|\ell f'\|_{[a,b],\infty}\left[\frac{1}{4}+\left(\frac{\ln\left(\frac{x}{G(a,b)}\right)}{\ln\left(\frac{b}{a}\right)}\right)^2\right]\ln\left(\frac{b}{a}\right); \\ \\ \frac{1}{2}\left[\|\ell f'\|_{[a,x],\infty}^{\alpha}+\|\ell f'\|_{[x,b],\infty}^{\alpha}\right]^{\frac{1}{\alpha}} \\ \times\left[\left(\frac{\ln\left(\frac{x}{G(a,b)}\right)}{\ln\left(\frac{b}{a}\right)}\right)^{2\beta}+\left(\frac{\ln\left(\frac{x}{G(a,b)}\right)}{\ln\left(\frac{b}{a}\right)}\right)^{2\beta}\right]^{\frac{1}{\beta}}\ln\left(\frac{b}{a}\right), \\ \text{where }\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \frac{1}{2}\left[\|\ell f'\|_{[a,x],\infty}+\|\ell f'\|_{[x,b],\infty}\right] \\ \times\left[\frac{1}{2}+\left|\frac{\ln\left(\frac{x}{G(a,b)}\right)}{\ln\left(\frac{b}{a}\right)}\right|\right]^2\ln\left(\frac{b}{a}\right)\end{cases}$$

for any $x\in[a,b]\subset(0,\infty)$.

In particular, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b\frac{f(t)}{t}dt\right| \tag{2.15}$$
$$\leq\frac{1}{8}\left[\|\ell f'\|_{[a,G(a,b)],\infty}+\|\ell f'\|_{[G(a,b),b],\infty}\right]\ln\left(\frac{b}{a}\right)$$
$$\leq\begin{cases}\frac{1}{4}\|\ell f'\|_{[a,b],\infty}\ln\left(\frac{b}{a}\right); \\ \\ \frac{1}{2^{\frac{3\beta-1}{\beta}}}\left[\|\ell f'\|_{[a,G(a,b)],\infty}^{\alpha}+\|\ell f'\|_{[G(a,b),b],\infty}^{\alpha}\right]^{\frac{1}{\alpha}}\ln\left(\frac{b}{a}\right), \\ \text{where }\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1.\end{cases}$$

b). If we take $g:[a,b]\subset\mathbb{R}\to(0,\infty)$, $g(t)=\exp t$, in (2.5) and assume that $\frac{f'}{\exp}\in L_{\infty}[a,b]$, then we get

$$\left|\frac{(\exp x-\exp a)f(a)+(\exp b-\exp x)f(b)}{\exp b-\exp a}\right. \tag{2.16}$$
$$\left.-\frac{1}{\exp b-\exp a}\int_a^b f(t)\exp t\,dt\right|$$
$$\leq\frac{1}{2}\left[\left\|\frac{f'}{\exp}\right\|_{[a,x],\infty}\left(\frac{\exp x-\exp a}{\exp b-\exp a}\right)^2+\left\|\frac{f'}{\exp}\right\|_{[x,b],\infty}\left(\frac{\exp b-\exp x}{\exp b-\exp a}\right)^2\right](\exp b-\exp a)$$

$$\leq \begin{cases} \left\|\frac{f'}{\exp}\right\|_{[a,b],\infty}\left[\frac{1}{4}+\left(\frac{\exp x-\frac{\exp a+\exp b}{2}}{\exp b-\exp a}\right)^2\right](\exp b-\exp a); \\ \\ \frac{1}{2}\left[\left\|\frac{f'}{\exp}\right\|_{[a,x],\infty}^{\alpha}+\left\|\frac{f'}{\exp}\right\|_{[x,b],\infty}^{\alpha}\right]^{\frac{1}{\alpha}} \\ \times\left[\left(\frac{\exp x-\exp a}{\exp b-\exp a}\right)^{2\beta}+\left(\frac{\exp b-\exp x}{\exp b-\exp a}\right)^{2\beta}\right]^{\frac{1}{\beta}}(\exp b-\exp a), \\ \text{where } \ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \frac{1}{2}\left[\left\|\frac{f'}{\exp}\right\|_{[a,x],\infty}+\left\|\frac{f'}{\exp}\right\|_{[x,b],\infty}\right] \\ \times\left[\frac{1}{2}+\left|\frac{\exp x-\frac{\exp a+\exp b}{2}}{\exp b-\exp a}\right|\right]^2(\exp b-\exp a) \end{cases}$$

for any $x\in[a,b]$.

In particular, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\exp b-\exp a}\int_a^b f(t)\exp t dt\right| \tag{2.17}$$

$$\leq \frac{1}{8}\left[\left\|\frac{f'}{\exp}\right\|_{[a,LME(a,b)],\infty}+\left\|\frac{f'}{\exp}\right\|_{[LME(a,b),b],\infty}\right](\exp b-\exp a)$$

$$\leq \begin{cases} \frac{1}{4}\left\|\frac{f'}{\exp}\right\|_{[a,b],\infty}(\exp b-\exp a); \\ \\ \frac{1}{2^{\frac{3\beta-1}{\beta}}}\left[\left\|\frac{f'}{\exp}\right\|_{[a,LME(a,b)],\infty}^{\alpha}+\left\|\frac{f'}{\exp}\right\|_{[LME(a,b),b],\infty}^{\alpha}\right]^{\frac{1}{\alpha}}(\exp b-\exp a), \\ \text{where } \ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1. \end{cases}$$

c). If we take $g:[a,b]\subset(0,\infty)\to\mathbb{R}$, $g(t)=t^r$, $r>0$ in (2.5) and assume that $\ell^{1-r}f'\in L_\infty[a,b]$ then we get

$$\left|\frac{(x^r-a^r)f(a)+(b^r-x^r)f(b)}{b^r-a^r}-\frac{r}{b^r-a^r}\int_a^b f(t)t^{r-1}dt\right| \tag{2.18}$$

$$\leq \frac{1}{2r}\left[\left\|\ell^{1-r}f'\right\|_{[a,x],\infty}\left(\frac{x^r-a^r}{b^r-a^r}\right)^2+\left\|\ell^{1-r}f'\right\|_{[x,b],\infty}\left(\frac{b^r-x^r}{b^r-a^r}\right)^2\right](b^r-a^r)$$

$$\leq \begin{cases} \frac{1}{r}\left\|\ell^{1-r}f'\right\|_{[a,b],\infty}\left[\frac{1}{4}+\left(\frac{x^r-\frac{a^r+b^r}{2}}{b^r-a^r}\right)^2\right](b^r-a^r); \\ \\ \frac{1}{2r}\left[\left\|\ell^{1-r}f'\right\|_{[a,x],\infty}^{\alpha}+\left\|\ell^{1-r}f'\right\|_{[x,b],\infty}^{\alpha}\right]^{\frac{1}{\alpha}} \\ \times\left[\left(\frac{x^r-a^r}{b^r-a^r}\right)^{2\beta}+\left(\frac{b^r-x^r}{b^r-a^r}\right)^{2\beta}\right]^{\frac{1}{\beta}}(b^r-a^r), \\ \text{where } \ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \frac{1}{2r}\left[\left\|\ell^{1-r}f'\right\|_{[a,x],\infty}+\left\|\ell^{1-r}f'\right\|_{[x,b],\infty}\right] \\ \times\left[\frac{1}{2}+\left|\frac{x^r-\frac{a^r+b^r}{2}}{b^r-a^r}\right|\right]^2(b^r-a^r) \end{cases}$$

for all $x\in[a,b]$.

In particular, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{r}{b^r-a^r}\int_a^b f(t)\,t^{r-1}dt\right| \tag{2.19}$$

$$\leq \frac{1}{8r}\left[\left\|\ell^{1-r}f'\right\|_{[a,M_r(a,b)],\infty}+\left\|\ell^{1-r}f'\right\|_{[M_r(a,b),b],\infty}\right](b^r-a^r)$$

$$\leq \begin{cases} \frac{1}{4r}\left\|\ell^{1-r}f'\right\|_{[a,b],\infty}(b^r-a^r); \\ \\ \frac{1}{2^{\frac{3\beta-1}{\beta}}r}\left[\left\|\ell^{1-r}f'\right\|_{[a,M_r(a,b)],\infty}^{\alpha}+\left\|\ell^{1-r}f'\right\|_{[M_r(a,b),b],\infty}^{\alpha}\right]^{\frac{1}{\alpha}}(b^r-a^r), \\ \text{where } \ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \end{cases}$$

where $M_r(a,b)=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, $r>0$.

d). If we take $g:[a,b]\subset(0,\infty)\to\mathbb{R}$, $g(t)=-t^{-r}$, $r>0$ in (2.5) and assume that $\ell^{r+1}f'\in L_\infty[a,b]$, then we get

$$\left|\frac{\left(\frac{x^ra^r}{x^r-a^r}\right)f(a)+\left(\frac{b^rx^r}{b^r-x^r}\right)f(b)}{\frac{b^ra^r}{b^r-a^r}}-\frac{rb^ra^r}{b^r-a^r}\int_a^b f(t)\,t^{-r-1}dt\right| \tag{2.20}$$

$$\leq \frac{b^ra^r}{2r(b^r-a^r)}$$

$$\times\left[\left\|\ell^{r+1}f'\right\|_{[a,x],\infty}\left(\frac{x^ra^r}{x^r-a^r}\right)^2+\left\|\ell^{r+1}f'\right\|_{[x,b],\infty}\left(\frac{b^rx^r}{b^r-x^r}\right)^2\right]$$

$$\leq \begin{cases} \frac{1}{r}\left\|\ell^{r+1}f'\right\|_{[a,b],\infty}\left[\frac{1}{4}+\left(\left(x^{-r}-\frac{a^{-r}+b^{-r}}{2}\right)\frac{b^{r}a^{r}}{(b^{r}-a^{r})}\right)^{2}\right]\left(\frac{b^{r}a^{r}}{b^{r}-a^{r}}\right); \\ \\ \frac{1}{2r}\left[\left\|\ell^{r+1}f'\right\|_{[a,x],\infty}^{\alpha}+\left\|\ell^{r+1}f'\right\|_{[x,b],\infty}^{\alpha}\right]^{\frac{1}{\alpha}} \\ \times\left[\left(\frac{x^{r}-a^{r}}{x^{r}}\frac{b^{r}}{b^{r}-a^{r}}\right)^{2\beta}+\left(\frac{b^{r}-x^{r}}{x^{r}}\frac{a^{r}}{b^{r}-a^{r}}\right)^{2\beta}\right]^{\frac{1}{\beta}}\left(\frac{b^{r}a^{r}}{b^{r}-a^{r}}\right), \\ \text{where } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \frac{1}{2r}\left[\left\|\ell^{r+1}f'\right\|_{[a,x],\infty}+\left\|\ell^{r+1}f'\right\|_{[x,b],\infty}\right] \\ \times\left[\frac{1}{2}+\left|\left(x^{-r}-\frac{a^{-r}+b^{-r}}{2}\right)\frac{b^{r}a^{r}}{(b^{r}-a^{r})}\right|\right]^{2}\left(\frac{b^{r}a^{r}}{b^{r}-a^{r}}\right) \end{cases}$$

for all $x\in[a,b]$.

In particular,

$$\left|\frac{f(a)+f(b)}{2}-\frac{rb^{r}a^{r}}{b^{r}-a^{r}}\int_{a}^{b}f(t)\,t^{-r-1}dt\right| \tag{2.21}$$
$$\leq\frac{1}{8r}\left[\left\|\ell^{r+1}f'\right\|_{[a,M_{-r}(a,b)],\infty}+\left\|\ell^{r+1}f'\right\|_{[M_{-r}(a,b),b],\infty}\right]\left(\frac{b^{r}a^{r}}{b^{r}-a^{r}}\right)$$
$$\leq\begin{cases}\frac{1}{4r}\left\|\ell^{r+1}f'\right\|_{[a,b],\infty}\left(\frac{b^{r}a^{r}}{b^{r}-a^{r}}\right); \\ \\ \frac{1}{2^{\frac{3\beta-1}{\beta}}r}\left[\left\|\ell^{r+1}f'\right\|_{[a,M_{-r}(a,b)],\infty}^{\alpha}+\left\|\ell^{r+1}f'\right\|_{[M_{-r}(a,b),b],\infty}^{\alpha}\right]^{\frac{1}{\alpha}}\left(\frac{b^{r}a^{r}}{b^{r}-a^{r}}\right), \\ \text{where } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\end{cases}$$

where

$$M_{-r}(a,b):=\left(\frac{a^{-r}+b^{-r}}{2}\right)^{-1/r}=\left(\frac{2a^{r}b^{r}}{b^{r}+a^{r}}\right)^{1/r}.$$

If we take $r=1$ in (2.21), then we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{ba}{b-a}\int_{a}^{b}\frac{f(t)}{t^{2}}dt\right| \tag{2.22}$$
$$\leq\frac{1}{8}\left[\left\|\ell^{2}f'\right\|_{[a,H(a,b)],\infty}+\left\|\ell^{2}f'\right\|_{[H(a,b),b],\infty}\right]\left(\frac{ba}{b-a}\right)$$
$$\leq\begin{cases}\frac{1}{4}\left\|\ell^{2}f'\right\|_{[a,b],\infty}\left(\frac{ba}{b-a}\right); \\ \\ \frac{1}{2^{\frac{3\beta-1}{\beta}}}\left[\left\|\ell^{2}f'\right\|_{[a,H(a,b)],\infty}^{\alpha}+\left\|\ell^{2}f'\right\|_{[H(a,b),b],\infty}^{\alpha}\right]^{\frac{1}{\alpha}}\left(\frac{ba}{b-a}\right), \\ \text{where } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\end{cases}$$

provided $\ell^{2}f'\in L_{\infty}[a,b]$, where

$$H(a,b):=\frac{2ab}{b+a}$$

is the *Harmonic mean* of a, $b>0$.

3. Weighted Integral Inequalities and Probability Distributions

If $w:[a,b]\to\mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\to[0,\infty)$, $W(x):=\int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x)=w(x)$ for any $x\in(a,b)$.

Proposition 1. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ and $f:[a,b]\to\mathbb{C}$ is absolutely continuous on $[a,b]$ with $\frac{f'}{w}\in L_\infty[a,b]$, then we have*

$$\left|\frac{f(a)\int_a^x w(s)\,ds+f(b)\int_x^b w(s)\,ds}{\int_a^b w(s)\,ds}-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(t)\,w(t)\,dt\right| \tag{3.1}$$
$$\le\frac{1}{2}\left[\left\|\frac{f'}{w}\right\|_{[a,x],\infty}\left(\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\right)^2+\left\|\frac{f'}{w}\right\|_{[x,b],\infty}\left(\frac{\int_x^b w(s)\,ds}{\int_a^b w(s)\,ds}\right)^2\right]\int_a^b w(s)\,ds$$
$$\le\begin{cases}\frac{1}{4}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\left[1+\left(\frac{\int_x^b w(s)ds-\int_a^x w(s)ds}{\int_a^b w(s)ds}\right)^2\right]\int_a^b w(s)\,ds;\\[2ex] \frac{1}{2}\left[\left\|\frac{f'}{w}\right\|_{[a,x],\infty}^\alpha+\left\|\frac{f'}{w}\right\|_{[x,b],\infty}^\alpha\right]^{\frac{1}{\alpha}}\\ \times\left[\left(\frac{\int_a^x w(s)ds}{\int_a^b w(s)ds}\right)^{2\beta}+\left(\frac{\int_x^b w(s)ds}{\int_a^b w(s)ds}\right)^{2\beta}\right]^{\frac{1}{\beta}}\int_a^b w(s)\,ds,\\ \text{where }\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\[2ex] \frac{1}{8}\left[\left\|\frac{f'}{w}\right\|_{[a,x],\infty}+\left\|\frac{f'}{w}\right\|_{[x,b],\infty}\right]\left[1+\left|\frac{\int_x^b w(s)ds-\int_a^x w(s)ds}{\int_a^b w(s)ds}\right|\right]^2\int_a^b w(s)\,ds\end{cases}$$

for all $x\in[a,b]$.

In particular, if

$$M_W(a,b):=W^{-1}\left(\frac{1}{2}\int_a^b w(s)\,ds\right),$$

then we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(t)\,w(t)\,dt\right| \tag{3.2}$$
$$\le\frac{1}{8}\left[\left\|\frac{f'}{w}\right\|_{[a,M_W(a,b)],\infty}+\left\|\frac{f'}{w}\right\|_{[M_W(a,b),b],\infty}\right]\int_a^b w(s)\,ds$$
$$\le\begin{cases}\frac{1}{4}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds;\\[2ex] \frac{1}{2^{\frac{3\beta-1}{\beta}}}\left[\left\|\frac{f'}{w}\right\|_{[a,M_W(a,b)],\infty}^\alpha+\left\|\frac{f'}{w}\right\|_{[M_W(a,b),b],\infty}^\alpha\right]^{\frac{1}{\alpha}}\int_a^b w(s)\,ds,\\ \text{where }\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1.\end{cases}$$

The above result can be extended for infinite intervals I by assuming that the function $f:I\to\mathbb{C}$ is locally absolutely continuous on I.

For instance, if $I=[a,\infty)$, $f:[a,\infty)\to\mathbb{C}$ is locally absolutely continuous on $[a,\infty)$ and $w(s)>0$ for $s\in[a,\infty)$ with $\int_a^\infty w(s)\,ds=1$, namely w is a probability density function on $[a,\infty)$, and if $\frac{f'}{w}\in L_{\infty}[a,\infty)$, then by (3.1) we get

$$\begin{aligned}(3.3)\quad & \left|f(a)W(x)+f(b)(1-W(x))-\int_a^{\infty}f(t)w(t)\,dt\right| \\ & \leq \frac{1}{2}\left[\left\|\frac{f'}{w}\right\|_{[a,x],\infty}W^2(x)+\left\|\frac{f'}{w}\right\|_{[x,\infty),\infty}(1-W(x))^2\right] \\ & \leq \begin{cases}\left\|\frac{f'}{w}\right\|_{[a,\infty),\infty}\left[\frac{1}{4}+\left(W(x)-\frac{1}{2}\right)^2\right]; \\ \frac{1}{2}\left[\left\|\frac{f'}{w}\right\|_{[a,x],\infty}^{\alpha}+\left\|\frac{f'}{w}\right\|_{[x,\infty),\infty}^{\alpha}\right]^{\frac{1}{\alpha}}\left[(W(x))^{2\beta}+(1-W(x))^{2\beta}\right]^{\frac{1}{\beta}}, \\ \text{where } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \frac{1}{2}\left[\left\|\frac{f'}{w}\right\|_{[a,x],\infty}+\left\|\frac{f'}{w}\right\|_{[x,\infty),\infty}\right]\left[\frac{1}{2}+\left|W(x)-\frac{1}{2}\right|\right]^2\end{cases}\end{aligned}$$

for any $x\in[a,\infty)$, where $W(x):=\int_a^x w(s)\,ds$ is the cumulative distribution function.

If $m\in(a,\infty)$ is the *median point* for w, namely $W(m)=\frac{1}{2}$, then by (3.3) we get

$$\begin{aligned}(3.4)\quad & \left|\frac{f(a)+f(b)}{2}-\int_a^{\infty}f(t)w(t)\,dt\right| \\ & \leq \frac{1}{8}\left[\left\|\frac{f'}{w}\right\|_{[a,m],\infty}+\left\|\frac{f'}{w}\right\|_{[m,b],\infty}\right] \\ & \leq \begin{cases}\frac{1}{4}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}; \\ \frac{1}{2^{\frac{3\beta-1}{\beta}}}\left[\left\|\frac{f'}{w}\right\|_{[a,m],\infty}^{\alpha}+\left\|\frac{f'}{w}\right\|_{[m,b],\infty}^{\alpha}\right]^{\frac{1}{\alpha}} \\ \text{where } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1.\end{cases}\end{aligned}$$

In probability theory and statistics, the *beta prime distribution* (also known as *inverted beta distribution* or *beta distribution of the second kind*) is an absolutely continuous probability distribution defined for $x>0$ with two parameters α and β, having the probability density function:

$$w_{\alpha,\beta}(x):=\frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)}$$

where B is *Beta function*

$$B(\alpha,\beta):=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1},\ \alpha,\ \beta>0.$$

The cumulative distribution function is

$$W_{\alpha,\beta}(x)=I_{\frac{x}{1+x}}(\alpha,\beta),$$

where I is the *regularized incomplete beta function* defined by

$$I_z(\alpha,\beta):=\frac{B(z;\alpha,\beta)}{B(\alpha,\beta)}.$$

Here $B(\cdot;\alpha,\beta)$ is the *incomplete beta function* defined by

$$B(z;\alpha,\beta):=\int_0^z t^{\alpha-1}(1-t)^{\beta-1}\,,\ \alpha,\ \beta,\ z>0.$$

Assume that $f:[0,\infty)\to\mathbb{C}$ is locally absolutely continuous on $[0,\infty)$ with $\frac{f'}{\ell^{\alpha-1}(1+\ell)^{-\alpha-\beta}}\in L_\infty[0,\infty)$, were $\ell(t)=t$. Using the inequality (3.3) we have for $x>0$ that

$$\begin{aligned}(3.5)\quad &\Big|f(a)\,I_{\frac{x}{1+x}}(\alpha,\beta)+f(b)\Big(1-I_{\frac{x}{1+x}}(\alpha,\beta)\Big)\\ &\qquad-\frac{1}{B(\alpha,\beta)}\int_0^\infty f(t)\,t^{\alpha-1}(1+t)^{-\alpha-\beta}\,dt\Big|\\ &\le\frac{1}{2}B(\alpha,\beta)\left[\left\|\frac{f'}{\ell^{\alpha-1}(1+\ell)^{-\alpha-\beta}}\right\|_{[a,x],\infty}\Big[I_{\frac{x}{1+x}}(\alpha,\beta)\Big]^2\right.\\ &\qquad\left.+\left\|\frac{f'}{\ell^{\alpha-1}(1+\ell)^{-\alpha-\beta}}\right\|_{[x,\infty),\infty}\Big(1-I_{\frac{x}{1+x}}(\alpha,\beta)\Big)^2\right]\end{aligned}$$

$$\le\begin{cases}B(\alpha,\beta)\left\|\frac{f'}{\ell^{\alpha-1}(1+\ell)^{-\alpha-\beta}}\right\|_{[a,\infty),\infty}\left[\frac{1}{4}+\Big(I_{\frac{x}{1+x}}(\alpha,\beta)-\frac{1}{2}\Big)^2\right];\\[2ex] \frac{1}{2}B(\alpha,\beta)\left[\left\|\frac{f'}{\ell^{\alpha-1}(1+\ell)^{-\alpha-\beta}}\right\|_{[a,x],\infty}^{\alpha}+\left\|\frac{f'}{\ell^{\alpha-1}(1+\ell)^{-\alpha-\beta}}\right\|_{[x,\infty),\infty}^{\alpha}\right]^{\frac{1}{\alpha}}\\ \times\left[\Big(I_{\frac{x}{1+x}}(\alpha,\beta)\Big)^{2\beta}+\Big(1-I_{\frac{x}{1+x}}(\alpha,\beta)\Big)^{2\beta}\right]^{\frac{1}{\beta}},\\ \text{where }\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\[2ex] \frac{1}{2}B(\alpha,\beta)\left[\left\|\frac{f'}{\ell^{\alpha-1}(1+\ell)^{-\alpha-\beta}}\right\|_{[a,x],\infty}+\left\|\frac{f'}{\ell^{\alpha-1}(1+\ell)^{-\alpha-\beta}}\right\|_{[x,\infty),\infty}\right]\\ \times\left[\frac{1}{2}+\left|I_{\frac{x}{1+x}}(\alpha,\beta)-\frac{1}{2}\right|\right]^2\end{cases}$$

for $\alpha,\ \beta>0$.

Similar results may be stated for the probability distributions that are supported on the whole axis $\mathbb{R}=(-\infty,\infty)$. Namely, if $I=(-\infty,\infty)$, $f:\mathbb{R}\to\mathbb{C}$ is locally absolutely continuous on $\mathbb{R}$ and $w(s)>0$ for $s\in\mathbb{R}$ with $\int_{-\infty}^{\infty}w(s)\,ds=1$, namely w is a probability density function on $(-\infty,\infty)$, and if $\frac{f'}{w}\in L_\infty(-\infty,\infty)$ then by (3.1) we get

$$\begin{aligned}(3.6)\quad &\left|f(a)\,W(x)+f(b)(1-W(x))-\int_{-\infty}^{\infty}f(t)\,w(t)\,dt\right|\\ &\le\frac{1}{2}\left[\left\|\frac{f'}{w}\right\|_{(-\infty,x],\infty}W^2(x)+\left\|\frac{f'}{w}\right\|_{[x,\infty),\infty}(1-W(x))^2\right]\end{aligned}$$

$$\leq \begin{cases} \left\|\frac{f'}{w}\right\|_{(-\infty,\infty),\infty}\left[\frac{1}{4}+\left(W(x)-\frac{1}{2}\right)^2\right]; \\ \\ \frac{1}{2}\left[\left\|\frac{f'}{w}\right\|_{(-\infty,x],\infty}^{\alpha}+\left\|\frac{f'}{w}\right\|_{[x,\infty),\infty}^{\alpha}\right]^{\frac{1}{\alpha}}\left[(W(x))^{2\beta}+(1-W(x))^{2\beta}\right]^{\frac{1}{\beta}}, \\ \text{where } \ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \frac{1}{2}\left[\left\|\frac{f'}{w}\right\|_{(-\infty,x],\infty}+\left\|\frac{f'}{w}\right\|_{[x,\infty),\infty}\right]\left[\frac{1}{2}+\left|W(x)-\frac{1}{2}\right|\right]^2 \end{cases}$$

for all $x\in(-\infty,\infty)$.

In particular, if $m\in\mathbb{R}$ is the *median point* for w, namely $W(m)=\frac{1}{2}$, then by (3.6) we get

$$\left|\frac{f(a)+f(b)}{2}-\int_{-\infty}^{\infty}f(t)\,w(t)\,dt\right|\leq\frac{1}{8}\left[\left\|\frac{f'}{w}\right\|_{(-\infty,m],\infty}+\left\|\frac{f'}{w}\right\|_{[m,\infty),\infty}\right] \tag{3.7}$$

$$\leq\begin{cases}\frac{1}{4}\left\|\frac{f'}{w}\right\|_{(-\infty,\infty),\infty} \\ \\ \frac{1}{2^{\frac{3\beta-1}{\beta}}}\left[\left\|\frac{f'}{w}\right\|_{(-\infty,m],\infty}^{\alpha}+\left\|\frac{f'}{w}\right\|_{[m,\infty),\infty}^{\alpha}\right]^{\frac{1}{\alpha}}, \\ \text{where } \ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1.\end{cases}$$

for any $x\in(-\infty,\infty)$, where $W(x):=\int_{-\infty}^{x}w(s)\,ds$ is the cumulative distribution function.

In what follows we give an example.

The probability density of the *normal distribution* on $(-\infty,\infty)$ is

$$w_{\mu,\sigma^2}(x):=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\ x\in\mathbb{R},$$

where μ is the *mean* or *expectation* of the distribution (and also its *median* and *mode*), σ is the *standard deviation*, and σ^2 is the *variance*.

The cumulative distribution function is

$$W_{\mu,\sigma^2}(x)=\frac{1}{2}+\frac{1}{2}\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right),$$

where the *error function* erf is defined by $\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x\exp\left(-t^2\right)dt$.

If $f:\mathbb{R}\to\mathbb{R}$ is locally absolutely continuous with $\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)f'\in L_\infty(-\infty,\infty)$, where $\ell(t)=t$, then from (3.6) we get

$$\begin{aligned}(3.8)\quad &\left|\frac{1}{2}\left[f(a)\left(1+\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)+f(b)\left(1-\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)\right]\right.\\ &\left.-\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}f(t)\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)dt\right|\\ &\le\frac{\sqrt{2\pi}\sigma}{8}\left[\left\|\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)f'\right\|_{(-\infty,x],\infty}\left(1+\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)^2\right.\\ &\left.+\left\|\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)f'\right\|_{[x,\infty),\infty}\left(1-\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)^2\right]\\ &\le\begin{cases}\frac{\sqrt{2\pi}\sigma}{4}\left\|\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)f'\right\|_{(-\infty,\infty),\infty}\left[1+\left[\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]^2\right];\\[2ex] \frac{\sqrt{2\pi}\sigma}{8}\left[\left\|\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)f'\right\|^{\alpha}_{(-\infty,x],\infty}+\left\|\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)f'\right\|^{\alpha}_{[x,\infty),\infty}\right]^{\frac{1}{\alpha}}\\ \times\left[\left(1+\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)^{2\beta}+\left(1-\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)^{2\beta}\right]^{\frac{1}{\beta}},\\ \text{where }\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\[2ex] \frac{\sqrt{2\pi}\sigma}{8}\left[\left\|\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)f'\right\|_{(-\infty,x],\infty}+\left\|\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)f'\right\|_{[x,\infty),\infty}\right]\\ \times\left[1+\left|\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right|\right]^2\end{cases}\end{aligned}$$

for any $x\in(-\infty,\infty)$.

In particular, we have

$$\begin{aligned}(3.9)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}f(t)\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)dt\right|\\ &\le\frac{\sqrt{2\pi}\sigma}{8}\left[\left\|\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)f'\right\|_{(-\infty,\mu],\infty}+\left\|\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)f'\right\|_{[\mu,\infty),\infty}\right]\\ &\le\begin{cases}\frac{\sqrt{2\pi}\sigma}{4}\left\|\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)f'\right\|_{(-\infty,\infty),\infty};\\[2ex] \frac{\sqrt{2\pi}\sigma}{2^{\frac{3\beta-1}{\beta}}}\left[\left\|\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)f'\right\|^{\alpha}_{(-\infty,\mu],\infty}+\left\|\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)f'\right\|^{\alpha}_{[\mu,\infty),\infty}\right]^{\frac{1}{\alpha}},\\ \text{where }\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1.\end{cases}\end{aligned}$$

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