Received 08/05/18

REVERSES OF JENSEN'S INTEGRAL INEQUALITY VIA A WEIGHTED ČEBYŠEV TYPE RESULT WITH APPLICATIONS FOR COMPOSITE CONVEX FUNCTIONS

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Abstract. In this paper we obtain some reverses of Jensen's integral inequality by employing a new weighted integral inequality of Čebyšev type. Applications for general composite convex functions with examples for AG, GA-convex functions and HA, AH-convex function are also given.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ-a.e. (almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega w(x)|f(x)|\, d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\, d\mu(x)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S. S. Dragomir obtained in 2002 [4] the following result:

Theorem 1. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. (almost everywhere) on Ω with $\int_\Omega w d\mu = 1$. Then we have the inequality:*

$$0 \leq \int_\Omega w(\Phi \circ f)\, d\mu - \Phi\left(\int_\Omega wf d\mu\right) \leq \int_\Omega w(\Phi' \circ f) f d\mu - \int_\Omega w(\Phi' \circ f)\, d\mu \int_\Omega wf d\mu. \tag{1.1}$$

Let $\Phi : [m, M] \to \mathbb{R}$ be a differentiable convex function on (m, M). If $x_i \in [m, M]$ and $w_i \geq 0$ $(i = 1, \ldots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then one has the reverse of

1991 *Mathematics Subject Classification.* 26D15; 26D10.

Key words and phrases. Convex functions, Jensen's inequality, Čebyšev's inequality, AG, AH-convex functions, GA, GG, GH-convex functions and HA, HG, HH-convex function, Integral inequalities.

RGMIA Res. Rep. Coll. 21 (2018), Art. 46, 16 pp.

Jensen's weighted discrete inequality:

$$0 \leq \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i\right) \leq \sum_{i=1}^{n} w_i \Phi'(x_i) x_i - \sum_{i=1}^{n} w_i \Phi'(x_i) \sum_{i=1}^{n} w_i x_i. \tag{1.2}$$

The inequality (1.2) was obtained in 1994 by Dragomir & Ionescu, see [18].

If $h,\ g : \Omega \to \mathbb{R}$ are μ-measurable functions and $h,\ g,\ hg \in L_w(\Omega, \mu)$, then we may consider the *Čebyšev functional*

$$T_w(h, g) := \int_\Omega whg d\mu - \int_\Omega wh d\mu \int_\Omega wg d\mu. \tag{1.3}$$

The following result is known in the literature as the *Grüss inequality*

$$|T_w(h, g)| \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta), \tag{1.4}$$

provided

$$-\infty < \gamma \leq h(x) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(x) \leq \Delta < \infty \tag{1.5}$$

for μ-a.e. a. $x \in \Omega$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

With the above assumptions, if $h \in L_{w,2}(\Omega, \mu)$ then we may define

$$D_w(h) := D_{w,1}(h) := \int_\Omega w \left| h - \int_\Omega wh d\mu \right| d\mu \tag{1.6}$$

and

$$D_{w,2}(h) := \left[\int_\Omega wh^2 d\mu - \left(\int_\Omega wh d\mu\right)^2\right]^{\frac{1}{2}}.$$

In 2002, Cerone & Dragomir [3] have obtained the following refinement of the Grüss inequality (1.4):

Theorem 2. *Let $w, h, g : \Omega \to \mathbb{R}$ be μ-measurable functions with $w \geq 0$ μ-a.e. (almost everywhere) on Ω and $\int_\Omega w d\mu = 1$. If $h,\ g,\ hg \in L_w(\Omega, \mu)$ and there exists the constants δ, Δ such that the condition (1.5) holds,*

$$|T_w(h, g)| \leq \frac{1}{2}(\Delta - \delta) D_w(h) \leq \frac{1}{2}(\Delta - \delta) D_{w,2}(h). \tag{1.7}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

Moreover, if h satisfies the condition (1.5), then

$$|T_w(h, g)| \leq \frac{1}{2}(\Delta - \delta) D_w(h) \leq \frac{1}{2}(\Delta - \delta) D_{w,2}(h) \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta). \tag{1.8}$$

On making use of Theorems 1 and 2 we can state the following result providing a sequence of bounds for the Jensen's gap [4]:

Theorem 3. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_w(\Omega, \mu)$, where $w \geq 0$*

μ-a.e. (almost everywhere) on Ω with $\int_\Omega w d\mu = 1$. Then we have the sequence of inequalities:

$$
\begin{aligned}
(1.9)\qquad 0 &\leq \int_\Omega w(\Phi\circ f)\,d\mu - \Phi\left(\int_\Omega wf d\mu\right) \\
&\leq \int_\Omega w(\Phi'\circ f) f d\mu - \int_\Omega w(\Phi'\circ f)\,d\mu \int_\Omega wf d\mu \\
&\leq \frac{1}{2}\begin{cases} \left[\Phi'_-(M)-\Phi'_+(m)\right]\int_\Omega w\left|f-\int_\Omega wfd\mu\right|d\mu \\ (M-m)\int_\Omega w\left|\Phi'\circ f-\int_\Omega w(\Phi'\circ f)\,d\mu\right|d\mu \end{cases} \\
&\leq \frac{1}{2}\begin{cases} \left[\Phi'_-(M)-\Phi'_+(m)\right]\left[\int_\Omega wf^2d\mu-\left(\int_\Omega wfd\mu\right)^2\right]^{\frac{1}{2}} \\ (M-m)\left[\int_\Omega w(\Phi'\circ f)^2\,d\mu-\left(\int_\Omega w(\Phi'\circ f)\,d\mu\right)^2\right]^{\frac{1}{2}} \end{cases} \\
&\leq \frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right].
\end{aligned}
$$

For other similar reverses of Jensen's integral inequality in the general setting of Lebesgue integral, see [6]-[8].

In the recent paper [17], by the use of a weighted version of Ostrowski's inequality, we obtained the following reverse of Jensen's integral inequality for functions of a real variable:

Theorem 4. *Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m,M), $w : [a,b] \to (0,\infty)$ be continuous on $[a,b]$ and $f : [a,b] \to [m,M]$ be absolutely continuous so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f) f \in L_w[a,b]$.*

(i) *If $\frac{f'}{w} \in L_\infty[a,b]$, then we have the inequality*

$$
\begin{aligned}
(1.10)\qquad 0 &\leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi\circ f)(t)\,dt - \Phi\left(\frac{\int_a^b w(t) f(t)\,dt}{\int_a^b w(s)\,ds}\right) \\
&\leq \frac{1}{8}\left[\Phi'_-(M)-\Phi'_+(m)\right]\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds.
\end{aligned}
$$

(ii) *If Φ is twice differentiable on (m,M) and $\frac{(\Phi''\circ f) f'}{w} \in L_\infty[a,b]$, then*

$$
\begin{aligned}
(1.11)\qquad 0 &\leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi\circ f)(t)\,dt - \Phi\left(\frac{\int_a^b w(t) f(t)\,dt}{\int_a^b w(s)\,ds}\right) \\
&\leq \frac{1}{8}(M-m)\left\|\frac{(\Phi''\circ f) f'}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds.
\end{aligned}
$$

This result has the following particular cases of interest:

Corollary 1. *Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m,M) and $f : [a,b] \to [m,M]$ be absolutely continuous so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f) f \in L[a,b]$.*

(i) *If* $f' \in L_\infty [a,b]$, *then we have the inequality*

$$0 \leq \frac{1}{b-a}\int_a^b (\Phi \circ f)(t)\,dt - \Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \leq \frac{1}{8}(b-a)\left[\Phi'_-(M) - \Phi'_+(m)\right]\|f'\|_{[a,b],\infty}. \tag{1.12}$$

(ii) *If* Φ *is twice differentiable on* (m,M) *and* $(\Phi'' \circ f) f' \in L_\infty [a,b]$, *then*

$$0 \leq \frac{1}{b-a}\int_a^b (\Phi \circ f)(t)\,dt - \Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \leq \frac{1}{8}(b-a)(M-m)\|(\Phi'' \circ f) f'\|_{[a,b],\infty}. \tag{1.13}$$

Corollary 2. *Let* $\Phi : [a,b] \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable convex function on* (a,b), $w : [a,b] \to (0,\infty)$ *be continuous on* $[a,b]$ *and* Φ, $\Phi' \in L_w [a,b]$.

(i) *If* $\frac{1}{w} \in L_\infty [a,b]$, *then we have the inequality*

$$0 \leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\,\Phi(t)\,dt - \Phi\left(\frac{\int_a^b t w(t)\,dt}{\int_a^b w(s)\,ds}\right) \leq \frac{1}{8}\left[\Phi'_-(b) - \Phi'_+(a)\right]\left\|\frac{1}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds. \tag{1.14}$$

(ii) *If* f Φ *is twice differentiable on* (m,M) *and* $\frac{\Phi''}{w} \in L_\infty [a,b]$, *then*

$$0 \leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\,\Phi(t)\,dt - \Phi\left(\frac{\int_a^b t w(t)\,dt}{\int_a^b w(s)\,ds}\right) \leq \frac{1}{8}(b-a)\left\|\frac{\Phi''}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds. \tag{1.15}$$

Motivated by the above results, in this paper we obtain some reverses of Jensen's integral inequality by employing a new weighted integral inequality of Čebyšev type. Applications for general composite convex functions with examples for AG, GA-convex functions and HA, AH-convex function are also given.

2. Reverses of Jensen's Inequality Via a Weighted Čebyšev Result

For two *Lebesgue integrable* functions f, $g : [a,b] \to \mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt. \tag{2.1}$$

In 1935, Grüss [20] showed that

$$|C(f,g)| \leq \frac{1}{4}(M-m)(N-n), \tag{2.2}$$

provided that there exists the real numbers m, M, n, N such that

$$m \leq f(t) \leq M \quad \text{and} \quad n \leq g(t) \leq N \quad \text{for a.e. } t \in [a,b]. \tag{2.3}$$

The constant $\frac{1}{4}$ is best possible in (2.1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [2], states that

$$|C(f,g)| \leq \frac{1}{12}\|f'\|_{\infty}\|g'\|_{\infty}(b-a)^2, \tag{2.4}$$

provided that f', g' exist and are continuous on $[a,b]$ and $\|f'\|_{\infty} = \sup_{t\in[a,b]}|f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (2.4) also holds if f, $g : [a,b] \to \mathbb{R}$ are assumed to be *absolutely continuous* and f', $g' \in L_{\infty}[a,b]$ while $\|f'\|_{\infty} = \operatorname{essup}_{t\in[a,b]}|f'(t)|$.

We can define, as above

$$\begin{aligned} C_{h'}(f,g) := & \frac{1}{h(b)-h(a)}\int_a^b f(t)\,g(t)\,h'(t)\,dt \\ & -\frac{1}{h(b)-h(a)}\int_a^b f(t)\,h'(t)\,dt\frac{1}{h(b)-h(a)}\int_a^b g(t)\,h'(t)\,dt, \end{aligned} \tag{2.5}$$

where h is absolutely continuous and f, g are Lebesgue measurable on $[a,b]$ and such that the above integrals exist.

The following weighted version of Čebyšev's inequality holds:

Lemma 1. *Let $h : [a,b] \to [h(a), h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If f, $g : [a,b] \to \mathbb{R}$ are absolutely continuous on $[a,b]$ and $\frac{f'}{h'}$, $\frac{g'}{h'}$ is essentially bounded, namely $\frac{f'}{h'}$, $\frac{g'}{h'} \in L_{\infty}[a,b]$, then we have*

$$|C_{h'}(f,g)| \leq \frac{1}{12}[h(b)-h(a)]^2\left\|\frac{f'}{h'}\right\|_{[a,b],\infty}\left\|\frac{g'}{h'}\right\|_{[a,b],\infty}. \tag{2.6}$$

The constant $\frac{1}{12}$ is best possible.

Proof. Assume that $[c,d] \subset [a,b]$. If $g : [c,d] \to \mathbb{C}$ is absolutely continuous on $[c,d]$, then $g \circ h^{-1} : [h(c), h(d)] \to \mathbb{C}$ is absolutely continuous on $[h(c), h(d)]$ and using the chain rule and the derivative of inverse functions we have

$$\left(g\circ h^{-1}\right)'(z) = \left(g'\circ h^{-1}\right)(z)\left(h^{-1}\right)'(z) = \frac{\left(g'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)} \tag{2.7}$$

for almost every (a.e.) $z \in [h(c), h(d)]$.

If $x \in [c,d]$, then by taking $z = h(x)$, we get

$$\left(g\circ h^{-1}\right)'(z) = \frac{\left(g'\circ h^{-1}\right)(h(x))}{\left(h'\circ h^{-1}\right)(h(x))} = \frac{g'(x)}{h'(x)}.$$

Therefore, since $\frac{g'}{h'} \in L_{\infty}[c,d]$, hence $\left(g\circ h^{-1}\right)' \in L_{\infty}[h(c), h(d)]$. Also

$$\left\|\left(g\circ h^{-1}\right)'\right\|_{[h(c),h(d)],\infty} = \left\|\frac{g'}{h'}\right\|_{[c,d],\infty}.$$

Now, if we use Čebyšev's inequality (2.4) for the functions $f\circ h^{-1}$ and $g\circ h^{-1}$ on the interval $[h(a),h(b)]$, then we get

$$\begin{aligned}(2.8)\quad &\left|\frac{1}{h(b)-h(a)}\int_{h(a)}^{h(b)} f\circ h^{-1}(u)g\circ h^{-1}(u)du\right.\\ &\left.-\frac{1}{[h(b)-h(a)]^2}\int_{h(a)}^{h(b)} f\circ h^{-1}(u)du\int_{h(a)}^{h(b)} g\circ h^{-1}(u)du\right|\\ &\leq \frac{1}{12}[h(b)-h(a)]^2\left\|\left(f\circ h^{-1}\right)'\right\|_{[h(a),h(b)],\infty}\left\|\left(g\circ h^{-1}\right)'\right\|_{[h(a),h(b)],\infty}.\end{aligned}$$

Observe also that, by the change of variable $t=h^{-1}(u)$, $u\in[g(a),g(b)]$, we have $u=h(t)$ that gives $du=h'(t)\,dt$ and

$$\int_{h(a)}^{h(b)}\left(f\circ h^{-1}\right)(u)\,du=\int_a^b f(t)\,h'(t)\,dt,$$

$$\int_{h(a)}^{h(b)} g\circ h^{-1}(u)du=\int_a^b g(t)\,h'(t)\,dt,$$

$$\int_{h(a)}^{h(b)} f\circ h^{-1}(u)g\circ h^{-1}(u)du=\int_a^b f(t)\,g(t)\,h'(t)\,dt,$$

$$\left\|\left(f\circ h^{-1}\right)'\right\|_{[h(a),h(b)],\infty}=\left\|\frac{f'}{h'}\right\|_{[a,b],\infty},$$

and

$$\left\|\left(g\circ h^{-1}\right)'\right\|_{[h(a),h(b)],\infty}=\left\|\frac{g'}{h'}\right\|_{[a,b],\infty}.$$

By making use of (2.8) we then get the desired result (2.6).

The best constant follows by Čebyšev's inequality (2.4). □

If $w:[a,b]\to\mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\to[0,\infty)$, $W(x):=\int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x)=w(x)$ for any $x\in(a,b)$.

Corollary 3. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$, f and g are absolutely continuous on $[a,b]$ with $\frac{f'}{w}$, $\frac{g'}{w}$ is essentially bounded, namely $\frac{f'}{w}$, $\frac{g'}{w}\in L_\infty[a,b]$, then we have*

$$|C_w(f,g)|\leq\frac{1}{12}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\left\|\frac{g'}{w}\right\|_{[a,b],\infty}\left(\int_a^b w(s)\,ds\right)^2. \tag{2.9}$$

The constant $\frac{1}{12}$ is best possible.

Remark 1. *Under the assumptions of Corollary 3 and if there exists a constant K, $L>0$ such that $|f'(t)|\leq Lw(t)$, $|g'(t)|\leq Kw(t)$ for a.e. $t\in[a,b]$, then by (2.9) we get*

$$|C_w(f,g)|\leq\frac{1}{12}LK\left(\int_a^b w(s)\,ds\right)^2. \tag{2.10}$$

We have the following reverse of Jensen's integral inequality:

Theorem 5. *Let* $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ *be a twice differentiable convex function on* (m,M), $w:[a,b]\to(0,\infty)$ *be continuous on* $[a,b]$ *and* $f:[a,b]\to[m,M]$ *be absolutely continuous so that* $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f)\,f\in L_w[a,b]$. *If* $\frac{f'}{w}\in L_\infty[a,b]$ *and* $\frac{(\Phi''\circ f)f'}{w}\in L_\infty[a,b]$, *then we have the inequality*

$$
\begin{aligned}
(2.11)\qquad 0 &\le \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi\circ f)(t)\,dt-\Phi\left(\frac{\int_a^b w(t)f(t)\,dt}{\int_a^b w(s)\,ds}\right)\\
&\le \frac{1}{12}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\left\|\frac{(\Phi''\circ f)f'}{w}\right\|_{[a,b],\infty}\left(\int_a^b w(s)\,ds\right)^2.
\end{aligned}
$$

Proof. By (4.14) we have

$$
\begin{aligned}
(2.12)\qquad 0 &\le \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi\circ f)(t)\,dt-\Phi\left(\frac{\int_a^b w(t)f(t)\,dt}{\int_a^b w(s)\,ds}\right)\\
&\le \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi'\circ f)(t)f(t)\,dt\\
&\quad-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi'\circ f)(t)\,dt\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)f(t)\,dt.
\end{aligned}
$$

Since Φ is twice differentiable on (a,b), then

$$(\Phi'\circ f)'(t)=(\Phi''\circ f)(t)\,f'(t)$$

for $t\in(a,b)$.

If we use the inequality (2.9), then we get

$$
\begin{aligned}
&\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi\circ f)(t)f(t)\,dt\\
&-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi'\circ f)(t)\,dt\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)f(t)\,dt\\
&\le \frac{1}{12}\left\|\frac{(\Phi'\circ f)'}{w}\right\|_{[a,b],\infty}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\left(\int_a^b w(s)\,ds\right)^2\\
&=\frac{1}{12}\left\|\frac{(\Phi''\circ f)f'}{w}\right\|_{[a,b],\infty}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\left(\int_a^b w(s)\,ds\right)^2
\end{aligned}
$$

which, together with (2.12), proves the required inequality (2.11). □

Corollary 4. *Let* $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ *be a twice differentiable convex function on* (m,M) *and* $f:[a,b]\to[m,M]$ *be absolutely continuous on* $[a,b]$. *If* $f'\in L_\infty[a,b]$ *and* $(\Phi''\circ f)\,f'\in L_\infty[a,b]$, *then we have the inequality*

$$
\begin{aligned}
(2.13)\qquad 0 &\le \frac{1}{b-a}\int_a^b(\Phi\circ f)(t)\,dt-\Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\\
&\le \frac{1}{12}\|f'\|_{[a,b],\infty}\|(\Phi''\circ f)f'\|_{[a,b],\infty}(b-a)^2.
\end{aligned}
$$

Corollary 5. *Let $\Phi : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on (a,b), $w : [a,b] \to (0,\infty)$ a continuous function on $[a,b]$ and $\Phi,\ \Phi' \in L_w[a,b]$. If $\frac{1}{w} \in L_\infty[a,b]$ and $\frac{\Phi''}{w} \in L_\infty[a,b]$, then we have the inequality*

$$0 \leq \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)\,\Phi(t)\,dt - \Phi\left(\frac{\int_a^b t w(t)\,dt}{\int_a^b w(s)\,ds}\right) \leq \frac{1}{12} \left\|\frac{1}{w}\right\|_{[a,b],\infty} \left\|\frac{\Phi''}{w}\right\|_{[a,b],\infty} \left(\int_a^b w(s)\,ds\right)^2. \tag{2.14}$$

Define the function $\ell(t) := t,\ t \in \mathbb{R}$.

a). Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on (m,M) and $f : [a,b] \subset (0,\infty) \to [m,M]$ be absolutely continuous and so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_{\ell^{-1}}[a,b]$. If $f'\ell \in L_\infty[a,b]$ and $(\Phi'' \circ f) f'\ell \in L_\infty[a,b]$ then by (2.11) for $w(t) = \frac{1}{t}$, we have

$$0 \leq \frac{1}{\ln\left(\frac{b}{a}\right)} \int_a^b \frac{(\Phi \circ f)(t)}{t} dt - \Phi\left(\frac{\int_a^b \frac{f(t)}{t} dt}{\ln\left(\frac{b}{a}\right)}\right) \leq \frac{1}{12} \|f'\ell\|_{[a,b],\infty} \|(\Phi'' \circ f) f'\ell\|_{[a,b],\infty} \left(\ln\left(\frac{b}{a}\right)\right)^2. \tag{2.15}$$

b). Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on (m,M) and $f : [a,b] \to [m,M]$ be absolutely continuous and so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_{\exp}[a,b]$. If $\frac{f'}{\exp} \in L_\infty[a,b]$ and $\frac{(\Phi'' \circ f) f'}{\exp} \in L_\infty[a,b]$, then by (2.11) for $w(t) = \exp(t)$, we have

$$0 \leq \frac{1}{\exp b - \exp a} \int_a^b (\Phi \circ f)(t) \exp t dt - \Phi\left(\frac{\int_a^b f(t) \exp t dt}{\exp b - \exp a}\right) \leq \frac{1}{12} \left\|\frac{f'}{\exp}\right\|_{[a,b],\infty} \left\|\frac{(\Phi'' \circ f) f'}{\exp}\right\|_{[a,b],\infty} (\exp b - \exp a)^2. \tag{2.16}$$

c). Consider the function $\ell^p(t) := t^p,\ t > 0,\ p \in \mathbb{R} \setminus \{-1\}$. Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on (m,M) and $f : [a,b] \subset (0,\infty) \to [m,M]$ be absolutely continuous and so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_{\ell^p}[a,b]$. If $f'\ell^{-p} \in L_\infty[a,b]$ and $(\Phi'' \circ f) f'\ell^{-p} \in L_\infty[a,b]$ then by (2.11) for $w(t) = \ell^p$, we have

$$0 \leq \frac{p+1}{b^{p+1} - a^{p+1}} \int_a^b t^p (\Phi \circ f)(t)\,dt - \Phi\left(\frac{(p+1)\int_a^b t^p f(t)\,dt}{b^{p+1} - a^{p+1}}\right) \leq \frac{1}{12(p+1)^2} \left\|f'\ell^{-p}\right\|_{[a,b],\infty} \left\|(\Phi'' \circ f) f'\ell^{-p}\right\|_{[a,b],\infty} \left(b^{p+1} - a^{p+1}\right)^2. \tag{2.17}$$

For $p = -2$, we get from (2.17) that

$$0 \leq \frac{ab}{b-a} \int_a^b \frac{(\Phi \circ f)(t)}{t^2} dt - \Phi\left(\frac{ab}{b-a} \int_a^b \frac{f(t)}{t^2} dt\right) \leq \frac{1}{12} \left(\frac{b-a}{ab}\right)^2 \left\|f'\ell^2\right\|_{[a,b],\infty} \left\|(\Phi'' \circ f) f'\ell^2\right\|_{[a,b],\infty}, \tag{2.18}$$

provided $f'\ell^2,\ (\Phi'' \circ f) f'\ell^2 \in L_\infty[a,b]$.

3. Inequalities for Composite Convexity

We have the following result for composite convexity:

Theorem 6. *Let $\Psi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on (m,M), $\gamma : [m,M] \to [\gamma(m), \gamma(M)]$ a strictly increasing, continuous and twice differentiable function on (m,M), $w : [a,b] \to (0,\infty)$ a continuous function on $[a,b]$ and $g : [a,b] \to [m,M]$ an absolutely continuous function on $[a,b]$. Assume that $\Psi \circ \gamma^{-1}$ is convex on $[\gamma(m), \gamma(M)]$ and $\Psi \circ g$, $\gamma \circ g \in L_w[a,b]$. Define*

$$\Delta(\Psi,\gamma,g)(t) := \frac{(\Psi'' \circ g)(t)(\gamma' \circ g)(t) - (\Psi' \circ g)(t)(\gamma'' \circ g)(t)}{[(\gamma' \circ g)(t)]^2} \tag{3.1}$$

for $t \in [a,b]$.

If $\frac{(\gamma' \circ g) g'}{w} \in L_\infty[a,b]$ and $\frac{\Delta(\Psi,\gamma,g)}{w} \in L_\infty[a,b]$, then

$$\begin{aligned} 0 &\le \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)(\Psi \circ g)(t)\,dt - \Psi \circ \gamma^{-1}\left(\frac{\int_a^b w(t)(\gamma \circ g)(t)\,dt}{\int_a^b w(s)\,ds}\right) \\ &\le \frac{1}{12}\left\|\frac{(\gamma' \circ g) g'}{w}\right\|_{[a,b],\infty}\left\|\frac{\Delta(\Psi,\gamma,g)}{w}\right\|_{[a,b],\infty}\left(\int_a^b w(s)\,ds\right)^2. \end{aligned} \tag{3.2}$$

Proof. If we write the inequality (2.11) for the convex function $\Phi = \Psi \circ \gamma^{-1}$ on $[\gamma(m), \gamma(M)]$ and for the function $f = \gamma \circ g$ on $[a,b]$, then we have

$$\begin{aligned} 0 \le \frac{1}{\int_a^b w(s)\,ds} &\int_a^b w(t)\left(\Psi \circ \gamma^{-1} \circ \gamma \circ g\right)(t)\,dt \\ &- \Psi \circ \gamma^{-1}\left(\frac{\int_a^b w(t)(\gamma \circ g)(t)\,dt}{\int_a^b w(s)\,ds}\right) \\ \le \frac{1}{12}\left\|\frac{(\gamma \circ g)'}{w}\right\|_{[a,b],\infty} &\left\|\frac{\left(\Psi \circ \gamma^{-1}\right)''((\gamma \circ g)) \cdot (\gamma' \circ g)}{w}\right\|_{[a,b],\infty}\left(\int_a^b w(s)\,ds\right)^2. \end{aligned} \tag{3.3}$$

Using the chain rule and the derivative of inverse functions we have

$$\left(\Psi \circ \gamma^{-1}\right)'(z) = \left(\Psi' \circ \gamma^{-1}\right)(z)\left(\gamma^{-1}\right)'(z) = \frac{\left(\Psi' \circ \gamma^{-1}\right)(z)}{\left(\gamma' \circ \gamma^{-1}\right)(z)} \tag{3.4}$$

for every $z \in (\gamma(m), \gamma(M))$.

We have by (3.4) that

$$\begin{aligned} \left(\Psi \circ \gamma^{-1}\right)''(z) &= \left(\frac{\left(\Psi' \circ \gamma^{-1}\right)(z)}{\left(\gamma' \circ \gamma^{-1}\right)(z)}\right)' \\ &= \frac{\left(\Psi' \circ \gamma^{-1}\right)'(z)\left(\gamma' \circ \gamma^{-1}\right)(z) - \left(\Psi' \circ \gamma^{-1}\right)(z)\left(\gamma' \circ \gamma^{-1}\right)'(z)}{\left[\left(\gamma' \circ \gamma^{-1}\right)(z)\right]^2} \\ &= \frac{\frac{\left(\Psi'' \circ \gamma^{-1}\right)(z)}{\left(\gamma' \circ \gamma^{-1}\right)(z)}\left(\gamma' \circ \gamma^{-1}\right)(z) - \left(\Psi' \circ \gamma^{-1}\right)(z)\frac{\left(\gamma'' \circ \gamma^{-1}\right)(z)}{\left(\gamma' \circ \gamma^{-1}\right)(z)}}{\left[\left(\gamma' \circ \gamma^{-1}\right)(z)\right]^2} \\ &= \frac{\left(\Psi'' \circ \gamma^{-1}\right)(z)\left(\gamma' \circ \gamma^{-1}\right)(z) - \left(\Psi' \circ \gamma^{-1}\right)(z)\left(\gamma'' \circ \gamma^{-1}\right)(z)}{\left[\left(\gamma' \circ \gamma^{-1}\right)(z)\right]^3} \end{aligned}$$

for every $z \in (\gamma(m), \gamma(M))$.

Therefore, for $f = \gamma \circ g$ we get

$$\left(\Psi \circ \gamma^{-1}\right)''((\gamma \circ g)(t)) = \frac{(\Psi'' \circ g)(t)(\gamma' \circ g)(t) - (\Psi' \circ g)(t)(\gamma'' \circ g)(t)}{[(\gamma' \circ g)(t)]^3}$$

and

$$\begin{aligned}&\left(\Psi \circ \gamma^{-1}\right)''((\gamma \circ g)(t))(\gamma' \circ g)(t) \\ &= \frac{(\Psi'' \circ g)(t)(\gamma' \circ g)(t) - (\Psi' \circ g)(t)(\gamma'' \circ g)(t)}{[(\gamma' \circ g)(t)]^2} = \Delta(\Psi, \gamma, g)(t)\end{aligned}$$

for any $t \in (a, b)$.

By employing the inequality (3.4) we then get the desired result (3.2). □

Corollary 6. *Let $\Psi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on (m, M), $\gamma : [m, M] \to [\gamma(m), \gamma(M)]$ a strictly increasing, continuous and twice differentiable function on (m, M), and $g : [a, b] \to [m, M]$ an absolutely continuous function on $[a, b]$. Assume that $\Psi \circ \gamma^{-1}$ is convex on $[\gamma(m), \gamma(M)]$ and $\Psi \circ g$, $\gamma \circ g \in L[a, b]$. If $(\gamma' \circ g) g' \in L_\infty[a, b]$ and $\Delta(\Psi, \gamma, g) \in L_\infty[a, b]$, then*

$$\begin{aligned}(3.5)\qquad 0 &\le \frac{1}{b-a}\int_a^b (\Psi \circ g)(t)\,dt - \Psi \circ \gamma^{-1}\left(\frac{1}{b-a}\int_a^b (\gamma \circ g)(t)\,dt\right) \\ &\le \frac{1}{12}(b-a)^2 \|(\gamma' \circ g) g'\|_{[a,b],\infty} \|\Delta(\Psi, \gamma, g)\|_{[a,b],\infty}.\end{aligned}$$

We also have:

Corollary 7. *Let $\Psi : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on (a, b), $\gamma : [a, b] \to [\gamma(a), \gamma(b)]$ a strictly increasing, continuous and differentiable function on (a, b), and $w : [a, b] \to (0, \infty)$ a continuous function on $[a, b]$. Assume that $\Psi \circ \gamma^{-1}$ is convex on $[\gamma(a), \gamma(b)]$ and Ψ, $\gamma \in L_w[a, b]$. Define, for $t \in (a, b)$,*

$$\Delta(\Psi, \gamma)(t) := \frac{\Psi''(t)\gamma'(t) - \Psi'(t)\gamma''(t)}{[\gamma'(t)]^2}$$

and assume that $\frac{\gamma'}{w} \in L_\infty[a, b]$ and $\frac{\Delta(\Psi,\gamma)}{w} \in L_\infty[a, b]$, then

$$\begin{aligned}(3.6)\qquad 0 &\le \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\Psi(t)\,dt - \Psi \circ \gamma^{-1}\left(\frac{\int_a^b w(t)\gamma(t)\,dt}{\int_a^b w(s)\,ds}\right) \\ &\le \frac{1}{12}\left\|\frac{\gamma'}{w}\right\|_{[a,b],\infty}\left\|\frac{\Delta(\Psi, \gamma)}{w}\right\|_{[a,b],\infty}\left(\int_a^b w(s)\,ds\right)^2.\end{aligned}$$

Remark 2. *Let $\Psi : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on (a, b) and $\gamma : [a, b] \to [\gamma(a), \gamma(b)]$ a strictly increasing, continuous and twice differentiable function on (a, b). Assume that $\Psi \circ \gamma^{-1}$ is convex on $[\gamma(a), \gamma(b)]$. If $\gamma' \in L_\infty[a, b]$ and $\Delta(\Psi, \gamma) \in L_\infty[a, b]$, then*

$$\begin{aligned}(3.7)\qquad 0 &\le \frac{1}{b-a}\int_a^b \Psi(t)\,dt - \Psi \circ \gamma^{-1}\left(\frac{1}{b-a}\int_a^b \gamma(t)\,dt\right) \\ &\le \frac{1}{12}(b-a)^2 \|\gamma'\|_{[a,b],\infty} \|\Delta(\Psi, \gamma)\|_{[a,b],\infty}.\end{aligned}$$

Also, if we take $w=\gamma'$ in (3.6), then we get

$$0\leq\frac{1}{\gamma(b)-\gamma(a)}\int_a^b\Psi(t)\gamma'(t)\,dt-\Psi\circ\gamma^{-1}\left(\frac{\gamma(b)+\gamma(a)}{2}\right)\leq\frac{1}{12}\left[\gamma(b)-\gamma(a)\right]^2\left\|\frac{\Delta(\Psi,\gamma)}{\gamma'}\right\|_{[a,b],\infty},\tag{3.8}$$

provided $\frac{\Delta(\Psi,\gamma)}{\gamma'}\in L_\infty[a,b]$.

4. Applications for Some Particular Convexities

Let $\gamma:[a,b]\to[\gamma(a),\gamma(b)]$ be a *continuous strictly increasing function* that is *differentiable* on (a,b).

Definition 1. *A function $\Psi:[a,b]\to\mathbb{R}$ will be called composite-γ^{-1} convex (concave) on $[a,b]$ if the composite function $\Psi\circ\gamma^{-1}:[\gamma(a),\gamma(b)]\to\mathbb{R}$ is convex (concave) in the usual sense on $[\gamma(a),\gamma(b)]$.*

In this way, any concept of convexity (log-convexity, harmonic convexity, trigonometric convexity, hyperbolic convexity, h-convexity, quasi-convexity, s-convexity, s-Godunova-Levin convexity etc...) can be extended to the corresponding *composite-γ^{-1} convexity.* The details however will not be presented here.

If $\Psi:[a,b]\to\mathbb{R}$ is composite-γ^{-1} convex on $[a,b]$ then we have the inequality

$$\Psi\circ\gamma^{-1}((1-\lambda)u+\lambda v)\leq(1-\lambda)\Psi\circ\gamma^{-1}(u)+\lambda\Psi\circ\gamma^{-1}(v)\tag{4.1}$$

for any $u,\ v\in[\gamma(a),\gamma(b)]$ and $\lambda\in[0,1]$.

This is equivalent to the condition

$$\Psi\circ\gamma^{-1}((1-\lambda)\gamma(t)+\lambda\gamma(s))\leq(1-\lambda)\Psi(t)+\lambda\Psi(s)\tag{4.2}$$

for any $t,\ s\in[a,b]$ and $\lambda\in[0,1]$.

If we take $\gamma(t)=\ln t,\ t\in[a,b]\subset(0,\infty)$, then the condition (4.2) becomes

$$\Psi\left(t^{1-\lambda}s^{\lambda}\right)\leq(1-\lambda)\Psi(t)+\lambda\Psi(s)\tag{4.3}$$

for any $t,\ s\in[a,b]$ and $\lambda\in[0,1]$, which is the concept of *GA-convexity* as considered in [1].

If we take $\gamma(t)=-\frac{1}{t},\ t\in[a,b]\subset(0,\infty)$, then (4.2) becomes

$$\Psi\left(\frac{ts}{(1-\lambda)s+\lambda t}\right)\leq(1-\lambda)\Psi(t)+\lambda\Psi(s)\tag{4.4}$$

for any $t,\ s\in[a,b]$ and $\lambda\in[0,1]$, which is the concept of *HA-convexity* as considered in [1].

If $p>0$ and we consider $\gamma(t)=t^p,\ t\in[a,b]\subset(0,\infty)$, then the condition (4.2) becomes

$$\Psi\left[\left((1-\lambda)t^p+\lambda s^p\right)^{1/p}\right]\leq(1-\lambda)\Psi(t)+\lambda\Psi(s)\tag{4.5}$$

for any $t,\ s\in[a,b]$ and $\lambda\in[0,1]$, which is the concept of *p-convexity* as considered in [27].

If we take $\gamma(t)=\exp t,\ t\in[a,b]$, then the condition (4.2) becomes

$$\Psi\left[\ln((1-\lambda)\exp(t)+\exp\gamma(s))\right]\leq(1-\lambda)\Psi(t)+\lambda\Psi(s)\tag{4.6}$$

which is the concept of *LogExp convex function* on $[a,b]$ as considered in [16].

Further, assume that $\Psi : [a,b] \to J$, J an interval of real numbers and $\delta : J \to \mathbb{R}$ a continuous function on J that is *strictly increasing (decreasing)* on J.

Definition 2. *We say that the function $\Psi : [a,b] \to J$ is δ-composite convex (concave) on $[a,b]$, if $\delta \circ \Psi$ is convex (concave) on $[a,b]$.*

In this way, any concept of convexity as mentioned above can be extended to the corresponding *δ-composite convexity.* The details however will not be presented here.

With $\gamma : [a,b] \to [\gamma(a), \gamma(b)]$ a *continuous strictly increasing function* that is *differentiable* on (a,b), $\Psi : [a,b] \to J$, J an interval of real numbers and $\delta : J \to \mathbb{R}$ a continuous function on J that is *strictly increasing (decreasing)* on J, we can also consider the following concept:

Definition 3. *We say that the function $\Psi : [a,b] \to J$ is δ-composite-γ^{-1} convex (concave) on $[a,b]$, if $\delta \circ \Psi \circ \gamma^{-1}$ is convex (concave) on $[\gamma(a), \gamma(b)]$.*

This definition is equivalent to the condition

$$\delta \circ \Psi \circ \gamma^{-1}((1-\lambda)\gamma(t) + \lambda\gamma(s)) \le (1-\lambda)(\delta \circ \Psi)(t) + \lambda(\delta \circ \Psi)(s) \tag{4.7}$$

for any $t, s \in [a,b]$ and $\lambda \in [0,1]$.

If $\delta : J \to \mathbb{R}$ is *strictly increasing (decreasing)* on J, then the condition (4.7) is equivalent to:

$$\Psi \circ \gamma^{-1}((1-\lambda)\gamma(t) + \lambda\gamma(s)) \le (\ge)\, \delta^{-1}[(1-\lambda)(\delta \circ \Psi)(t) + \lambda(\delta \circ \Psi)(s)] \tag{4.8}$$

for any $t, s \in [a,b]$ and $\lambda \in [0,1]$.

If $\delta(t) = \ln t$, $t > 0$ and $\Psi : [a,b] \to (0,\infty)$, then the fact that Ψ is δ-composite convex on $[a,b]$ is equivalent to the fact that Ψ is *log-convex* or *multiplicatively convex* or *AG*-convex, namely, for all $x, y \in I$ and $t \in [0,1]$ one has the inequality:

$$\Psi(tx + (1-t)y) \le [\Psi(x)]^t [\Psi(y)]^{1-t}. \tag{4.9}$$

A function $\Psi : I \to \mathbb{R}\setminus\{0\}$ is called *AH-convex (concave)* on the interval I if the following inequality holds [1]

$$\Psi((1-\lambda)x + \lambda y) \le (\ge) \frac{1}{(1-\lambda)\frac{1}{\Psi(x)} + \lambda\frac{1}{\Psi(y)}} = \frac{\Psi(x)\Psi(y)}{(1-\lambda)\Psi(y) + \lambda\Psi(x)} \tag{4.10}$$

for any $x, y \in I$ and $\lambda \in [0,1]$.

An important case that provides many examples is that one in which the function is assumed to be positive for any $x \in I$. In that situation the inequality (4.10) is equivalent to

$$(1-\lambda)\frac{1}{\Psi(x)} + \lambda\frac{1}{\Psi(y)} \le (\ge) \frac{1}{\Psi((1-\lambda)x + \lambda y)}$$

for any $x, y \in I$ and $\lambda \in [0,1]$.

Taking into account this fact, we can conclude that the function $\Psi : I \to (0,\infty)$ is *AH*-convex (concave) on I if and only if Ψ is δ-composite concave (convex) on I with $\delta : (0,\infty) \to (0,\infty)$, $\delta(t) = \frac{1}{t}$.

Following [1], we can introduce the concept of *GH-convex (concave)* function $\Psi : I \subset (0,\infty) \to \mathbb{R}$ on an interval of positive numbers I as satisfying the condition

$$\Psi\left(x^{1-\lambda}y^{\lambda}\right) \le (\ge) \frac{1}{(1-\lambda)\frac{1}{\Psi(x)} + \lambda\frac{1}{\Psi(y)}} = \frac{\Psi(x)\Psi(y)}{(1-\lambda)\Psi(y) + \lambda\Psi(x)}. \tag{4.11}$$

Since

$$\Psi\left(x^{1-\lambda}y^{\lambda}\right)=\Psi\circ\exp\left[(1-\lambda)\ln x+\lambda\ln y\right]$$

and

$$\frac{\Psi(x)\Psi(y)}{(1-\lambda)\Psi(y)+\lambda\Psi(x)}=\frac{\Psi\circ\exp(\ln x)\Psi\circ\exp(\ln y)}{(1-\lambda)\Psi\circ\exp(y)+\lambda\Psi\circ\exp(x)}$$

then $\Psi:I\subset(0,\infty)\rightarrow\mathbb{R}$ is GH-convex (concave) on I if and only if $\Psi\circ\exp$ is *AH-convex (concave)* on $\ln I:=\{x|\ x=\ln t,\ t\in I\}$. This is equivalent to the fact that Ψ is δ-composite-γ^{-1} concave (convex) on I with $\delta:(0,\infty)\rightarrow(0,\infty)$, $\delta(t)=\frac{1}{t}$ and $\gamma(t)=\ln t,\ t\in I.$

Following [1], we say that the function $\Psi:I\subset\mathbb{R}\setminus\{0\}\rightarrow(0,\infty)$ is *HH-convex* if

$$\Psi\left(\frac{xy}{tx+(1-t)y}\right)\leq\frac{\Psi(x)\Psi(y)}{(1-t)\Psi(y)+t\Psi(x)} \tag{4.12}$$

for all $x,\ y\in I$ and $t\in[0,1]$. If the inequality in (4.12) is reversed, then Ψ is said to be *HH-concave*.

We observe that the inequality (4.12) is equivalent to

$$(1-t)\frac{1}{\Psi(x)}+t\frac{1}{\Psi(y)}\leq\frac{1}{\Psi\left(\frac{xy}{tx+(1-t)y}\right)} \tag{4.13}$$

for all $x,\ y\in I$ and $t\in[0,1]$.

This is equivalent to the fact that Ψ is δ-composite-γ^{-1} concave on $[a,b]$ with $\delta:(0,\infty)\rightarrow(0,\infty)$, $\delta(t)=\frac{1}{t}$ and $\gamma(t)=-\frac{1}{t},\ t\in[a,b]$.

The function $\Psi:I\subset(0,\infty)\rightarrow(0,\infty)$ is called *GG-convex* on the interval I of real umbers $\mathbb{R}$ if [1]

$$\Psi\left(x^{1-\lambda}y^{\lambda}\right)\leq[\Psi(x)]^{1-\lambda}[\Psi(y)]^{\lambda} \tag{4.14}$$

for any $x,\ y\in I$ and $\lambda\in[0,1]$. If the inequality is reversed in (4.14) then the function is called *GG-concave*.

This concept was introduced in 1928 by P. Montel [23], however, the roots of the research in this area can be traced long before him [24]. It is easy to see that [24], the function $\Psi:[a,b]\subset(0,\infty)\rightarrow(0,\infty)$ is *GG-convex* if and only if the the function $\gamma:[\ln a,\ln b]\rightarrow\mathbb{R}$, $\gamma=\ln\circ\Psi\circ\exp$ is convex on $[\ln a,\ln b]$. This is equivalent to the fact that Ψ is δ-composite-γ^{-1} convex on $[a,b]$ with $\delta:(0,\infty)\rightarrow\mathbb{R}$, $\delta(t)=\ln t$ and $\gamma(t)=\ln t,\ t\in[a,b]$.

Following [1] we say that the function $\Psi:I\subset\mathbb{R}\setminus\{0\}\rightarrow(0,\infty)$ is *HG-convex* if

$$\Psi\left(\frac{xy}{tx+(1-t)y}\right)\leq[\Psi(x)]^{1-t}[\Psi(y)]^{t} \tag{4.15}$$

for all $x,\ y\in I$ and $t\in[0,1]$. If the inequality in (4.2) is reversed, then Ψ is said to be *HG-concave*.

Let $\Psi:[a,b]\subset(0,\infty)\rightarrow(0,\infty)$ and define the associated functions $G_{\Psi}:\left[\frac{1}{b},\frac{1}{a}\right]\rightarrow\mathbb{R}$ defined by $G_{\Psi}(t)=\ln\Psi\left(\frac{1}{t}\right)$. Then Ψ is *HG-convex* on $[a,b]$ iff G_{Ψ} is convex on $\left[\frac{1}{b},\frac{1}{a}\right]$. This is equivalent to the fact that Ψ is δ-composite-γ^{-1} convex on $[a,b]$ with $\delta:(0,\infty)\rightarrow\mathbb{R}$, $\delta(t)=\ln t$ and $\gamma(t)=-\frac{1}{t},\ t\in[a,b]$.

Following [26], we say that the function $\Psi:[a,b]\rightarrow(0,\infty)$ is r-convex, for $r\neq0$, if

$$\Psi((1-\lambda)x+\lambda y)\leq[(1-\lambda)\Psi^{r}(y)+\lambda\Psi^{r}(x)]^{1/r} \tag{4.16}$$

for any $x,\, y \in [a,b]$ and $\lambda \in [0,1]$.

If $r > 0$, then the condition (4.16) is equivalent to

$$\Psi^r\left((1-\lambda)x+\lambda y\right) \le (1-\lambda)\Psi^r(y)+\lambda\Psi^r(x)$$

namely Ψ is δ-composite convex on $[a,b]$ where $\delta(t)=t^r$, $t\ge 0$.

If $r < 0$, then the condition (4.16) is equivalent to

$$\Psi^r\left((1-\lambda)x+\lambda y\right) \ge (1-\lambda)\Psi^r(y)+\lambda\Psi^r(x)$$

namely Ψ is δ-composite concave on $[a,b]$ where $\delta(t)=t^r$, $t>0$.

For some results related to these concepts of convexity, see [9]-[15].

We assume in the following that $w:[a,b]\to(0,\infty)$ is a continuous function on $[a,b]$ and $g:[a,b]\to[m,M]$ is absolutely continuous on $[a,b]$.

If Ψ is *log convex* on $[m,M]$, then Ψ is δ-composite-γ^{-1} convex on $[a,b]$ with $\delta:(0,\infty)\to\mathbb{R}$, $\delta(t)=\ln t$ and $\gamma(t)=\ell(t)=t$, $t\in[a,b]$. If we use the inequality (2.11) and assume that Ψ is twice differentiable on (m,M), then we have

$$\begin{aligned}(4.17)\qquad 0 &\le \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\ln(\Psi\circ g)(t)\,dt-\ln\left[\Psi\left(\frac{\int_a^b w(t)g(t)\,dt}{\int_a^b w(s)\,ds}\right)\right]\\ &\le \frac{1}{12}\left\|\frac{g'}{w}\right\|_{[a,b],\infty}\left\|\frac{\Delta(\ln\Psi,g)}{w}\right\|_{[a,b],\infty}\left(\int_a^b w(s)\,ds\right)^2,\end{aligned}$$

where

$$\Delta(\ln\Psi,g)(t)=\frac{(\Psi''\circ g)(t)(\Psi\circ g)(t)-\left((\Psi'\circ g)(t)\right)^2}{\left((\Psi\circ g)(t)\right)^2},\ t\in[a,b]$$

and provided that $\frac{g'}{w}\in L_\infty[a,b]$ and $\frac{\Delta(\ln\Psi,g)}{w}\in L_\infty[a,b]$.

If Ψ is *GA-convex* on $[a,b]\subset(0,\infty)$, then Ψ is δ-composite-γ^{-1} convex on $[a,b]$ with $\gamma:(0,\infty)\to\mathbb{R}$, $\gamma(t)=\ln t$ and $\delta(t)=\ell(t)=t$, $t\in[a,b]$. If we use the inequality (2.11) and assume that Ψ is twice differentiable on (m,M), then we have

$$\begin{aligned}(4.18)\qquad 0 &\le \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Psi\circ g)(t)\,dt-\Psi\left[\exp\left(\frac{\int_a^b w(t)\ln g(t)\,dt}{\int_a^b w(s)\,ds}\right)\right]\\ &\le \frac{1}{12}\left\|\frac{g'}{wg}\right\|_{[a,b],\infty}\left\|\frac{\Delta(\Psi,\ln,g)}{w}\right\|_{[a,b],\infty}\left(\int_a^b w(s)\,ds\right)^2,\end{aligned}$$

where

$$\Delta(\Psi,\ln,g)(t)=(\Psi''\circ g)(t)\,g(t)+(\Psi'\circ g)(t)$$

and provided that $\frac{g'}{wg},\ \frac{\Delta(\Psi,\ln,g)}{w}\in L_\infty[a,b]$.

The function $\Psi:[a,b]\to(0,\infty)$ is *AH-convex* on $[a,b]$ if and only if Ψ is δ-composite-γ^{-1} concave on $[a,b]$ with $\delta:(0,\infty)\to(0,\infty)$, $\delta(t)=\frac{1}{t}$ and and $\gamma(t)=\ell(t)=t$, $t\in[a,b]$. If we use the inequality (2.11) for the convex function

$-\Psi^{-1}$ and assume that Ψ is twice differentiable on (m,M), then we have

$$0 \le \left[\Psi\left(\frac{\int_a^b w(t)\,g(t)\,dt}{\int_a^b w(s)\,ds}\right)\right]^{-1} - \frac{1}{\int_a^b w(s)\,ds}\int_a^b \frac{w(t)}{(\Psi\circ g)(t)}dt \le \frac{1}{12}\left\|\frac{g'}{w}\right\|_{[a,b],\infty}\left\|\frac{\Delta\left(-\Psi^{-1},g\right)}{w}\right\|_{[a,b],\infty}\left(\int_a^b w(s)\,ds\right)^2, \tag{4.19}$$

where

$$\Delta\left(-\Psi^{-1},g\right)(t) := \frac{(\Psi''\circ g)(t)\,(\Psi\circ g)(t) - 2\left((\Psi'\circ g)(t)\right)^2}{\left((\Psi\circ g)(t)\right)^3}$$

and provided that $\frac{g'}{w}$, $\frac{\Delta\left(-\Psi^{-1},g\right)}{w} \in L_\infty[a,b]$.

If the function Ψ is HA-*convex* $[a,b]$, then Ψ is δ-composite-γ^{-1} convex on $[a,b]$ with $\gamma:(0,\infty)\to\mathbb{R}$, $\gamma(t)=-t^{-1}$ and $\delta(t)=\ell(t)=t$, $t\in[a,b]$. If we use the inequality (2.17) and assume that Ψ is twice differentiable on (m,M), then we have

$$0 \le \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\,(\Psi\circ g)(t)\,dt - \Psi\left(\frac{\int_a^b w(s)\,ds}{\int_a^b \frac{w(t)}{g(t)}dt}\right) \le \frac{1}{12}\left\|\frac{g'}{wg^2}\right\|_{[a,b],\infty}\left\|\frac{\Delta\left(\Psi,-\ell^{-1},g\right)}{w}\right\|_{[a,b],\infty}\left(\int_a^b w(s)\,ds\right)^2, \tag{4.20}$$

where

$$\Delta\left(\Psi,-\ell^{-1},g\right)(t) := (\Psi''\circ g)(t)\,g^2(t) + 2g(t)\,(\Psi'\circ g)(t)$$

and provided that $\frac{g'}{wg^2}$, $\frac{\Delta\left(\Psi,-\ell^{-1},g\right)}{w} \in L_\infty[a,b]$.

Similar results may be stated for the other concepts of convexity as presented above, however the details are omitted.

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