Received 08/05/18

REVERSES OF JENSEN'S INTEGRAL INEQUALITY VIA A WEIGHTED OSTROWSKI TYPE RESULT WITH APPLICATIONS FOR COMPOSITE CONVEX FUNCTIONS

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ABSTRACT. In this paper we obtain some reverses of Jensen's integral inequality by employing a new weighted integral inequality of Ostrowski type. Applications for general composite convex functions with examples for AG, GA-convex functions and HA, AH-convex function are also given.

1. INTRODUCTION

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ-a.e. (almost every) $x \in \Omega$, consider the *Lebesgue space*

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega w(x)|f(x)|\,d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S. S. Dragomir obtained in 2002 [4] the following result:

Theorem 1. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. (almost everywhere) on Ω with $\int_\Omega w d\mu = 1$. Then we have the inequality:*

$$0 \leq \int_\Omega w(\Phi \circ f)\,d\mu - \Phi\left(\int_\Omega w f d\mu\right) \leq \int_\Omega w(\Phi' \circ f) f d\mu - \int_\Omega w(\Phi' \circ f)\,d\mu \int_\Omega w f d\mu. \tag{1.1}$$

Let $\Phi : [m, M] \to \mathbb{R}$ be a differentiable convex function on (m, M). If $x_i \in [m, M]$ and $w_i \geq 0$ $(i = 1, \ldots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then one has the reverse of

1991 *Mathematics Subject Classification.* 26D15; 26D10.

Key words and phrases. Convex functions, Jensen's inequality, Ostrowski's inequality, AG, AH-convex functions, GA, GG, GH-convex functions and HA, HG, HH-convex function, Integral inequalities.

RGMIA Res. Rep. Coll. 21 (2018), Art. 45, 20 pp.

Jensen's weighted discrete inequality:

$$\begin{aligned}(1.2)\qquad 0 &\leq \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i\right) \\ &\leq \sum_{i=1}^{n} w_i \Phi'(x_i) x_i - \sum_{i=1}^{n} w_i \Phi'(x_i) \sum_{i=1}^{n} w_i x_i.\end{aligned}$$

The inequality (1.2) was obtained in 1994 by Dragomir & Ionescu, see [17].

If $h,\ g:\Omega\to\mathbb{R}$ are μ-measurable functions and $h,\ g,\ hg\in L_w(\Omega,\mu)$, then we may consider the *Čebyšev functional*

$$T_w(h,g) := \int_\Omega whg\,d\mu - \int_\Omega wh\,d\mu \int_\Omega wg\,d\mu. \tag{1.3}$$

The following result is known in the literature as the *Grüss inequality*

$$|T_w(h,g)| \leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta), \tag{1.4}$$

provided

$$-\infty<\gamma\leq h(x)\leq\Gamma<\infty,\quad -\infty<\delta\leq g(x)\leq\Delta<\infty \tag{1.5}$$

for μ-a.e. $x\in\Omega$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

With the above assumptions, if $h\in L_{w,2}(\Omega,\mu)$ then we may define

$$D_w(h) := D_{w,1}(h) := \int_\Omega w\left|h-\int_\Omega wh\,d\mu\right| d\mu \tag{1.6}$$

and

$$D_{w,2}(h) := \left[\int_\Omega wh^2 d\mu - \left(\int_\Omega wh\,d\mu\right)^2\right]^{\frac{1}{2}}.$$

In 2002, Cerone & Dragomir [3] obtained the following refinement of the Grüss inequality (1.4):

Theorem 2. *Let $w,\ h,\ g:\Omega\to\mathbb{R}$ be μ-measurable functions with $w\geq 0$ μ-a.e. (almost everywhere) on Ω and $\int_\Omega w\,d\mu=1$. If $h,\ g,\ hg\in L_w(\Omega,\mu)$ and there exists the constants $\delta,\ \Delta$ such that the condition (1.5) holds,*

$$|T_w(h,g)| \leq \frac{1}{2}(\Delta-\delta) D_w(h) \leq \frac{1}{2}(\Delta-\delta) D_{w,2}(h). \tag{1.7}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

Moreover, if h satisfies the condition (1.5), then

$$|T_w(h,g)| \leq \frac{1}{2}(\Delta-\delta) D_w(h) \leq \frac{1}{2}(\Delta-\delta) D_{w,2}(h) \leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta). \tag{1.8}$$

On making use of Theorems 1 and 2 we can state the following result providing a sequence of bounds for the Jensen's gap, see also [4]:

Theorem 3. *Let $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable convex function on (m,M) and $f:\Omega\to[m,M]$ so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f)f\in L_w(\Omega,\mu)$, where $w\geq 0$*

μ-a.e. (almost everywhere) on Ω with $\int_\Omega w d\mu = 1$. Then we have the sequence of inequalities:

$$
\begin{aligned}
(1.9)\qquad 0 &\leq \int_\Omega w\,(\Phi\circ f)\,d\mu - \Phi\left(\int_\Omega wfd\mu\right)\\
&\leq \int_\Omega w\,(\Phi'\circ f)\,fd\mu - \int_\Omega w\,(\Phi'\circ f)\,d\mu \int_\Omega wfd\mu\\
&\leq \frac{1}{2}\begin{cases} \left[\Phi'_-(M)-\Phi'_+(m)\right]\int_\Omega w\left|f-\int_\Omega wfd\mu\right|d\mu \\ \\ (M-m)\int_\Omega w\left|\Phi'\circ f-\int_\Omega w\,(\Phi'\circ f)\,d\mu\right|d\mu\end{cases}\\
&\leq \frac{1}{2}\begin{cases} \left[\Phi'_-(M)-\Phi'_+(m)\right]\left[\int_\Omega wf^2d\mu-\left(\int_\Omega wfd\mu\right)^2\right]^{\frac{1}{2}} \\ \\ (M-m)\left[\int_\Omega w\,(\Phi'\circ f)^2\,d\mu-\left(\int_\Omega w\,(\Phi'\circ f)\,d\mu\right)^2\right]^{\frac{1}{2}}\end{cases}\\
&\leq \frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right].
\end{aligned}
$$

For other similar reverses of Jensen's integral inequality in the general setting of Lebesgue integral on measurable spaces, see [6]-[8].

If $\Omega = I$ is a finite or infinite interval of real numbers, $w \geq 0$ a.e. on I with $\int_I w(t)\,dt = 1$, $\Phi : [m,M]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable convex function on (m,M) and $f : I\to[m,M]$ so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f)\,f\in L_w(I)$, then we have the inequalities

$$
\begin{aligned}
(1.10)\qquad 0 &\leq \int_I w(t)\,(\Phi\circ f)(t)\,dt - \Phi\left(\int_I w(t)\,f(t)\,dt\right)\\
&\leq \int_I w(t)\,(\Phi'\circ f)(t)\,f(t)\,dt - \int_I w(t)\,(\Phi'\circ f)(t)\,dt\int_I w(t)\,f(t)\,dt\\
&\leq \frac{1}{2}\begin{cases} \left[\Phi'_-(M)-\Phi'_+(m)\right]\int_I w(t)\left|f(t)-\int_I w(s)\,f(s)\,ds\right|dt \\ \\ (M-m)\int_I w(t)\left|(\Phi'\circ f)(t)-\int_I w(s)\,(\Phi'\circ f)(s)\,ds\right|dt\end{cases}\\
&\leq \frac{1}{2}\begin{cases} \left[\Phi'_-(M)-\Phi'_+(m)\right]\left[\int_I w(t)\,f^2(t)\,dt-\left(\int_I w(t)\,f(t)\,dt\right)^2\right]^{\frac{1}{2}} \\ \\ (M-m)\left[\int_I w(t)\,(\Phi'\circ f)^2(t)\,dt-\left(\int_I w(t)\,(\Phi'\circ f)(t)\,dt\right)^2\right]^{\frac{1}{2}}\end{cases}\\
&\leq \frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right].
\end{aligned}
$$

In probability theory and statistics, the *beta prime distribution* (also known as *inverted beta distribution* or *beta distribution of the second kind*) is an absolutely continuous probability distribution defined for $x>0$ with two parameters α and β, having the probability density function:

$$w_{\alpha,\beta}(x) := \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)}$$

where B is *Beta function*

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1},\ \alpha,\ \beta>0.$$

The cumulative distribution function is

$$W_{\alpha,\beta}(x)=I_{\frac{x}{1+x}}(\alpha,\beta),$$

where I is the *regularized incomplete beta function* defined by

$$I_{z}(\alpha,\beta):=\frac{B(z;\alpha,\beta)}{B(\alpha,\beta)}.$$

Here $B(\cdot;\alpha,\beta)$ is the *incomplete beta function* defined by

$$B(z;\alpha,\beta):=\int_0^z t^{\alpha-1}(1-t)^{\beta-1},\ \alpha,\ \beta,\ z>0.$$

If we take $I=(0,\infty)$ and $w=w_{\alpha,\beta}(x)$ and assume that $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable convex function on (m,M) and $f:(0,\infty)\to[m,M]$ so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f)f\in L_{w_{\alpha,\beta}}(0,\infty)$, then (1.10) holds for the infinite interval $I=(0,\infty)$ and for the probability distribution $w=w_{\alpha,\beta}(x)$.

The probability density of the *normal distribution* on $(-\infty,\infty)$ is

$$w_{\mu,\sigma^2}(x):=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\ x\in\mathbb{R},$$

where μ is the *mean* or *expectation* of the distribution (and also its *median* and *mode*), σ is the *standard deviation*, and σ^2 is the *variance*.

The cumulative distribution function is

$$W_{\mu,\sigma^2}(x)=\frac{1}{2}+\frac{1}{2}\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right),$$

where the *error function* erf is defined by

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x\exp\left(-t^2\right)dt.$$

If we take $I=(-\infty,\infty)$ and $w=w_{\mu,\sigma^2}(x)$ and assume that $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable convex function on (m,M) and $f:(-\infty,\infty)\to[m,M]$ so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f)f\in L_{w_{\mu,\sigma^2}}(-\infty,\infty)$, then (1.10) holds for the infinite interval $I=(-\infty,\infty)$ and for the probability distribution $w=w_{\mu,\sigma^2}$.

Motivated by the above results, in this paper we obtain some reverses of Jensen's integral inequality by employing a new weighted integral inequality of Ostrowski type. Applications for general composite convex functions with examples for AG, GA-convex functions and HA, AH-convex function are also given.

2. Reverses of Jensen's Inequality Via Ostrowski's Result

For two *Lebesgue integrable* functions $f,\ g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g):=\frac{1}{b-a}\int_a^b f(t)g(t)dt-\frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt. \tag{2.1}$$

In 1935, Grüss [19] showed that

$$|C(f,g)|\leq\frac{1}{4}(M-m)(N-n), \tag{2.2}$$

provided that there exists the real numbers $m,\ M,\ n,\ N$ such that

$$m\leq f(t)\leq M\quad\text{and}\quad n\leq g(t)\leq N\quad\text{for a.e. } t\in[a,b]. \tag{2.3}$$

The constant $\frac{1}{4}$ is best possible in (2.1) in the sense that it cannot be replaced by a smaller quantity.

The following inequality was obtained by Ostrowski in 1970, [24]:

$$|C(f,g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_{\infty}, \tag{2.4}$$

provided that f is *Lebesgue integrable* and satisfies (2.3) while g is absolutely continuous and $g' \in L_{\infty}[a,b]$. The constant $\frac{1}{8}$ is best possible in (2.4).

Consider now the *weighted Čebyšev functional*

$$\begin{aligned} C_w(f,g) :=& \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f(t) g(t)\,dt \\ &- \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f(t)\,dt \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) g(t)\,dt \end{aligned} \tag{2.5}$$

where $f,\ g,\ w : [a,b] \to \mathbb{R}$ and $w(t) \geq 0$ for a.e. $t \in [a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t)\,dt > 0$.

We can also define, as above,

$$\begin{aligned} C_{h'}(f,g) :=& \frac{1}{h(b)-h(a)}\int_a^b f(t) g(t) h'(t)\,dt \\ &- \frac{1}{h(b)-h(a)}\int_a^b f(t) h'(t)\,dt \frac{1}{h(b)-h(a)}\int_a^b g(t) h'(t)\,dt, \end{aligned} \tag{2.6}$$

where h is absolutely continuous and $f,\ g$ are Lebesgue measurable on $[a,b]$ and such that the above integrals exist.

The following weighted version of Ostrowski's inequality holds:

Lemma 1. *Let $h : [a,b] \to [h(a), h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If f is Lebesgue integrable and satisfies the condition $m \leq f(t) \leq M$ for $t \in [a,b]$ and $g : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ and $\frac{g'}{h'}$ is essentially bounded, namely $\frac{g'}{h'} \in L_{\infty}[a,b]$, then we have*

$$|C_{h'}(f,g)| \leq \frac{1}{8}[h(b)-h(a)](M-m)\left\|\frac{g'}{h'}\right\|_{[a,b],\infty}. \tag{2.7}$$

The constant $\frac{1}{8}$ is best possible.

Proof. Assume that $[c,d] \subset [a,b]$. If $g : [c,d] \to \mathbb{C}$ is absolutely continuous on $[c,d]$, then $g \circ h^{-1} : [h(c), h(d)] \to \mathbb{C}$ is absolutely continuous on $[h(c), h(d)]$ and using the chain rule and the derivative of inverse functions we have

$$\left(g \circ h^{-1}\right)'(z) = \left(g' \circ h^{-1}\right)(z)\left(h^{-1}\right)'(z) = \frac{\left(g' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \tag{2.8}$$

for almost every (a.e.) $z \in [h(c), h(d)]$.

If $x \in [c,d]$, then by taking $z = h(x)$, we get

$$\left(g \circ h^{-1}\right)'(z) = \frac{\left(g' \circ h^{-1}\right)(h(x))}{\left(h' \circ h^{-1}\right)(h(x))} = \frac{g'(x)}{h'(x)}.$$

Therefore, since $\frac{g'}{h'} \in L_\infty[c,d]$, hence $\left(g\circ h^{-1}\right)' \in L_\infty[h(c),h(d)]$. Also

$$\left\|\left(g\circ h^{-1}\right)'\right\|_{[h(c),h(d)],\infty} = \left\|\frac{g'}{h'}\right\|_{[c,d],\infty}.$$

Now, if we use the Ostrowski's inequality (2.4) for the functions $f\circ h^{-1}$ and $g\circ h^{-1}$ on the interval $[h(a),h(b)]$, then we get

$$\begin{aligned}(2.9)\quad &\left|\frac{1}{h(b)-h(a)}\int_{h(a)}^{h(b)} f\circ h^{-1}(u) g\circ h^{-1}(u)du\right.\\ &\left.-\frac{1}{[h(b)-h(a)]^2}\int_{h(a)}^{h(b)} f\circ h^{-1}(u)du\int_{h(a)}^{h(b)} g\circ h^{-1}(u)du\right|\\ &\leq \frac{1}{8}[h(b)-h(a)](M-m)\left\|\left(g\circ h^{-1}\right)'\right\|_{[h(a),h(b)],\infty}\end{aligned}$$

since $m\leq f\circ h^{-1}(u)\leq M$ for all $u\in[h(a),h(b)]$.

Observe also that, by the change of variable $t=h^{-1}(u)$, $u\in[g(a),g(b)]$, we have $u=h(t)$ that gives $du=h'(t)\,dt$ and

$$\int_{h(a)}^{h(b)}\left(f\circ h^{-1}\right)(u)\,du = \int_a^b f(t)\,h'(t)\,dt,$$

$$\int_{h(a)}^{h(b)} g\circ h^{-1}(u)du = \int_a^b g(t)\,h'(t)\,dt,$$

$$\int_{h(a)}^{h(b)} f\circ h^{-1}(u) g\circ h^{-1}(u)du = \int_a^b f(t)\,g(t)\,h'(t)\,dt$$

and

$$\left\|\left(g\circ h^{-1}\right)'\right\|_{[h(a),h(b)],\infty} = \left\|\frac{g'}{h'}\right\|_{[a,b],\infty}.$$

By making use of (2.9) we then get the desired result (2.7).

The best constant follows by Ostrowski's inequality (2.4). □

If $w:[a,b]\to\mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\to[0,\infty)$, $W(x):=\int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x)=w(x)$ for any $x\in(a,b)$.

Corollary 1. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$, f is Lebesgue integrable and satisfies the condition $m\leq f(t)\leq M$ for $t\in[a,b]$ and $g:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ with $\frac{g'}{w}$ is essentially bounded, namely $\frac{g'}{w}\in L_\infty[a,b]$, then we have*

$$(2.10)\qquad |C_w(f,g)| \leq \frac{1}{8}(M-m)\left\|\frac{g'}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds.$$

The constant $\frac{1}{8}$ is best possible.

Remark 1. *Under the assumptions of Corollary 1 and if there exists a constant $K>0$ such that $|g'(t)|\leq Kw(t)$ for a.e. $t\in[a,b]$, then by (2.10) we get*

$$(2.11)\qquad |C_w(f,g)| \leq \frac{1}{8}(M-m)K\int_a^b w(s)\,ds.$$

We have the following reverse of Jensen's inequality:

Theorem 4. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M), $w : [a, b] \to (0, \infty)$ be continuous on $[a, b]$ and $f : [a, b] \to [m, M]$ is absolutely continuous so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_w [a, b]$.*

(i) *If $\frac{f'}{w} \in L_\infty [a, b]$, then we have the inequality*

$$0 \leq \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)(\Phi \circ f)(t)\,dt - \Phi\left(\frac{\int_a^b w(t) f(t)\,dt}{\int_a^b w(s)\,ds}\right) \leq \frac{1}{8}\left[\Phi'_-(M) - \Phi'_+(m)\right] \left\| \frac{f'}{w} \right\|_{[a,b],\infty} \int_a^b w(s)\,ds. \tag{2.12}$$

(ii) *If Φ is twice differentiable on (m, M) and $\frac{(\Phi'' \circ f) f'}{w} \in L_\infty [a, b]$, then*

$$0 \leq \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)(\Phi \circ f)(t)\,dt - \Phi\left(\frac{\int_a^b w(t) f(t)\,dt}{\int_a^b w(s)\,ds}\right) \leq \frac{1}{8}(M - m) \left\| \frac{(\Phi'' \circ f) f'}{w} \right\|_{[a,b],\infty} \int_a^b w(s)\,ds. \tag{2.13}$$

Proof. (i) By (4.14) we have

$$\begin{aligned} 0 &\leq \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)(\Phi \circ f)(t)\,dt - \Phi\left(\frac{\int_a^b w(t) f(t)\,dt}{\int_a^b w(s)\,ds}\right) \\ &\leq \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)(\Phi' \circ f)(t) f(t)\,dt \\ &\quad - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)(\Phi' \circ f)(t)\,dt \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t) f(t)\,dt. \end{aligned} \tag{2.14}$$

Since Φ is differentiable convex on (m, M), hence

$$\Phi'_+(m) \leq (\Phi' \circ f)(t) \leq \Phi'_-(M)$$

for $t \in [a, b]$.

If we use the inequality (2.10), then we get

$$\begin{aligned} &\frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)(\Phi \circ f)(t) f(t)\,dt \\ &- \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)(\Phi' \circ f)(t)\,dt \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t) f(t)\,dt \\ &\leq \frac{1}{8}\left[\Phi'_-(M) - \Phi'_+(m)\right] \left\| \frac{f'}{w} \right\|_{[a,b],\infty} \int_a^b w(s)\,ds, \end{aligned}$$

which, together with (2.14), proves the required inequality (2.12).

(ii) If Φ is twice differentiable on (a, b), then

$$(\Phi' \circ f)'(t) = (\Phi'' \circ f)(t) f'(t)$$

for $t \in (a, b)$.

Since $m \leq f(t) \leq M$ for $t \in [a,b]$ and

$$\frac{(\Phi'' \circ f) f'}{w} \in L_\infty [a,b],$$

then by using the inequality (2.10) we also have

$$\begin{aligned}
&\frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t) (\Phi \circ f)(t) f(t)\,dt \\
&- \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t) (\Phi' \circ f)(t)\,dt \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t) f(t)\,dt \\
&\leq \frac{1}{8} (M-m) \left\| \frac{(\Phi'' \circ f) f'}{w} \right\|_{[a,b],\infty} \int_a^b w(s)\,ds,
\end{aligned}$$

which, together with (2.14), proves (2.13). □

Corollary 2. *Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m,M) and $f : [a,b] \to [m,M]$ be absolutely continuous so that $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f) f \in L[a,b]$.*

(i) *If $f' \in L_\infty [a,b]$, then we have the inequality*

$$\begin{aligned}
0 &\leq \frac{1}{b-a} \int_a^b (\Phi \circ f)(t)\,dt - \Phi\left(\frac{1}{b-a} \int_a^b f(t)\,dt \right) \\
&\leq \frac{1}{8} (b-a) \left[\Phi'_-(M) - \Phi'_+(m) \right] \|f'\|_{[a,b],\infty}.
\end{aligned} \tag{2.15}$$

(ii) *If Φ is twice differentiable on (m,M) and $(\Phi'' \circ f) f' \in L_\infty [a,b]$, then*

$$\begin{aligned}
0 &\leq \frac{1}{b-a} \int_a^b (\Phi \circ f)(t)\,dt - \Phi\left(\frac{1}{b-a} \int_a^b f(t)\,dt \right) \\
&\leq \frac{1}{8} (b-a)(M-m) \|(\Phi'' \circ f) f'\|_{[a,b],\infty}.
\end{aligned} \tag{2.16}$$

Corollary 3. *Let $\Phi : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (a,b), $w : [a,b] \to (0,\infty)$ be continuous on $[a,b]$ and Φ, $\Phi' \in L_w [a,b]$.*

(i) *If $\frac{1}{w} \in L_\infty [a,b]$, then we have the inequality*

$$\begin{aligned}
0 &\leq \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t) \Phi(t)\,dt - \Phi\left(\frac{\int_a^b t w(t)\,dt}{\int_a^b w(s)\,ds} \right) \\
&\leq \frac{1}{8} \left[\Phi'_-(b) - \Phi'_+(a) \right] \left\| \frac{1}{w} \right\|_{[a,b],\infty} \int_a^b w(s)\,ds.
\end{aligned} \tag{2.17}$$

(ii) *If $f\,\Phi$ is twice differentiable on (m,M) and $\frac{\Phi''}{w} \in L_\infty [a,b]$, then*

$$\begin{aligned}
0 &\leq \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t) \Phi(t)\,dt - \Phi\left(\frac{\int_a^b t w(t)\,dt}{\int_a^b w(s)\,ds} \right) \\
&\leq \frac{1}{8} (b-a) \left\| \frac{\Phi''}{w} \right\|_{[a,b],\infty} \int_a^b w(s)\,ds.
\end{aligned} \tag{2.18}$$

We observe that, if either in Corollary 2 or 3 we take the weight $w \equiv 1$, then we get the known result

$$0 \leq \frac{1}{b-a} \int_a^b \Phi(t)\, dt - \Phi\left(\frac{a+b}{2}\right) \leq \frac{1}{8}(b-a)\left[\Phi'_-(b) - \Phi'_+(a)\right] \tag{2.19}$$

with $\frac{1}{8}$ as the best possible constant.

Define the function $\ell(t) := t$, $t \in \mathbb{R}$.

a). Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : [a, b] \subset (0, \infty) \to [m, M]$ be absolutely continuous and so that $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f) f \in L_{\ell^{-1}}[a, b]$. If $f'\ell \in L_\infty[a, b]$, then by the statement (i) of Theorem 4 we have the inequality

$$\begin{aligned} 0 &\leq \frac{1}{\ln\left(\frac{b}{a}\right)} \int_a^b \frac{(\Phi \circ f)(t)}{t} dt - \Phi\left(\frac{\int_a^b \frac{f(t)}{t} dt}{\ln\left(\frac{b}{a}\right)}\right) \\ &\leq \frac{1}{8}\left[\Phi'_-(M) - \Phi'_+(m)\right] \ln\left(\frac{b}{a}\right) \|\ell f'\|_{[a,b],\infty}. \end{aligned} \tag{2.20}$$

If Φ is twice differentiable on (m, M) and $(\Phi'' \circ f) f'\ell \in L_\infty[a, b]$, then by the statement (ii) of Theorem 4 we have the inequality

$$\begin{aligned} 0 &\leq \frac{1}{\ln\left(\frac{b}{a}\right)} \int_a^b \frac{(\Phi \circ f)(t)}{t} dt - \Phi\left(\frac{\int_a^b \frac{f(t)}{t} dt}{\ln\left(\frac{b}{a}\right)}\right) \\ &\leq \frac{1}{8}(M - m) \|(\Phi'' \circ f) f'\ell\|_{[a,b],\infty} \ln\left(\frac{b}{a}\right). \end{aligned} \tag{2.21}$$

b). Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : [a, b] \to [m, M]$ be absolutely continuous and so that $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f) f \in L_{\exp}[a, b]$. If $\frac{f'}{\exp} \in L_\infty[a, b]$, then by the statement (i) of Theorem 4 we have the inequality

$$\begin{aligned} 0 &\leq \frac{1}{\exp b - \exp a} \int_a^b (\Phi \circ f)(t) \exp t dt - \Phi\left(\frac{\int_a^b f(t) \exp t dt}{\exp b - \exp a}\right) \\ &\leq \frac{1}{8}\left[\Phi'_-(M) - \Phi'_+(m)\right] \left\|\frac{f'}{\exp}\right\|_{[a,b],\infty} (\exp b - \exp a). \end{aligned} \tag{2.22}$$

If Φ is twice differentiable on (m, M) and $\frac{(\Phi'' \circ f) f'}{\exp} \in L_\infty[a, b]$, then by the statement (ii) of Theorem 4 we have the inequality

$$\begin{aligned} 0 &\leq \frac{1}{\exp b - \exp a} \int_a^b (\Phi \circ f)(t) \exp t dt - \Phi\left(\frac{\int_a^b f(t) \exp t d}{\exp b - \exp a} t\right) \\ &\leq \frac{1}{8}(M - m) \left\|\frac{(\Phi'' \circ f) f'}{\exp}\right\|_{[a,b],\infty} (\exp b - \exp a). \end{aligned} \tag{2.23}$$

c). Consider the function $\ell^p(t) := t^p$, $t > 0$, $p \in \mathbb{R} \setminus \{-1\}$. Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : [a, b] \subset (0, \infty) \to [m, M]$ be absolutely continuous and so that $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f) f \in L_{\ell^p}[a, b]$.

If $f'\ell^{-p} \in L_\infty[a,b]$, then by the statement (i) of Theorem 4 we have the inequality

$$0 \leq \frac{p+1}{b^{p+1}-a^{p+1}} \int_a^b t^p (\Phi \circ f)(t)\,dt - \Phi\left(\frac{(p+1)\int_a^b t^p f(t)\,dt}{b^{p+1}-a^{p+1}}\right) \leq \frac{1}{8(p+1)}\left[\Phi'_-(M) - \Phi'_+(m)\right]\left(b^{p+1}-a^{p+1}\right)\left\|f'\ell^{-p}\right\|_{[a,b],\infty}. \tag{2.24}$$

If Φ is twice differentiable on (m,M) and $(\Phi'' \circ f) f'\ell^{-p} \in L_\infty[a,b]$, then by the statement (ii) of Theorem 4 we have the inequality

$$0 \leq \frac{p+1}{b^{p+1}-a^{p+1}} \int_a^b t^p (\Phi \circ f)(t)\,dt - \Phi\left(\frac{(p+1)\int_a^b t^p f(t)\,dt}{b^{p+1}-a^{p+1}}\right) \leq \frac{1}{8(p+1)}(M-m)\left(b^{p+1}-a^{p+1}\right)\left\|(\Phi'' \circ f) f'\ell^{-p}\right\|_{[a,b],\infty}. \tag{2.25}$$

For $p=-2$, we get from (2.24) that

$$0 \leq \frac{ab}{b-a}\int_a^b \frac{(\Phi \circ f)(t)}{t^2}dt - \Phi\left(\frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt\right) \leq \frac{1}{8}\left[\Phi'_-(M) - \Phi'_+(m)\right]\left(\frac{b-a}{ab}\right)\left\|f'\ell^2\right\|_{[a,b],\infty}, \tag{2.26}$$

provided $f'\ell^2 \in L_\infty[a,b]$, while from (2.25) we obtain

$$0 \leq \frac{ab}{b-a}\int_a^b \frac{(\Phi \circ f)(t)}{t^2}dt - \Phi\left(\frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt\right) \leq \frac{1}{8}(M-m)\left(\frac{b-a}{ab}\right)\left\|(\Phi'' \circ f) f'\ell^2\right\|_{[a,b],\infty}, \tag{2.27}$$

provided $(\Phi'' \circ f) f'\ell^2 \in L_\infty[a,b]$.

3. Inequalities for Composite Convexity

We have the following result for composite convexity:

Theorem 5. *Let $\Psi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on (m,M), $\gamma : [m,M] \to [\gamma(m), \gamma(M)]$ a strictly increasing, continuous and differentiable function on (m,M), $w : [a,b] \to (0,\infty)$ a continuous function on $[a,b]$ and $g : [a,b] \to [m,M]$ an absolutely continuous on $[a,b]$. Assume that $\Psi \circ \gamma^{-1}$ is convex on $[\gamma(m), \gamma(M)]$ and $\Psi \circ g$, $\gamma \circ g \in L_w[a,b]$.*

(i) *If $\frac{(\gamma' \circ g) g'}{w} \in L_\infty[a,b]$, then*

$$0 \leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Psi \circ g)(t)\,dt - \Psi \circ \gamma^{-1}\left(\frac{\int_a^b w(t)(\gamma \circ g)(t)\,dt}{\int_a^b w(s)\,ds}\right) \leq \frac{1}{8}\left[\frac{\Psi'_-(M)}{\gamma'_-(M)} - \frac{\Psi'_+(m)}{\gamma'_+(m)}\right]\left\|\frac{(\gamma' \circ g) g'}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds. \tag{3.1}$$

(ii) *If Ψ and γ are twice differentiable, define for $t \in [a,b]$,*

$$\Delta(\Psi, \gamma, g)(t) := \frac{(\Psi'' \circ g)(t)(\gamma' \circ g)(t) - (\Psi' \circ g)(t)(\gamma'' \circ g)(t)}{\left[(\gamma' \circ g)(t)\right]^2}$$

and assume that $\frac{\Delta(\Psi,\gamma,g)}{w} \in L_\infty[a,b]$, *then*

$$0 \leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Psi\circ g)(t)\,dt - \Psi\circ\gamma^{-1}\left(\frac{\int_a^b w(t)(\gamma\circ g)(t)\,dt}{\int_a^b w(s)\,ds}\right) \leq \frac{1}{8}\left[\gamma(M)-\gamma(m)\right]\left\|\frac{\Delta(\Psi,\gamma,g)}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds. \tag{3.2}$$

Proof. (i) If we write the inequality (2.12) for the convex function $\Phi=\Psi\circ\gamma^{-1}$ on $[\gamma(m),\gamma(M)]$ and for the function $f=\gamma\circ g$ on $[a,b]$, then we have

$$\begin{aligned}0 \leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\left(\Psi\circ\gamma^{-1}\circ\gamma\circ g\right)(t)\,dt &\\ - \Psi\circ\gamma^{-1}\left(\frac{\int_a^b w(t)(\gamma\circ g)(t)\,dt}{\int_a^b w(s)\,ds}\right)&\\ \leq \frac{1}{8}\left[\left(\Psi\circ\gamma^{-1}\right)'_-(\gamma(M)) - \left(\Psi\circ\gamma^{-1}\right)'_+(\gamma(m))\right]\left\|\frac{(\gamma\circ g)'}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds.&\end{aligned} \tag{3.3}$$

Using the chain rule and the derivative of inverse functions we have

$$\left(\Psi\circ\gamma^{-1}\right)'(z) = \left(\Psi'\circ\gamma^{-1}\right)(z)\left(\gamma^{-1}\right)'(z) = \frac{\left(\Psi'\circ\gamma^{-1}\right)(z)}{\left(\gamma'\circ\gamma^{-1}\right)(z)} \tag{3.4}$$

for every $z\in(\gamma(m),\gamma(M))$,

$$\left(\Psi\circ\gamma^{-1}\right)'_-(\gamma(M)) = \frac{\Psi'_-(M)}{\gamma'_-(M)} \tag{3.5}$$

and

$$\left(\Psi\circ\gamma^{-1}\right)'_+(m) = \frac{\Psi'_+(m)}{\gamma'_+(m)}. \tag{3.6}$$

Therefore by (3.3) we obtain the desired result (3.1).

(ii) If we write the inequality (2.13) for the function $\Phi=\Psi\circ\gamma^{-1}$ on $[\gamma(m),\gamma(M)]$ and the function $f=\gamma\circ g$ on $[a,b]$, then we have

$$\begin{aligned}0 \leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\left(\Psi\circ\gamma^{-1}\circ\gamma\circ g\right)(t)\,dt &\\ - \Psi\circ\gamma^{-1}\left(\frac{\int_a^b w(t)(\gamma\circ g)(t)\,dt}{\int_a^b w(s)\,ds}\right)&\\ \leq \frac{1}{8}\left[\gamma(M)-\gamma(m)\right]\left\|\frac{\left(\Psi\circ\gamma^{-1}\right)''((\gamma\circ g))\cdot(\gamma'\circ g)}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds.&\end{aligned} \tag{3.7}$$

We have by (3.4) that

$$\begin{aligned}\left(\Psi\circ\gamma^{-1}\right)''(z) &= \left(\frac{\left(\Psi'\circ\gamma^{-1}\right)(z)}{\left(\gamma'\circ\gamma^{-1}\right)(z)}\right)'\\ &= \frac{\left(\Psi'\circ\gamma^{-1}\right)'(z)\left(\gamma'\circ\gamma^{-1}\right)(z)-\left(\Psi'\circ\gamma^{-1}\right)(z)\left(\gamma'\circ\gamma^{-1}\right)'(z)}{\left[\left(\gamma'\circ\gamma^{-1}\right)(z)\right]^2}\\ &= \frac{\frac{\left(\Psi''\circ\gamma^{-1}\right)(z)}{\left(\gamma'\circ\gamma^{-1}\right)(z)}\left(\gamma'\circ\gamma^{-1}\right)(z)-\left(\Psi'\circ\gamma^{-1}\right)(z)\frac{\left(\gamma''\circ\gamma^{-1}\right)(z)}{\left(\gamma'\circ\gamma^{-1}\right)(z)}}{\left[\left(\gamma'\circ\gamma^{-1}\right)(z)\right]^2}\\ &= \frac{\left(\Psi''\circ\gamma^{-1}\right)(z)\left(\gamma'\circ\gamma^{-1}\right)(z)-\left(\Psi'\circ\gamma^{-1}\right)(z)\left(\gamma''\circ\gamma^{-1}\right)(z)}{\left[\left(\gamma'\circ\gamma^{-1}\right)(z)\right]^3}\end{aligned}$$

for every $z\in(\gamma(m),\gamma(M))$.

Therefore, for $f=\gamma\circ g$ we get

$$\left(\Psi\circ\gamma^{-1}\right)''((\gamma\circ g)(t))=\frac{(\Psi''\circ g)(t)(\gamma'\circ g)(t)-(\Psi'\circ g)(t)(\gamma''\circ g)(t)}{[(\gamma'\circ g)(t)]^3}$$

and

$$\begin{aligned}&\left(\Psi\circ\gamma^{-1}\right)''((\gamma\circ g)(t))(\gamma'\circ g)(t)\\ &=\frac{(\Psi''\circ g)(t)(\gamma'\circ g)(t)-(\Psi'\circ g)(t)(\gamma''\circ g)(t)}{[(\gamma'\circ g)(t)]^2}=\Delta(\Psi,\gamma,g)(t)\end{aligned}$$

for any $t\in(a,b)$.

By employing the inequality (3.7) we then get the desired result (3.2). $\square$

Corollary 4. *Let $\Psi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on (m,M), $\gamma:[m,M]\to[\gamma(m),\gamma(M)]$ a strictly increasing, continuous and differentiable function on (m,M), and $g:[a,b]\to[m,M]$ an absolutely continuous function on $[a,b]$. Assume that $\Psi\circ\gamma^{-1}$ is convex on $[\gamma(m),\gamma(M)]$ and $\Psi\circ g$, $\gamma\circ g\in L[a,b]$.*

(i) *If $(\gamma'\circ g)g'\in L_\infty[a,b]$, then*

$$\begin{aligned}0&\le\frac{1}{b-a}\int_a^b(\Psi\circ g)(t)\,dt-\Psi\circ\gamma^{-1}\left(\frac{1}{b-a}\int_a^b(\gamma\circ g)(t)\,dt\right)\\ &\le\frac{1}{8}\left[\frac{\Psi'_-(M)}{\gamma'_-(M)}-\frac{\Psi'_+(m)}{\gamma'_+(m)}\right](b-a)\left\|(\gamma'\circ g)g'\right\|_{[a,b],\infty}.\end{aligned}\tag{3.8}$$

(ii) *If Ψ and γ are twice differentiable and $\Delta(\Psi,\gamma,g)\in L_\infty[a,b]$, then*

$$\begin{aligned}0&\le\frac{1}{b-a}\int_a^b(\Psi\circ g)(t)\,dt-\Psi\circ\gamma^{-1}\left(\frac{1}{b-a}\int_a^b(\gamma\circ g)(t)\,dt\right)\\ &\le\frac{1}{8}[\gamma(M)-\gamma(m)](b-a)\left\|\Delta(\Psi,\gamma,g)\right\|_{[a,b],\infty}.\end{aligned}\tag{3.9}$$

We also have:

Corollary 5. *Let $\Psi:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on (a,b), $\gamma:[a,b]\to[\gamma(a),\gamma(b)]$ a strictly increasing, continuous and differentiable function on (a,b), and $w:[a,b]\to(0,\infty)$ a continuous function on $[a,b]$. Assume that $\Psi\circ\gamma^{-1}$ is convex on $[\gamma(a),\gamma(b)]$ and Ψ, $\gamma\in L_w[a,b]$.*

(i) *If* $\frac{\gamma'}{w} \in L_\infty [a,b]$, *then*

$$0 \leq \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)\,\Psi(t)\,dt - \Psi \circ \gamma^{-1}\left(\frac{\int_a^b w(t)\,\gamma(t)\,dt}{\int_a^b w(s)\,ds}\right) \leq \frac{1}{8}\left[\frac{\Psi'_-(b)}{\gamma'_-(b)} - \frac{\Psi'_+(a)}{\gamma'_+(a)}\right]\left\|\frac{\gamma'}{w}\right\|_{[a,b],\infty} \int_a^b w(s)\,ds. \tag{3.10}$$

(ii) *If* Ψ *and* γ *are twice differentiable, define for* $t \in (a,b)$,

$$\Delta(\Psi,\gamma)(t) := \frac{\Psi''(t)\,\gamma'(t) - \Psi'(t)\,\gamma''(t)}{[\gamma'(t)]^2}$$

and assume that $\frac{\Delta(\Psi,\gamma)}{w} \in L_\infty [a,b]$, *then*

$$0 \leq \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)\,\Psi(t)\,dt - \Psi \circ \gamma^{-1}\left(\frac{\int_a^b w(t)\,\gamma(t)\,dt}{\int_a^b w(s)\,ds}\right) \leq \frac{1}{8}[\gamma(b) - \gamma(a)]\left\|\frac{\Delta(\Psi,\gamma)}{w}\right\|_{[a,b],\infty} \int_a^b w(s)\,ds. \tag{3.11}$$

Remark 2. *Let* $\Psi : [a,b] \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* (a,b) *and* $\gamma : [a,b] \to [\gamma(a), \gamma(b)]$ *a strictly increasing, continuous and differentiable function on* (a,b). *Assume that* $\Psi \circ \gamma^{-1}$ *is convex on* $[\gamma(a), \gamma(b)]$.

If $\gamma' \in L_\infty [a,b]$, *then*

$$0 \leq \frac{1}{b-a}\int_a^b \Psi(t)\,dt - \Psi \circ \gamma^{-1}\left(\frac{1}{b-a}\int_a^b \gamma(t)\,dt\right) \leq \frac{1}{8}\left[\frac{\Psi'_-(b)}{\gamma'_-(b)} - \frac{\Psi'_+(a)}{\gamma'_+(a)}\right](b-a)\,\|\gamma'\|_{[a,b],\infty}. \tag{3.12}$$

If Ψ *and* γ *are twice differentiable and* $\Delta(\Psi,\gamma) \in L_\infty [a,b]$, *then*

$$0 \leq \frac{1}{b-a}\int_a^b \Psi(t)\,dt - \Psi \circ \gamma^{-1}\left(\frac{1}{b-a}\int_a^b \gamma(t)\,dt\right) \leq \frac{1}{8}[\gamma(b) - \gamma(a)](b-a)\,\|\Delta(\Psi,\gamma)\|_{[a,b],\infty}. \tag{3.13}$$

Also, if we take $w = \gamma'$ *in (3.10), then we get*

$$0 \leq \frac{1}{\gamma(b) - \gamma(a)}\int_a^b \Psi(t)\,\gamma'(t)\,dt - \Psi \circ \gamma^{-1}\left(\frac{\gamma(b) + \gamma(a)}{2}\right) \leq \frac{1}{8}\left[\frac{\Psi'_-(b)}{\gamma'_-(b)} - \frac{\Psi'_+(a)}{\gamma'_+(a)}\right][\gamma(b) - \gamma(a)]\,\|\gamma'\|_{[a,b],\infty}, \tag{3.14}$$

while from (3.11) we get

$$0 \leq \frac{1}{\gamma(b) - \gamma(a)}\int_a^b \Psi(t)\,\gamma'(t)\,dt - \Psi \circ \gamma^{-1}\left(\frac{\gamma(b) + \gamma(a)}{2}\right) \leq \frac{1}{8}[\gamma(b) - \gamma(a)]^2\left\|\frac{\Delta(\Psi,\gamma)}{\gamma'}\right\|_{[a,b],\infty}, \tag{3.15}$$

provided $\frac{\Delta(\Psi,\gamma)}{\gamma'} \in L_\infty [a,b]$.

4. Applications for Some Particular Convexities

Let $\gamma : [a,b] \to [\gamma(a), \gamma(b)]$ be a *continuous strictly increasing function* that is *differentiable* on (a,b).

Definition 1. *A function $\Psi : [a,b] \to \mathbb{R}$ will be called composite-γ^{-1} convex (concave) on $[a,b]$ if the composite function $\Psi \circ \gamma^{-1} : [\gamma(a), \gamma(b)] \to \mathbb{R}$ is convex (concave) in the usual sense on $[\gamma(a), \gamma(b)]$.*

In this way, any concept of convexity (log-convexity, harmonic convexity, trigonometric convexity, hyperbolic convexity, h-convexity, quasi-convexity, s-convexity, s-Godunova-Levin convexity etc...) can be extended to the corresponding *composite-γ^{-1} convexity.* The details however will not be presented here.

If $\Psi : [a,b] \to \mathbb{R}$ is composite-γ^{-1} convex on $[a,b]$ then we have the inequality

$$\Psi \circ \gamma^{-1}((1-\lambda)u + \lambda v) \le (1-\lambda)\Psi \circ \gamma^{-1}(u) + \lambda \Psi \circ \gamma^{-1}(v) \tag{4.1}$$

for any $u, v \in [\gamma(a), \gamma(b)]$ and $\lambda \in [0,1]$.

This is equivalent to the condition

$$\Psi \circ \gamma^{-1}((1-\lambda)\gamma(t) + \lambda\gamma(s)) \le (1-\lambda)\Psi(t) + \lambda\Psi(s) \tag{4.2}$$

for any $t, s \in [a,b]$ and $\lambda \in [0,1]$.

If we take $\gamma(t) = \ln t$, $t \in [a,b] \subset (0,\infty)$, then the condition (4.2) becomes

$$\Psi\left(t^{1-\lambda}s^{\lambda}\right) \le (1-\lambda)\Psi(t) + \lambda\Psi(s) \tag{4.3}$$

for any $t, s \in [a,b]$ and $\lambda \in [0,1]$, which is the concept of *GA-convexity* as considered in [1].

If we take $\gamma(t) = -\frac{1}{t}$, $t \in [a,b] \subset (0,\infty)$, then (4.2) becomes

$$\Psi\left(\frac{ts}{(1-\lambda)s + \lambda t}\right) \le (1-\lambda)\Psi(t) + \lambda\Psi(s) \tag{4.4}$$

for any $t, s \in [a,b]$ and $\lambda \in [0,1]$, which is the concept of *HA-convexity* as considered in [1].

If $p > 0$ and we consider $\gamma(t) = t^p$, $t \in [a,b] \subset (0,\infty)$, then the condition (4.2) becomes

$$\Psi\left[((1-\lambda)t^p + \lambda s^p)^{1/p}\right] \le (1-\lambda)\Psi(t) + \lambda\Psi(s) \tag{4.5}$$

for any $t, s \in [a,b]$ and $\lambda \in [0,1]$, which is the concept of *p-convexity* as considered in [26].

If we take $\gamma(t) = \exp t$, $t \in [a,b]$, then the condition (4.2) becomes

$$\Psi[\ln((1-\lambda)\exp(t) + \exp\gamma(s))] \le (1-\lambda)\Psi(t) + \lambda\Psi(s) \tag{4.6}$$

which is the concept of *LogExp convex function* on $[a,b]$ as considered in [16].

Further, assume that $\Psi : [a,b] \to J$, J an interval of real numbers and $\delta : J \to \mathbb{R}$ a continuous function on J that is *strictly increasing (decreasing)* on J.

Definition 2. *We say that the function $\Psi : [a,b] \to J$ is δ-composite convex (concave) on $[a,b]$, if $\delta \circ \Psi$ is convex (concave) on $[a,b]$.*

In this way, any concept of convexity as mentioned above can be extended to the corresponding *δ-composite convexity.* The details however will not be presented here.

With $\gamma : [a,b] \to [\gamma(a), \gamma(b)]$ a *continuous strictly increasing function* that is *differentiable* on (a,b), $\Psi : [a,b] \to J$, J an interval of real numbers and $\delta : J \to \mathbb{R}$ a continuous function on J that is *strictly increasing (decreasing)* on J, we can also consider the following concept:

Definition 3. *We say that the function $\Psi : [a,b] \to J$ is δ-composite-γ^{-1} convex (concave) on $[a,b]$, if $\delta \circ \Psi \circ \gamma^{-1}$ is convex (concave) on $[\gamma(a), \gamma(b)]$.*

This definition is equivalent to the condition

$$\delta \circ \Psi \circ \gamma^{-1}((1-\lambda)\gamma(t) + \lambda\gamma(s)) \le (1-\lambda)(\delta \circ \Psi)(t) + \lambda(\delta \circ \Psi)(s) \tag{4.7}$$

for any t, $s \in [a,b]$ and $\lambda \in [0,1]$.

If $\delta : J \to \mathbb{R}$ is *strictly increasing (decreasing)* on J, then the condition (4.7) is equivalent to:

$$\Psi \circ \gamma^{-1}((1-\lambda)\gamma(t) + \lambda\gamma(s)) \le (\ge)\, \delta^{-1}[(1-\lambda)(\delta \circ \Psi)(t) + \lambda(\delta \circ \Psi)(s)] \tag{4.8}$$

for any t, $s \in [a,b]$ and $\lambda \in [0,1]$.

If $\delta(t) = \ln t$, $t > 0$ and $\Psi : [a,b] \to (0,\infty)$, then the fact that Ψ is δ-composite convex on $[a,b]$ is equivalent to the fact that Ψ is *log-convex* or *multiplicatively convex* or AG-convex, namely, for all x, $y \in I$ and $t \in [0,1]$ one has the inequality:

$$\Psi(tx + (1-t)y) \le [\Psi(x)]^t [\Psi(y)]^{1-t}. \tag{4.9}$$

A function $\Psi : I \to \mathbb{R}\backslash\{0\}$ is called *AH-convex (concave)* on the interval I if the following inequality holds [1]

$$\Psi((1-\lambda)x + \lambda y) \le (\ge) \frac{1}{(1-\lambda)\frac{1}{\Psi(x)} + \lambda\frac{1}{\Psi(y)}} = \frac{\Psi(x)\Psi(y)}{(1-\lambda)\Psi(y) + \lambda\Psi(x)} \tag{4.10}$$

for any x, $y \in I$ and $\lambda \in [0,1]$.

An important case that provides many examples is that one in which the function is assumed to be positive for any $x \in I$. In that situation the inequality (4.10) is equivalent to

$$(1-\lambda)\frac{1}{\Psi(x)} + \lambda\frac{1}{\Psi(y)} \le (\ge) \frac{1}{\Psi((1-\lambda)x + \lambda y)}$$

for any x, $y \in I$ and $\lambda \in [0,1]$.

Taking into account this fact, we can conclude that the function $\Psi : I \to (0,\infty)$ is AH-convex (concave) on I if and only if Ψ is δ-composite concave (convex) on I with $\delta : (0,\infty) \to (0,\infty)$, $\delta(t) = \frac{1}{t}$.

Following [1], we can introduce the concept of *GH-convex (concave)* function $\Psi : I \subset (0,\infty) \to \mathbb{R}$ on an interval of positive numbers I as satisfying the condition

$$\Psi\left(x^{1-\lambda}y^{\lambda}\right) \le (\ge) \frac{1}{(1-\lambda)\frac{1}{\Psi(x)} + \lambda\frac{1}{\Psi(y)}} = \frac{\Psi(x)\Psi(y)}{(1-\lambda)\Psi(y) + \lambda\Psi(x)}. \tag{4.11}$$

Since

$$\Psi\left(x^{1-\lambda}y^{\lambda}\right) = \Psi \circ \exp[(1-\lambda)\ln x + \lambda \ln y]$$

and

$$\frac{\Psi(x)\Psi(y)}{(1-\lambda)\Psi(y) + \lambda\Psi(x)} = \frac{\Psi \circ \exp(\ln x)\,\Psi \circ \exp(\ln y)}{(1-\lambda)\Psi \circ \exp(y) + \lambda\Psi \circ \exp(x)}$$

then $\Psi : I \subset (0,\infty) \to \mathbb{R}$ is GH-convex (concave) on I if and only if $\Psi \circ \exp$ is *AH-convex (concave)* on $\ln I := \{x \mid x = \ln t,\ t \in I\}$. This is equivalent to the fact that Ψ is δ-composite-γ^{-1} concave (convex) on I with $\delta : (0,\infty) \to (0,\infty)$, $\delta(t) = \frac{1}{t}$ and $\gamma(t) = \ln t$, $t \in I$.

Following [1], we say that the function $\Psi : I \subset \mathbb{R} \setminus \{0\} \to (0,\infty)$ is *HH-convex* if

$$\Psi\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{\Psi(x)\Psi(y)}{(1-t)\Psi(y) + t\Psi(x)} \tag{4.12}$$

for all x, $y \in I$ and $t \in [0,1]$. If the inequality in (4.12) is reversed, then Ψ is said to be *HH-concave.*

We observe that the inequality (4.12) is equivalent to

$$(1-t)\frac{1}{\Psi(x)} + t\frac{1}{\Psi(y)} \leq \frac{1}{\Psi\left(\frac{xy}{tx+(1-t)y}\right)} \tag{4.13}$$

for all x, $y \in I$ and $t \in [0,1]$.

This is equivalent to the fact that Ψ is δ-composite-γ^{-1} concave on $[a,b]$ with $\delta : (0,\infty) \to (0,\infty)$, $\delta(t) = \frac{1}{t}$ and $\gamma(t) = -\frac{1}{t}$, $t \in [a,b]$.

The function $\Psi : I \subset (0,\infty) \to (0,\infty)$ is called *GG-convex* on the interval I of real umbers $\mathbb{R}$ if [1]

$$\Psi\left(x^{1-\lambda}y^{\lambda}\right) \leq [\Psi(x)]^{1-\lambda}[\Psi(y)]^{\lambda} \tag{4.14}$$

for any x, $y \in I$ and $\lambda \in [0,1]$. If the inequality is reversed in (4.14) then the function is called *GG-concave.*

This concept was introduced in 1928 by P. Montel [22], however, the roots of the research in this area can be traced long before him [23]. It is easy to see that [23], the function $\Psi : [a,b] \subset (0,\infty) \to (0,\infty)$ is *GG-convex* if and only if the the function $\gamma : [\ln a, \ln b] \to \mathbb{R}$, $\gamma = \ln \circ \Psi \circ \exp$ is convex on $[\ln a, \ln b]$. This is equivalent to the fact that Ψ is δ-composite-γ^{-1} convex on $[a,b]$ with $\delta : (0,\infty) \to \mathbb{R}$, $\delta(t) = \ln t$ and $\gamma(t) = \ln t$, $t \in [a,b]$.

Following [1] we say that the function $\Psi : I \subset \mathbb{R} \setminus \{0\} \to (0,\infty)$ is *HG-convex* if

$$\Psi\left(\frac{xy}{tx + (1-t)y}\right) \leq [\Psi(x)]^{1-t}[\Psi(y)]^{t} \tag{4.15}$$

for all x, $y \in I$ and $t \in [0,1]$. If the inequality in (4.2) is reversed, then Ψ is said to be *HG-concave.*

Let $\Psi : [a,b] \subset (0,\infty) \to (0,\infty)$ and define the associated functions $G_\Psi : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$ defined by $G_\Psi(t) = \ln \Psi\left(\frac{1}{t}\right)$. Then Ψ is *HG-convex* on $[a,b]$ iff G_Ψ is convex on $\left[\frac{1}{b}, \frac{1}{a}\right]$. This is equivalent to the fact that Ψ is δ-composite-γ^{-1} convex on $[a,b]$ with $\delta : (0,\infty) \to \mathbb{R}$, $\delta(t) = \ln t$ and $\gamma(t) = -\frac{1}{t}$, $t \in [a,b]$.

Following [25], we say that the function $\Psi : [a,b] \to (0,\infty)$ is r-convex, for $r \neq 0$, if

$$\Psi((1-\lambda)x + \lambda y) \leq [(1-\lambda)\Psi^r(y) + \lambda\Psi^r(x)]^{1/r} \tag{4.16}$$

for any x, $y \in [a,b]$ and $\lambda \in [0,1]$.

If $r > 0$, then the condition (4.16) is equivalent to

$$\Psi^r((1-\lambda)x + \lambda y) \leq (1-\lambda)\Psi^r(y) + \lambda\Psi^r(x)$$

namely Ψ is δ-composite convex on $[a,b]$ where $\delta(t) = t^r$, $t \geq 0$.

If $r < 0$, then the condition (4.16) is equivalent to

$$\Psi^r((1-\lambda)x + \lambda y) \geq (1-\lambda)\Psi^r(y) + \lambda\Psi^r(x)$$

namely Ψ is δ-composite concave on $[a,b]$ where $\delta(t)=t^r$, $t>0$.

For some results related to these concepts of convexity, see [9]-[15].

We assume in the following that $w:[a,b]\to(0,\infty)$ is a continuous function on $[a,b]$ and $g:[a,b]\to[m,M]$ is absolutely continuous on $[a,b]$.

If Ψ is *log convex* on $[m,M]$, then Ψ is δ-composite-γ^{-1} convex on $[a,b]$ with $\delta:(0,\infty)\to\mathbb{R}$, $\delta(t)=\ln t$ and $\gamma(t)=\ell(t)=t$, $t\in[a,b]$. If we use the inequality (3.1), then we have

$$\begin{aligned}(4.17)\qquad 0 &\le \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\ln(\Psi\circ g)(t)\,dt-\ln\left[\Psi\left(\frac{\int_a^b w(t)\,g(t)\,dt}{\int_a^b w(s)\,ds}\right)\right]\\ &\le \frac{1}{8}\left[\frac{\Psi'_-(M)}{\Psi(M)}-\frac{\Psi'_+(m)}{\Psi(m)}\right]\left\|\frac{g'}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds,\end{aligned}$$

provided $\frac{g'}{w}\in L_\infty[a,b]$.

We have

$$\Delta(\ln\Psi,g)(t)=\frac{(\Psi''\circ g)(t)\,(\Psi\circ g)(t)-((\Psi'\circ g)(t))^2}{((\Psi\circ g)(t))^2},\ t\in[a,b]$$

and if we assume that $\frac{\Delta(\ln\Psi,g)}{w}\in L_\infty[a,b]$, then by the inequality (3.2)

$$\begin{aligned}(4.18)\qquad 0 &\le \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\ln(\Psi\circ g)(t)\,dt-\ln\left[\Psi\left(\frac{\int_a^b w(t)\,g(t)\,dt}{\int_a^b w(s)\,ds}\right)\right]\\ &\le \frac{1}{8}(M-m)\left\|\frac{\Delta(\ln\Psi,g)}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds.\end{aligned}$$

If Ψ is *GA-convex* on $[a,b]\subset(0,\infty)$, then Ψ is δ-composite-γ^{-1} convex on $[a,b]$ with $\gamma:(0,\infty)\to\mathbb{R}$, $\gamma(t)=\ln t$ and $\delta(t)=\ell(t)=t$, $t\in[a,b]$. If we use the inequality (3.1), then we have

$$\begin{aligned}(4.19)\qquad 0 &\le \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Psi\circ g)(t)\,dt-\Psi\left[\exp\left(\frac{\int_a^b w(t)\ln g(t)\,dt}{\int_a^b w(s)\,ds}\right)\right]\\ &\le \frac{1}{8}\left(\Psi'_-(M)\,M-\Psi'_+(m)\,m\right)\left\|\frac{g'}{wg}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds.\end{aligned}$$

If Ψ is twice differentiable, define for $t\in[a,b]$,

$$\Delta(\Psi,\ln,g)(t)=(\Psi''\circ g)(t)\,g(t)+(\Psi'\circ g)(t)$$

and assume that $\frac{\Delta(\Psi,\ln,g)}{w}\in L_\infty[a,b]$, then

$$\begin{aligned}(4.20)\qquad 0 &\le \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Psi\circ g)(t)\,dt-\Psi\left[\exp\left(\frac{\int_a^b w(t)\ln g(t)\,dt}{\int_a^b w(s)\,ds}\right)\right]\\ &\le \frac{1}{8}\ln\left(\frac{M}{m}\right)\left\|\frac{\Delta(\Psi,\ln,g)}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds.\end{aligned}$$

The function $\Psi:[a,b]\to(0,\infty)$ is *AH-convex* on $[a,b]$ if and only if Ψ is δ-composite-γ^{-1} concave on $[a,b]$ with $\delta:(0,\infty)\to(0,\infty)$, $\delta(t)=\frac{1}{t}$ and $\gamma(t)=$

$\ell(t) = t,\ t \in [a,b]$. If we use the inequality (3.1) for the convex function $-\Psi^{-1}$, then we have

$$0 \leq \left[\Psi\left(\frac{\int_a^b w(t) g(t) dt}{\int_a^b w(s) ds}\right)\right]^{-1} - \frac{1}{\int_a^b w(s) ds} \int_a^b \frac{w(t)}{(\Psi \circ g)(t)} dt \tag{4.21}$$
$$\leq \frac{1}{8}\left[\frac{\Psi'_+(M)}{\Psi^2(M)} - \frac{\Psi'_+(m)}{\Psi^2(m)}\right] \left\|\frac{g'}{w}\right\|_{[a,b],\infty} \int_a^b w(s) ds,$$

provided $\frac{g'}{w} \in L_\infty[a,b]$.

If Ψ is twice differentiable, define for $t \in [a,b]$,

$$\Delta\left(-\Psi^{-1}, g\right)(t) := \frac{(\Psi'' \circ g)(t)(\Psi \circ g)(t) - 2\left((\Psi' \circ g)(t)\right)^2}{\left((\Psi \circ g)(t)\right)^3}.$$

If we use the inequality (3.2) for the convex function $-\Psi^{-1}$, then we have

$$0 \leq \left[\Psi\left(\frac{\int_a^b w(t) g(t) dt}{\int_a^b w(s) ds}\right)\right]^{-1} - \frac{1}{\int_a^b w(s) ds} \int_a^b \frac{w(t)}{(\Psi \circ g)(t)} dt \tag{4.22}$$
$$\leq \frac{1}{8}(M-m)\left\|\frac{\Delta\left(-\Psi^{-1}, g\right)}{w}\right\|_{[a,b],\infty} \int_a^b w(s) ds,$$

provided that $\frac{\Delta\left(-\Psi^{-1}, g\right)}{w} \in L_\infty[a,b]$.

If the function Ψ is *HA-convex* $[a,b]$, this means that Ψ is δ-composite-γ^{-1} convex on $[a,b]$ with $\gamma : (0,\infty) \to \mathbb{R}$, $\gamma(t) = -t^{-1}$ and $\delta(t) = \ell(t) = t$, $t \in [a,b]$. If we use the inequality (3.1), then we have

$$0 \leq \frac{1}{\int_a^b w(s) ds} \int_a^b w(t)(\Psi \circ g)(t) dt - \Psi\left(\frac{\int_a^b w(s) ds}{\int_a^b \frac{w(t)}{g(t)} dt}\right) \tag{4.23}$$
$$\leq \frac{1}{8}\left(\Psi'_-(M) M^2 - \Psi'_+(m) m^2\right)\left\|\frac{g'}{wg^2}\right\|_{[a,b],\infty} \int_a^b w(s) ds,$$

provided $\frac{g'}{wg^2} \in L_\infty[a,b]$.

If Ψ is differentiable, define for $t \in [a,b]$,

$$\Delta\left(\Psi, -\ell^{-1}, g\right)(t) := (\Psi'' \circ g)(t) g^2(t) + 2g(t)(\Psi' \circ g)(t).$$

By using the inequality (3.2) we have

$$0 \leq \frac{1}{\int_a^b w(s) ds} \int_a^b w(t)(\Psi \circ g)(t) dt - \Psi\left(\frac{\int_a^b w(s) ds}{\int_a^b \frac{w(t)}{g(t)} dt}\right) \tag{4.24}$$
$$\leq \frac{1}{8}(M-m)\left\|\frac{\Delta\left(\Psi, -\ell^{-1}, g\right)}{w}\right\|_{[a,b],\infty} \int_a^b w(s) ds,$$

provided $\frac{\Delta\left(\Psi, -\ell^{-1}, g\right)}{w} \in L_\infty[a,b]$.

Similar results may be stated for the other concepts of convexity as presented above, however the details are omitted.

References

[1] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Generalized convexity and inequalities, *J. Math. Anal. Appl.* **335** (2007) 1294–1308.

[2] P. L. Chebyshev, Sur les expressions approximatives des intègrals dèfinis par les outres prises entre les même limites, *Proc. Math. Soc. Charkov,* **2** (1882), 93-98.

[3] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications. *Tamkang J. Math.* **38** (2007), no. 1, 37–49. Preprint *RGMIA Res. Rep. Coll.* **5** (2002), No. 2, Art. 14. `[Online http://rgmia.org/papers/v5n2/RGIApp.pdf]`.

[4] S. S. Dragomir, A Grüss type inequality for isotonic linear functionals and applications. *Demonstratio Math.* **36** (2003), no. 3, 551–562. Preprint RGMIA *Res. Rep. Coll.* **5** (2002), Suplement, Art. 12. `[Online http://rgmia.org/papers/v5e/GTIILFApp.pdf]`.

[5] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results. *Aust. J. Math. Anal. Appl.* **14** (2017), no. 1, Art. 1, 283 pp. `[Online http://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex]`.

[6] S. S. Dragomir, Some reverses of the Jensen inequality with applications. Bull. Aust. Math. Soc. 87 (2013), no. 2, 177–194.

[7] S. S. Dragomir, Reverses of the Jensen inequality in terms of first derivative and applications. Acta Math. Vietnam. 38 (2013), no. 3, 429–446.

[8] S. S. Dragomir, A refinement and a divided difference reverse of Jensen's inequality with applications. Rev. Colombiana Mat. 50 (2016), no. 1, 17–39.

[9] S. S. Dragomir, New inequalities of Hermite-Hadamard type for log-convex functions. *Khayyam J. Math.* **3** (2017), no. 2, 98–115..

[10] S. S. Dragomir, Inequalities of Hermite-Hadamard type for GA-convex functions, to appear in *Annales Mathematicae Silesianae,* Preprint *RGMIA Res. Rep. Coll.* **18** **(**2015**),** Art. 30. `[Online http://rgmia.org/papers/v18/v18a30.pdf]`.

[11] S. S. Dragomir, Inequalities of Hermite-Hadamard type for GG-convex functions, Preprint *RGMIA, Research Report Collection*, **18** (2015), Art. 71, 15 pp., `[Online http://rgmia.org/papers/v18/v18a71.pdf]`.

[12] S. S. Dragomir, Some integral inequalities of Hermite-Hadamard type for GG-convex functions, *Mathematica* (Cluj), **59** (82), No 1-2, 2017, pp. 47-64. Preprint *RGMIA, Research Report Collection*, **18** (2015), Art. 74. `[Online http://rgmia.org/papers/v18/v18a74.pdf]`.

[13] S. S. Dragomir, Inequalities of Hermite-Hadamard type for HA-convex functions, *Maroccan J. Pure & Appl. Analysis*, Volume **3** (1), 2017, Pages 83-101. Preprint, *RGMIA Res. Rep. Coll.* **18** (2015), Art. 38. `[Online http://rgmia.org/papers/v18/v18a38.pdf]`.

[14] S. S. Dragomir, Inequalities of Hermite-Hadamard type for HG-convex functions, *Probl. Anal. Issues Anal.* Vol. **6** (**24**), No. 2, 2017 1-17. Preprint, *RGMIA Res. Rep. Coll.* **18** (2015), Art. 79. `[Online http://rgmia.org/papers/v18/v18a79.pdf]`.

[15] S. S. Dragomir, Inequalities of Hermite-Hadamard type for HH-convex functions, to appear in *Acta et Commentationes Universitatis Tartuensis de Mathematica,* Preprint, *RGMIA Res. Rep. Coll.* **18** (2015), Art. 80. `[Online http://rgmia.org/papers/v18/v18a80.pdf]`.

[16] S. S. Dragomir, Inequalities for a generalized finite Hilbert transform of convex functions, Preprint *RGMIA Res. Rep. Coll.* **21** (2018), Art

[17] S. S. Dragomir and N. M. Ionescu, Some converse of Jensen's inequality and applications. *Rev. Anal. Numér. Théor. Approx.* **23** (1994), no. 1, 71–78.

[18] L. Fejér, Über die Fourierreihen, II, (In Hungarian) *Math. Naturwiss, Anz. Ungar. Akad. Wiss.*, **24** (1906), 369-390.

[19] G. Grüss, Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$, *Math. Z.* , **39**(1935), 215-226.

[20] A. Lupaş, The best constant in an integral inequality, *Mathematica* (Cluj, Romania), **15**(**38**)(2) (1973), 219-222.

[21] D. S. Mitrinović and I. B. Lacković, Hermite and convexity, *Aequationes Math.* **28** (1985), 229–232.

[22] P. Montel, Sur les functions convexes et les fonctions sousharmoniques, *Journal de Math.*, **9** (1928), 7, 29–60.

[23] C. P. Niculescu, Convexity according to the geometric mean, *Math. Inequal. Appl.,* **3**, (2000), 2, 155–167.

[24] A. M. Ostrowski, On an integral inequality, *Aequat. Math.,* **4** (1970), 358-373.

[25] C. E. M. Pearce, J. Pečarić and V. Šimić, Stolarsky means and Hadamard's inequality. *J. Math. Anal. Appl.* **220**, 99-109 (1998)

[26] K. S. Zhang and J. P. Wan, p-convex functions and their properties. *Pure Appl. Math.* **23**(1), 130-133 (2007).

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