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REVERSES OF JENSEN'S INTEGRAL INEQUALITY VIA A WEIGHTED LUPAŞ TYPE RESULT WITH APPLICATIONS FOR COMPOSITE CONVEX FUNCTIONS

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Abstract. In this paper we obtain some reverses of Jensen's integral inequality by employing a new weighted integral inequality of Lupaş type. Applications for general composite convex functions with examples for AG, GA-convex functions and HA, AH-convex function are also given.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ-a.e. (almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega w(x)|f(x)|\,d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S. S. Dragomir obtained in 2002 [4] the following result:

Theorem 1. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. (almost everywhere) on Ω with $\int_\Omega w d\mu = 1$. Then we have the inequality:*

$$0 \leq \int_\Omega w(\Phi \circ f)\,d\mu - \Phi\left(\int_\Omega w f d\mu\right) \leq \int_\Omega w(\Phi' \circ f) f d\mu - \int_\Omega w(\Phi' \circ f)\,d\mu \int_\Omega w f d\mu. \tag{1.1}$$

Let $\Phi : [m, M] \to \mathbb{R}$ be a differentiable convex function on (m, M). If $x_i \in [m, M]$ and $w_i \geq 0$ $(i = 1, \ldots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then one has the reverse of

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Jensen's weighted discrete inequality:

$$0 \leq \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i\right) \leq \sum_{i=1}^{n} w_i \Phi'(x_i) x_i - \sum_{i=1}^{n} w_i \Phi'(x_i) \sum_{i=1}^{n} w_i x_i. \tag{1.2}$$

The inequality (1.2) was obtained in 1994 by Dragomir & Ionescu, see [19].

The following result providing a sequence of bounds for the Jensen's gap [4]:

Theorem 2. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. (almost everywhere) on Ω with $\int_\Omega w d\mu = 1$. Then we have the sequence of inequalities:*

$$\begin{aligned} 0 &\leq \int_\Omega w (\Phi \circ f) d\mu - \Phi\left(\int_\Omega w f d\mu\right) \\ &\leq \int_\Omega w (\Phi' \circ f) f d\mu - \int_\Omega w (\Phi' \circ f) d\mu \int_\Omega w f d\mu \\ &\leq \frac{1}{2} \begin{cases} \left[\Phi'_-(M) - \Phi'_+(m)\right] \int_\Omega w \left| f - \int_\Omega w f d\mu \right| d\mu \\ (M - m) \int_\Omega w \left| \Phi' \circ f - \int_\Omega w (\Phi' \circ f) d\mu \right| d\mu \end{cases} \\ &\leq \frac{1}{2} \begin{cases} \left[\Phi'_-(M) - \Phi'_+(m)\right] \left[\int_\Omega w f^2 d\mu - \left(\int_\Omega w f d\mu\right)^2\right]^{\frac{1}{2}} \\ (M - m) \left[\int_\Omega w (\Phi' \circ f)^2 d\mu - \left(\int_\Omega w (\Phi' \circ f) d\mu\right)^2\right]^{\frac{1}{2}} \end{cases} \\ &\leq \frac{1}{4} (M - m) \left[\Phi'_-(M) - \Phi'_+(m)\right]. \end{aligned} \tag{1.3}$$

For other similar reverses of Jensen's integral inequality in the general setting of Lebesgue integral, see [6]-[8].

In the recent paper [17], by the use of a weighted version of Ostrowski's inequality, we obtained the following reverse of Jensen's integral inequality for functions of a real variable:

Theorem 3. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M), $w : [a, b] \to (0, \infty)$ be continuous on $[a, b]$ and $f : [a, b] \to [m, M]$ be absolutely continuous so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_w[a, b]$.*

(i) *If $\frac{f'}{w} \in L_\infty[a, b]$, then we have the inequality*

$$\begin{aligned} 0 &\leq \frac{1}{\int_a^b w(s) ds} \int_a^b w(t) (\Phi \circ f)(t) dt - \Phi\left(\frac{\int_a^b w(t) f(t) dt}{\int_a^b w(s) ds}\right) \\ &\leq \frac{1}{8} \left[\Phi'_-(M) - \Phi'_+(m)\right] \left\| \frac{f'}{w} \right\|_{[a,b],\infty} \int_a^b w(s) ds. \end{aligned} \tag{1.4}$$

(ii) *If Φ is twice differentiable on (m,M) and $\frac{(\Phi''\circ f)f'}{w} \in L_\infty[a,b]$, then*

$$0 \leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi\circ f)(t)\,dt - \Phi\left(\frac{\int_a^b w(t) f(t)\,dt}{\int_a^b w(s)\,ds}\right) \leq \frac{1}{8}(M-m)\left\|\frac{(\Phi''\circ f) f'}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds. \tag{1.5}$$

This result has the following particular cases of interest:

Corollary 1. *Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m,M) and $f : [a,b] \to [m,M]$ be absolutely continuous so that $\Phi\circ f$, f, $\Phi'\circ f$, $(\Phi'\circ f) f \in L[a,b]$.*

(i) *If $f' \in L_\infty[a,b]$, then we have the inequality*

$$0 \leq \frac{1}{b-a}\int_a^b (\Phi\circ f)(t)\,dt - \Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \leq \frac{1}{8}(b-a)\left[\Phi'_-(M) - \Phi'_+(m)\right]\|f'\|_{[a,b],\infty}. \tag{1.6}$$

(ii) *If Φ is twice differentiable on (m,M) and $(\Phi''\circ f) f' \in L_\infty[a,b]$, then*

$$0 \leq \frac{1}{b-a}\int_a^b (\Phi\circ f)(t)\,dt - \Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \leq \frac{1}{8}(b-a)(M-m)\|(\Phi''\circ f) f'\|_{[a,b],\infty}. \tag{1.7}$$

Corollary 2. *Let $\Phi : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (a,b), $w : [a,b] \to (0,\infty)$ be continuous on $[a,b]$ and Φ, $\Phi' \in L_w[a,b]$.*

(i) *If $\frac{1}{w} \in L_\infty[a,b]$, then we have the inequality*

$$0 \leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\Phi(t)\,dt - \Phi\left(\frac{\int_a^b t w(t)\,dt}{\int_a^b w(s)\,ds}\right) \leq \frac{1}{8}\left[\Phi'_-(b) - \Phi'_+(a)\right]\left\|\frac{1}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds. \tag{1.8}$$

(ii) *If f Φ is twice differentiable on (m,M) and $\frac{\Phi''}{w} \in L_\infty[a,b]$, then*

$$0 \leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\Phi(t)\,dt - \Phi\left(\frac{\int_a^b t w(t)\,dt}{\int_a^b w(s)\,ds}\right) \leq \frac{1}{8}(b-a)\left\|\frac{\Phi''}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds. \tag{1.9}$$

By employing a new weighted integral inequality of Čebyšev type, in the recent paper [18], we obtained the following reverse of Jensen's integral inequality:

Theorem 4. *Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on (m,M), $w : [a,b] \to (0,\infty)$ be continuous on $[a,b]$ and $f : [a,b] \to [m,M]$ be* *absolutely continuous so that* $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_w [a,b]$. *If* $\frac{f'}{w} \in L_\infty [a,b]$ *and* $\frac{(\Phi'' \circ f) f'}{w} \in L_\infty [a,b]$, *then we have the inequality*

$$0 \le \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t) (\Phi \circ f)(t)\,dt - \Phi\left(\frac{\int_a^b w(t) f(t)\,dt}{\int_a^b w(s)\,ds} \right) \le \frac{1}{12} \left\| \frac{f'}{w} \right\|_{[a,b],\infty} \left\| \frac{(\Phi'' \circ f) f'}{w} \right\|_{[a,b],\infty} \left(\int_a^b w(s)\,ds \right)^2 . \tag{1.10}$$

The following particular cases are of interest:

Corollary 3. *Let* $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ *be a twice differentiable convex function on* (m,M) *and* $f : [a,b] \to [m,M]$ *be absolutely continuous on* $[a,b]$. *If* $f' \in L_\infty [a,b]$ *and* $(\Phi'' \circ f) f' \in L_\infty [a,b]$, *then we have the inequality*

$$0 \le \frac{1}{b-a} \int_a^b (\Phi \circ f)(t)\,dt - \Phi\left(\frac{1}{b-a} \int_a^b f(t)\,dt \right) \le \frac{1}{12} \|f'\|_{[a,b],\infty} \|(\Phi'' \circ f) f'\|_{[a,b],\infty} (b-a)^2 . \tag{1.11}$$

Corollary 4. *Let* $\Phi : [a,b] \subset \mathbb{R} \to \mathbb{R}$ *be a twice differentiable convex function on* (a,b), $w : [a,b] \to (0,\infty)$ *a continuous function on* $[a,b]$ *and* $\Phi,\ \Phi' \in L_w [a,b]$. *If* $\frac{1}{w} \in L_\infty [a,b]$ *and* $\frac{\Phi''}{w} \in L_\infty [a,b]$, *then we have the inequality*

$$0 \le \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t) \Phi(t)\,dt - \Phi\left(\frac{\int_a^b t w(t)\,dt}{\int_a^b w(s)\,ds} \right) \le \frac{1}{12} \left\| \frac{1}{w} \right\|_{[a,b],\infty} \left\| \frac{\Phi''}{w} \right\|_{[a,b],\infty} \left(\int_a^b w(s)\,ds \right)^2 . \tag{1.12}$$

Motivated by the above results, in this paper we obtain some reverses of Jensen's integral inequality by employing a new weighted integral inequality of Lupaş type. Applications for general composite convex functions with examples for AG, GA-convex functions and HA, AH-convex function are also given.

2. Reverses of Jensen's Inequality Via a Weighted Lupaş Result

For two *Lebesgue integrable* functions $f,\ g : [a,b] \to \mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g) := \frac{1}{b-a} \int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2} \int_a^b f(t)dt \int_a^b g(t)dt. \tag{2.1}$$

In 1935, Grüss [21] showed that

$$|C(f,g)| \le \frac{1}{4} (M-m)(N-n), \tag{2.2}$$

provided that there exists the real numbers m, M, n, N such that

$$m \le f(t) \le M \quad \text{and} \quad n \le g(t) \le N \quad \text{for a.e. } t \in [a,b]. \tag{2.3}$$

The constant $\frac{1}{4}$ is best possible in (2.1) in the sense that it cannot be replaced by a smaller quantity.

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [22] in which he proved that

$$|C(f,g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a), \tag{2.4}$$

provided that f, g are absolutely continuous and $f', g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Consider the functional:

$$\begin{aligned} C_{h'}(f,g) := & \frac{1}{h(b)-h(a)} \int_a^b f(t)\, g(t)\, h'(t)\, dt \\ & - \frac{1}{h(b)-h(a)} \int_a^b f(t)\, h'(t)\, dt \frac{1}{h(b)-h(a)} \int_a^b g(t)\, h'(t)\, dt, \end{aligned} \tag{2.5}$$

where h is absolutely continuous and f, g are Lebesgue measurable on $[a,b]$ and such that the above integrals exist.

We also have the following weighted version of Lupaş inequality:

Lemma 1. *Let $h:[a,b] \to [h(a), h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If f, $g:[a,b] \to \mathbb{R}$ are absolutely continuous on $[a,b]$ and $\frac{f'}{(h')^{1/2}}, \frac{g'}{(h')^{1/2}} \in L_2[a,b]$, then we have*

$$|C_{h'}(f,g)| \leq \frac{1}{\pi^2} \left\| \frac{f'}{(h')^{1/2}} \right\|_{[a,b],2} \left\| \frac{g'}{(h')^{1/2}} \right\|_{[a,b],2} [h(b)-h(a)]. \tag{2.6}$$

The constant $\frac{1}{\pi^2}$ is best possible.

Proof. Assume that $[c,d] \subset [a,b]$. If $g:[c,d] \to \mathbb{C}$ is absolutely continuous on $[c,d]$, then $g \circ h^{-1}:[h(c), h(d)] \to \mathbb{C}$ is absolutely continuous on $[h(c), h(d)]$ and using the chain rule and the derivative of inverse functions we have

$$\left(g \circ h^{-1}\right)'(z) = \left(g' \circ h^{-1}\right)(z) \left(h^{-1}\right)'(z) = \frac{\left(g' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \tag{2.7}$$

for almost every (a.e.) $z \in [h(c), h(d)]$.

Using the identity (2.7) above, we have

$$\int_{h(a)}^{h(b)} \left|\left(g \circ h^{-1}\right)'(u)\right|^2 du = \int_{h(a)}^{h(b)} \left|\frac{\left(g' \circ h^{-1}\right)(u)}{\left(h' \circ h^{-1}\right)(u)}\right|^2 du.$$

By the change of variable $t = h^{-1}(u)$, $u \in [h(a), h(b)]$, we have $u = h(t)$ that gives $du = h'(t)\, dt$. Therefore

$$\begin{aligned} \int_{h(a)}^{h(b)} \left|\frac{\left(g' \circ h^{-1}\right)(u)}{\left(h' \circ h^{-1}\right)(u)}\right|^2 du &= \int_b^b \left|\frac{g'(t)}{h'(t)}\right|^2 h'(t)\, dt \\ &= \int_b^b \left|\frac{g'(t)}{[h'(t)]^{1/2}}\right|^2 dt = \left\|\frac{g'}{(h')^{1/2}}\right\|_{[a,b],2}^2. \end{aligned}$$

In a similar way, we also have

$$\int_{h(a)}^{h(b)} \left|\frac{\left(f' \circ h^{-1}\right)(u)}{\left(h' \circ h^{-1}\right)(u)}\right|^2 du = \left\|\frac{f'}{(h')^{1/2}}\right\|_{[a,b],2}^2.$$

This mean that

$$\left\| \left(f\circ h^{-1}\right)' \right\|_{[h(a),h(b)],2} = \left\| \frac{f'}{(h')^{1/2}} \right\|_{[a,b],2}$$

and

$$\left\| \left(g\circ h^{-1}\right)' \right\|_{[h(a),h(b)],2} = \left\| \frac{g'}{(h')^{1/2}} \right\|_{[a,b],2}.$$

By making use of Lupaş inequality (2.4) for the functions $f\circ h^{-1}$ and $g\circ h^{-1}$ on the interval $[h(a), h(b)]$ we get

$$\begin{aligned} (2.8)\quad & \left| \frac{1}{h(b)-h(a)} \int_{h(a)}^{h(b)} f\circ h^{-1}(u) g\circ h^{-1}(u) du \right. \\ & \left. - \frac{1}{[h(b)-h(a)]^2} \int_{h(a)}^{h(b)} f\circ h^{-1}(u) du \int_{h(a)}^{h(b)} g\circ h^{-1}(u) du \right| \\ & \leq \frac{1}{\pi^2} \left\| \left(f\circ h^{-1}\right)' \right\|_{[h(a),h(b)],2} \left\| \left(g\circ h^{-1}\right)' \right\|_{[h(a),h(b)],2} [h(b)-h(a)]. \end{aligned}$$

Observe also that, by the change of variable $t = h^{-1}(u)$, $u \in [g(a), g(b)]$, we have $u = h(t)$ that gives $du = h'(t)\, dt$ and

$$\int_{h(a)}^{h(b)} \left(f\circ h^{-1}\right)(u)\, du = \int_a^b f(t) h'(t)\, dt,$$

$$\int_{h(a)}^{h(b)} g\circ h^{-1}(u) du = \int_a^b g(t) h'(t)\, dt,$$

$$\int_{h(a)}^{h(b)} f\circ h^{-1}(u) g\circ h^{-1}(u) du = \int_a^b f(t) g(t) h'(t)\, dt,$$

which together with (2.8) produces the desired result (2.6). □

Corollary 5. *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$. If $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous on $[a,b]$ and $\frac{f'}{w^{1/2}}, \frac{g'}{w^{1/2}} \in L_2[a,b]$, then we have*

$$(2.9)\qquad |C_w(f,g)| \leq \frac{1}{\pi^2} \left\| \frac{f'}{w^{1/2}} \right\|_{[a,b],2} \left\| \frac{g'}{w^{1/2}} \right\|_{[a,b],2} \int_a^b w(s)\, ds.$$

We have the following reverse of Jensen's integral inequality:

Theorem 5. *Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on (m,M), $w : [a,b] \to (0,\infty)$ be continuous on $[a,b]$ and $f : [a,b] \to [m,M]$ be absolutely continuous so that $\Phi\circ f$, f, $\Phi'\circ f$, $(\Phi'\circ f) f \in L_w[a,b]$. If $\frac{f'}{w^{1/2}} \in L_2[a,b]$ and $\frac{(\Phi''\circ f)f'}{w^{1/2}} \in L_2[a,b]$, then we have the inequality*

$$\begin{aligned} (2.10)\qquad 0 & \leq \frac{1}{\int_a^b w(s)\, ds} \int_a^b w(t) (\Phi\circ f)(t)\, dt - \Phi\left(\frac{\int_a^b w(t) f(t)\, dt}{\int_a^b w(s)\, ds} \right) \\ & \leq \frac{1}{\pi^2} \left\| \frac{f'}{w^{1/2}} \right\|_{[a,b],2} \left\| \frac{(\Phi''\circ f) f'}{w^{1/2}} \right\|_{[a,b],2} \int_a^b w(s)\, ds. \end{aligned}$$

Proof. By (4.14) we have

$$
\begin{aligned}
0 &\le \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi\circ f)(t)\,dt - \Phi\left(\frac{\int_a^b w(t) f(t)\,dt}{\int_a^b w(s)\,ds}\right) \\
&\le \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi'\circ f)(t) f(t)\,dt \\
&\quad - \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi'\circ f)(t)\,dt \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t) f(t)\,dt.
\end{aligned} \tag{2.11}
$$

Since Φ is twice differentiable on (a,b), then

$$
(\Phi'\circ f)'(t) = (\Phi''\circ f)(t) f'(t)
$$

for $t\in(a,b)$.

If we use the inequality (2.9), then we get

$$
\begin{aligned}
&\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi\circ f)(t) f(t)\,dt \\
&- \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi'\circ f)(t)\,dt \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t) f(t)\,dt \\
&\le \frac{1}{\pi^2}\left\|\frac{(\Phi'\circ f)'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{f'}{w^{1/2}}\right\|_{[a,b],2}\int_a^b w(s)\,ds \\
&= \frac{1}{\pi^2}\left\|\frac{(\Phi''\circ f) f'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{f'}{w^{1/2}}\right\|_{[a,b],2}\int_a^b w(s)\,ds,
\end{aligned}
$$

which, together with (2.11), proves the required inequality (2.10). □

Corollary 6. *Let $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable convex function on (m,M) and $f:[a,b]\to[m,M]$ be absolutely continuous on $[a,b]$. If $f'\in L_2[a,b]$ and $(\Phi''\circ f) f'\in L_2[a,b]$, then we have the inequality*

$$
\begin{aligned}
0 &\le \frac{1}{b-a}\int_a^b (\Phi\circ f)(t)\,dt - \Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \\
&\le \frac{1}{\pi^2}\|f'\|_{[a,b],2}\|(\Phi''\circ f) f'\|_{[a,b],2}(b-a).
\end{aligned} \tag{2.12}
$$

Corollary 7. *Let $\Phi:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable convex function on (a,b), $w:[a,b]\to(0,\infty)$ a continuous function on $[a,b]$ and Φ, $\Phi'\in L_w[a,b]$. If $\frac{1}{w^{1/2}}\in L_2[a,b]$ and $\frac{\Phi''}{w^{1/2}}\in L_2[a,b]$, then we have the inequality*

$$
\begin{aligned}
0 &\le \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\Phi(t)\,dt - \Phi\left(\frac{\int_a^b t w(t)\,dt}{\int_a^b w(s)\,ds}\right) \\
&\le \frac{1}{\pi^2}\left\|\frac{1}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{\Phi''}{w^{1/2}}\right\|_{[a,b],2}\int_a^b w(s)\,ds.
\end{aligned} \tag{2.13}
$$

Define the function $\ell(t):=t$, $t\in\mathbb{R}$.

a). Let $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable convex function on (m,M) and $f:[a,b]\subset(0,\infty)\to[m,M]$ be absolutely continuous and so that $\Phi\circ f$, f,

$\Phi' \circ f,\ (\Phi' \circ f) f \in L_{\ell^{-1}}[a,b]$. If $f'\ell^{1/2} \in L_2[a,b]$ and $(\Phi'' \circ f) f'\ell^{1/2} \in L_2[a,b]$, then by (2.10) for $w(t) = \frac{1}{t}$, we have

$$
\begin{aligned}
(2.14)\qquad 0 &\leq \frac{1}{\ln\left(\frac{b}{a}\right)} \int_a^b \frac{(\Phi \circ f)(t)}{t} dt - \Phi\left(\frac{\int_a^b \frac{f(t)}{t} dt}{\ln\left(\frac{b}{a}\right)}\right) \\
&\leq \frac{1}{\pi^2} \left\| f'\ell^{1/2} \right\|_{[a,b],2} \left\| (\Phi'' \circ f) f'\ell^{1/2} \right\|_{[a,b],2} \ln\left(\frac{b}{a}\right).
\end{aligned}
$$

b). Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on (m,M) and $f : [a,b] \to [m,M]$ be absolutely continuous and so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_{\exp}[a,b]$. If $\frac{f'}{\exp^{1/2}} \in L_2[a,b]$ and $\frac{(\Phi'' \circ f) f'}{\exp^{1/2}} \in L_2[a,b]$, then by (2.10) for $w(t) = \exp(t)$, we have

$$
\begin{aligned}
(2.15)\qquad 0 &\leq \frac{1}{\exp b - \exp a} \int_a^b (\Phi \circ f)(t) \exp t dt - \Phi\left(\frac{\int_a^b f(t) \exp t dt}{\exp b - \exp a}\right) \\
&\leq \frac{1}{\pi^2} \left\| \frac{f'}{\exp^{1/2}} \right\|_{[a,b],2} \left\| \frac{(\Phi'' \circ f) f'}{\exp^{1/2}} \right\|_{[a,b],2} (\exp b - \exp a).
\end{aligned}
$$

c). Consider the function $\ell^p(t) := t^p,\ t > 0,\ p \in \mathbb{R} \setminus \{-1\}$. Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on (m,M) and $f : [a,b] \subset (0,\infty) \to [m,M]$ be absolutely continuous and so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_{\ell^p}[a,b]$. If $f'\ell^{-p/2} \in L_2[a,b]$ and $(\Phi'' \circ f) f'\ell^{-p/2} \in L_2[a,b]$, then by (2.10) for $w(t) = \ell^p$, we have

$$
\begin{aligned}
(2.16)\qquad 0 &\leq \frac{p+1}{b^{p+1} - a^{p+1}} \int_a^b t^p (\Phi \circ f)(t) dt - \Phi\left(\frac{(p+1)\int_a^b t^p f(t) dt}{b^{p+1} - a^{p+1}}\right) \\
&\leq \frac{1}{\pi^2 (p+1)} \left\| f'\ell^{-p/2} \right\|_{[a,b],2} \left\| (\Phi'' \circ f) f'\ell^{-p/2} \right\|_{[a,b],2} \left(b^{p+1} - a^{p+1}\right).
\end{aligned}
$$

For $p = -2$, we get from (2.16) that

$$
\begin{aligned}
(2.17)\qquad 0 &\leq \frac{ab}{b-a} \int_a^b \frac{(\Phi \circ f)(t)}{t^2} dt - \Phi\left(\frac{ab}{b-a} \int_a^b \frac{f(t)}{t^2} dt\right) \\
&\leq \frac{1}{\pi^2} \left(\frac{b-a}{ab}\right) \| f'\ell \|_{[a,b],2} \| (\Phi'' \circ f) f'\ell \|_{[a,b],2},
\end{aligned}
$$

provided $f'\ell,\ (\Phi'' \circ f) f'\ell \in L_2[a,b]$.

3. Inequalities for Composite Convexity

We have the following result for composite convexity:

Theorem 6. *Let $\Psi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on (m,M), $\gamma : [m,M] \to [\gamma(m), \gamma(M)]$ a strictly increasing, continuous and twice differentiable function on (m,M), $w : [a,b] \to (0,\infty)$ a continuous function on $[a,b]$ and $g : [a,b] \to [m,M]$ an absolutely continuous function on $[a,b]$. Assume that $\Psi \circ \gamma^{-1}$ is convex on $[\gamma(m), \gamma(M)]$ and $\Psi \circ g,\ \gamma \circ g \in L_w[a,b]$. Define*

$$
(3.1)\qquad \Delta(\Psi, \gamma, g)(t) := \frac{(\Psi'' \circ g)(t)(\gamma' \circ g)(t) - (\Psi' \circ g)(t)(\gamma'' \circ g)(t)}{[(\gamma' \circ g)(t)]^2}
$$

for $t \in [a,b]$.

If $\frac{(\gamma'\circ g)g'}{w^{1/2}} \in L_2[a,b]$ *and* $\frac{\Delta(\Psi,\gamma,g)}{w^{1/2}} \in L_2[a,b]$, *then*

$$0 \le \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Psi\circ g)(t)\,dt - \Psi\circ\gamma^{-1}\left(\frac{\int_a^b w(t)(\gamma\circ g)(t)\,dt}{\int_a^b w(s)\,ds}\right) \le \frac{1}{\pi^2}\left\|\frac{(\gamma'\circ g)g'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{\Delta(\Psi,\gamma,g)}{w^{1/2}}\right\|_{[a,b],2}\int_a^b w(s)\,ds. \tag{3.2}$$

Proof. If we write the inequality (2.10) for the convex function $\Phi = \Psi\circ\gamma^{-1}$ on $[\gamma(m),\gamma(M)]$ and for the function $f = \gamma\circ g$ on $[a,b]$, then we have

$$\begin{aligned} 0 &\le \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\left(\Psi\circ\gamma^{-1}\circ\gamma\circ g\right)(t)\,dt \\ &\qquad - \Psi\circ\gamma^{-1}\left(\frac{\int_a^b w(t)(\gamma\circ g)(t)\,dt}{\int_a^b w(s)\,ds}\right) \\ &\le \frac{1}{\pi^2}\left\|\frac{(\gamma\circ g)'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{\left(\Psi\circ\gamma^{-1}\right)''((\gamma\circ g))\cdot(\gamma'\circ g)}{w^{1/2}}\right\|_{[a,b],2}\int_a^b w(s)\,ds. \end{aligned} \tag{3.3}$$

Using the chain rule and the derivative of inverse functions we have

$$\left(\Psi\circ\gamma^{-1}\right)'(z) = \left(\Psi'\circ\gamma^{-1}\right)(z)\left(\gamma^{-1}\right)'(z) = \frac{\left(\Psi'\circ\gamma^{-1}\right)(z)}{\left(\gamma'\circ\gamma^{-1}\right)(z)} \tag{3.4}$$

for every $z\in(\gamma(m),\gamma(M))$.

We have by (3.4) that

$$\begin{aligned} \left(\Psi\circ\gamma^{-1}\right)''(z) &= \left(\frac{\left(\Psi'\circ\gamma^{-1}\right)(z)}{\left(\gamma'\circ\gamma^{-1}\right)(z)}\right)' \\ &= \frac{\left(\Psi'\circ\gamma^{-1}\right)'(z)\left(\gamma'\circ\gamma^{-1}\right)(z) - \left(\Psi'\circ\gamma^{-1}\right)(z)\left(\gamma'\circ\gamma^{-1}\right)'(z)}{\left[\left(\gamma'\circ\gamma^{-1}\right)(z)\right]^2} \\ &= \frac{\frac{\left(\Psi''\circ\gamma^{-1}\right)(z)}{\left(\gamma'\circ\gamma^{-1}\right)(z)}\left(\gamma'\circ\gamma^{-1}\right)(z) - \left(\Psi'\circ\gamma^{-1}\right)(z)\frac{\left(\gamma''\circ\gamma^{-1}\right)(z)}{\left(\gamma'\circ\gamma^{-1}\right)(z)}}{\left[\left(\gamma'\circ\gamma^{-1}\right)(z)\right]^2} \\ &= \frac{\left(\Psi''\circ\gamma^{-1}\right)(z)\left(\gamma'\circ\gamma^{-1}\right)(z) - \left(\Psi'\circ\gamma^{-1}\right)(z)\left(\gamma''\circ\gamma^{-1}\right)(z)}{\left[\left(\gamma'\circ\gamma^{-1}\right)(z)\right]^3} \end{aligned}$$

for every $z\in(\gamma(m),\gamma(M))$.

Therefore, for $f=\gamma\circ g$ we get

$$\left(\Psi\circ\gamma^{-1}\right)''((\gamma\circ g)(t)) = \frac{(\Psi''\circ g)(t)(\gamma'\circ g)(t) - (\Psi'\circ g)(t)(\gamma''\circ g)(t)}{\left[(\gamma'\circ g)(t)\right]^3}$$

and

$$\begin{aligned} &\left(\Psi\circ\gamma^{-1}\right)''((\gamma\circ g)(t))(\gamma'\circ g)(t) \\ &= \frac{(\Psi''\circ g)(t)(\gamma'\circ g)(t) - (\Psi'\circ g)(t)(\gamma''\circ g)(t)}{\left[(\gamma'\circ g)(t)\right]^2} = \Delta(\Psi,\gamma,g)(t) \end{aligned}$$

for any $t\in(a,b)$.

By employing the inequality (3.3) we then get the desired result (3.2). □

Corollary 8. *Let $\Psi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on (m, M), $\gamma : [m, M] \to [\gamma(m), \gamma(M)]$ a strictly increasing, continuous and twice differentiable function on (m, M), and $g : [a, b] \to [m, M]$ an absolutely continuous function on $[a, b]$. Assume that $\Psi \circ \gamma^{-1}$ is convex on $[\gamma(m), \gamma(M)]$ and $\Psi \circ g$, $\gamma \circ g \in L[a, b]$. If $(\gamma' \circ g) g' \in L_2[a, b]$ and $\Delta(\Psi, \gamma, g) \in L_2[a, b]$, then*

$$0 \le \frac{1}{b-a}\int_a^b (\Psi \circ g)(t)\, dt - \Psi \circ \gamma^{-1}\left(\frac{1}{b-a}\int_a^b (\gamma \circ g)(t)\, dt\right) \tag{3.5}$$
$$\le \frac{1}{\pi^2}(b-a)\left\|(\gamma' \circ g) g'\right\|_{[a,b],2}\left\|\Delta(\Psi, \gamma, g)\right\|_{[a,b],2}.$$

We also have:

Corollary 9. *Let $\Psi : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on (a, b), $\gamma : [a, b] \to [\gamma(a), \gamma(b)]$ a strictly increasing, continuous and differentiable function on (a, b), and $w : [a, b] \to (0, \infty)$ a continuous function on $[a, b]$. Assume that $\Psi \circ \gamma^{-1}$ is convex on $[\gamma(a), \gamma(b)]$ and Ψ, $\gamma \in L_w[a, b]$. Define, for $t \in (a, b)$,*

$$\Delta(\Psi, \gamma)(t) := \frac{\Psi''(t)\gamma'(t) - \Psi'(t)\gamma''(t)}{[\gamma'(t)]^2}$$

and assume that $\frac{\gamma'}{w^{1/2}} \in L_2[a, b]$ and $\frac{\Delta(\Psi,\gamma)}{w^{1/2}} \in L_2[a, b]$, then

$$0 \le \frac{1}{\int_a^b w(s)\, ds}\int_a^b w(t)\Psi(t)\, dt - \Psi \circ \gamma^{-1}\left(\frac{\int_a^b w(t)\gamma(t)\, dt}{\int_a^b w(s)\, ds}\right) \tag{3.6}$$
$$\le \frac{1}{\pi^2}\left\|\frac{\gamma'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{\Delta(\Psi, \gamma)}{w^{1/2}}\right\|_{[a,b],2}\int_a^b w(s)\, ds.$$

Remark 1. *Let $\Psi : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on (a, b) and $\gamma : [a, b] \to [\gamma(a), \gamma(b)]$ a strictly increasing, continuous and twice differentiable function on (a, b). Assume that $\Psi \circ \gamma^{-1}$ is convex on $[\gamma(a), \gamma(b)]$. If $\gamma' \in L_2[a, b]$ and $\Delta(\Psi, \gamma) \in L_2[a, b]$, then*

$$0 \le \frac{1}{b-a}\int_a^b \Psi(t)\, dt - \Psi \circ \gamma^{-1}\left(\frac{1}{b-a}\int_a^b \gamma(t)\, dt\right) \tag{3.7}$$
$$\le \frac{1}{\pi^2}(b-a)\left\|\gamma'\right\|_{[a,b],2}\left\|\Delta(\Psi, \gamma)\right\|_{[a,b],2}.$$

Also, if we take $w = \gamma'$ in (3.6), then we get

$$0 \le \frac{1}{\gamma(b) - \gamma(a)}\int_a^b \Psi(t)\gamma'(t)\, dt - \Psi \circ \gamma^{-1}\left(\frac{\gamma(b) + \gamma(a)}{2}\right) \tag{3.8}$$
$$\le \frac{1}{\pi^2}[\gamma(b) - \gamma(a)]\left\|\frac{\Delta(\Psi, \gamma)}{(\gamma')^{1/2}}\right\|_{[a,b],2},$$

provided $\frac{\Delta(\Psi,\gamma)}{(\gamma')^{1/2}} \in L_2[a, b]$.

4. Applications for Some Particular Convexities

Let $\gamma : [a, b] \to [\gamma(a), \gamma(b)]$ be a *continuous strictly increasing function* that is *differentiable* on (a, b).

Definition 1. *A function $\Psi : [a,b] \to \mathbb{R}$ will be called composite-γ^{-1} convex (concave) on $[a,b]$ if the composite function $\Psi \circ \gamma^{-1} : [\gamma(a), \gamma(b)] \to \mathbb{R}$ is convex (concave) in the usual sense on $[\gamma(a), \gamma(b)]$.*

In this way, any concept of convexity (log-convexity, harmonic convexity, trigonometric convexity, hyperbolic convexity, h-convexity, quasi-convexity, s-convexity, s-Godunova-Levin convexity etc...) can be extended to the corresponding *composite-γ^{-1} convexity.* The details however will not be presented here.

If $\Psi : [a,b] \to \mathbb{R}$ is composite-γ^{-1} convex on $[a,b]$ then we have the inequality

$$\Psi \circ \gamma^{-1}((1-\lambda)u + \lambda v) \le (1-\lambda)\Psi \circ \gamma^{-1}(u) + \lambda \Psi \circ \gamma^{-1}(v) \tag{4.1}$$

for any $u, v \in [\gamma(a), \gamma(b)]$ and $\lambda \in [0,1]$.

This is equivalent to the condition

$$\Psi \circ \gamma^{-1}((1-\lambda)\gamma(t) + \lambda\gamma(s)) \le (1-\lambda)\Psi(t) + \lambda\Psi(s) \tag{4.2}$$

for any $t, s \in [a,b]$ and $\lambda \in [0,1]$.

If we take $\gamma(t) = \ln t$, $t \in [a,b] \subset (0,\infty)$, then the condition (4.2) becomes

$$\Psi\left(t^{1-\lambda}s^{\lambda}\right) \le (1-\lambda)\Psi(t) + \lambda\Psi(s) \tag{4.3}$$

for any $t, s \in [a,b]$ and $\lambda \in [0,1]$, which is the concept of *GA-convexity* as considered in [1].

If we take $\gamma(t) = -\frac{1}{t}$, $t \in [a,b] \subset (0,\infty)$, then (4.2) becomes

$$\Psi\left(\frac{ts}{(1-\lambda)s + \lambda t}\right) \le (1-\lambda)\Psi(t) + \lambda\Psi(s) \tag{4.4}$$

for any $t, s \in [a,b]$ and $\lambda \in [0,1]$, which is the concept of *HA-convexity* as considered in [1].

If $p > 0$ and we consider $\gamma(t) = t^p$, $t \in [a,b] \subset (0,\infty)$, then the condition (4.2) becomes

$$\Psi\left[\left((1-\lambda)t^p + \lambda s^p\right)^{1/p}\right] \le (1-\lambda)\Psi(t) + \lambda\Psi(s) \tag{4.5}$$

for any $t, s \in [a,b]$ and $\lambda \in [0,1]$, which is the concept of *p-convexity* as considered in [28].

If we take $\gamma(t) = \exp t$, $t \in [a,b]$, then the condition (4.2) becomes

$$\Psi[\ln((1-\lambda)\exp(t) + \exp\gamma(s))] \le (1-\lambda)\Psi(t) + \lambda\Psi(s) \tag{4.6}$$

which is the concept of *LogExp convex function* on $[a,b]$ as considered in [16].

Further, assume that $\Psi : [a,b] \to J$, J an interval of real numbers and $\delta : J \to \mathbb{R}$ a continuous function on J that is *strictly increasing (decreasing)* on J.

Definition 2. *We say that the function $\Psi : [a,b] \to J$ is δ-composite convex (concave) on $[a,b]$, if $\delta \circ \Psi$ is convex (concave) on $[a,b]$.*

In this way, any concept of convexity as mentioned above can be extended to the corresponding *δ-composite convexity.* The details however will not be presented here.

With $\gamma : [a,b] \to [\gamma(a), \gamma(b)]$ a *continuous strictly increasing function* that is *differentiable* on (a,b), $\Psi : [a,b] \to J$, J an interval of real numbers and $\delta : J \to \mathbb{R}$ a continuous function on J that is *strictly increasing (decreasing)* on J, we can also consider the following concept:

Definition 3. *We say that the function $\Psi : [a,b] \to J$ is δ-composite-γ^{-1} convex (concave) on $[a,b]$, if $\delta \circ \Psi \circ \gamma^{-1}$ is convex (concave) on $[\gamma(a), \gamma(b)]$.*

This definition is equivalent to the condition

$$\delta \circ \Psi \circ \gamma^{-1}\left((1-\lambda)\gamma(t) + \lambda\gamma(s)\right) \le (1-\lambda)(\delta \circ \Psi)(t) + \lambda(\delta \circ \Psi)(s) \tag{4.7}$$

for any $t, s \in [a,b]$ and $\lambda \in [0,1]$.

If $\delta : J \to \mathbb{R}$ is *strictly increasing (decreasing)* on J, then the condition (4.7) is equivalent to:

$$\Psi \circ \gamma^{-1}\left((1-\lambda)\gamma(t) + \lambda\gamma(s)\right) \le (\ge)\, \delta^{-1}\left[(1-\lambda)(\delta \circ \Psi)(t) + \lambda(\delta \circ \Psi)(s)\right] \tag{4.8}$$

for any $t, s \in [a,b]$ and $\lambda \in [0,1]$.

If $\delta(t) = \ln t$, $t > 0$ and $\Psi : [a,b] \to (0,\infty)$, then the fact that Ψ is δ-composite convex on $[a,b]$ is equivalent to the fact that Ψ is *log-convex* or *multiplicatively convex* or AG-convex, namely, for all $x, y \in I$ and $t \in [0,1]$ one has the inequality:

$$\Psi(tx + (1-t)y) \le [\Psi(x)]^t [\Psi(y)]^{1-t}. \tag{4.9}$$

A function $\Psi : I \to \mathbb{R}\setminus\{0\}$ is called *AH-convex (concave)* on the interval I if the following inequality holds [1]

$$\Psi((1-\lambda)x + \lambda y) \le (\ge) \frac{1}{(1-\lambda)\frac{1}{\Psi(x)} + \lambda\frac{1}{\Psi(y)}} = \frac{\Psi(x)\Psi(y)}{(1-\lambda)\Psi(y) + \lambda\Psi(x)} \tag{4.10}$$

for any $x, y \in I$ and $\lambda \in [0,1]$.

An important case that provides many examples is that one in which the function is assumed to be positive for any $x \in I$. In that situation the inequality (4.10) is equivalent to

$$(1-\lambda)\frac{1}{\Psi(x)} + \lambda\frac{1}{\Psi(y)} \le (\ge) \frac{1}{\Psi((1-\lambda)x + \lambda y)}$$

for any $x, y \in I$ and $\lambda \in [0,1]$.

Taking into account this fact, we can conclude that the function $\Psi : I \to (0,\infty)$ is AH-convex (concave) on I if and only if Ψ is δ-composite concave (convex) on I with $\delta : (0,\infty) \to (0,\infty)$, $\delta(t) = \frac{1}{t}$.

Following [1], we can introduce the concept of *GH-convex (concave)* function $\Psi : I \subset (0,\infty) \to \mathbb{R}$ on an interval of positive numbers I as satisfying the condition

$$\Psi\left(x^{1-\lambda}y^{\lambda}\right) \le (\ge) \frac{1}{(1-\lambda)\frac{1}{\Psi(x)} + \lambda\frac{1}{\Psi(y)}} = \frac{\Psi(x)\Psi(y)}{(1-\lambda)\Psi(y) + \lambda\Psi(x)}. \tag{4.11}$$

Since

$$\Psi\left(x^{1-\lambda}y^{\lambda}\right) = \Psi \circ \exp\left[(1-\lambda)\ln x + \lambda \ln y\right]$$

and

$$\frac{\Psi(x)\Psi(y)}{(1-\lambda)\Psi(y) + \lambda\Psi(x)} = \frac{\Psi \circ \exp(\ln x)\, \Psi \circ \exp(\ln y)}{(1-\lambda)\Psi \circ \exp(y) + \lambda\Psi \circ \exp(x)}$$

then $\Psi : I \subset (0,\infty) \to \mathbb{R}$ is GH-convex (concave) on I if and only if $\Psi \circ \exp$ is *AH-convex (concave)* on $\ln I := \{x \mid x = \ln t,\ t \in I\}$. This is equivalent to the fact that Ψ is δ-composite-γ^{-1} concave (convex) on I with $\delta : (0,\infty) \to (0,\infty)$, $\delta(t) = \frac{1}{t}$ and $\gamma(t) = \ln t$, $t \in I$.

Following [1], we say that the function $\Psi : I \subset \mathbb{R} \setminus \{0\} \to (0, \infty)$ is *HH-convex* if

$$\Psi\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{\Psi(x)\Psi(y)}{(1-t)\Psi(y) + t\Psi(x)} \tag{4.12}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (4.12) is reversed, then Ψ is said to be *HH-concave*.

We observe that the inequality (4.12) is equivalent to

$$(1-t)\frac{1}{\Psi(x)} + t\frac{1}{\Psi(y)} \leq \frac{1}{\Psi\left(\frac{xy}{tx+(1-t)y}\right)} \tag{4.13}$$

for all $x, y \in I$ and $t \in [0,1]$.

This is equivalent to the fact that Ψ is δ-composite-γ^{-1} concave on $[a,b]$ with $\delta : (0,\infty) \to (0,\infty)$, $\delta(t) = \frac{1}{t}$ and $\gamma(t) = -\frac{1}{t}$, $t \in [a,b]$.

The function $\Psi : I \subset (0,\infty) \to (0,\infty)$ is called *GG-convex* on the interval I of real umbers $\mathbb{R}$ if [1]

$$\Psi\left(x^{1-\lambda}y^{\lambda}\right) \leq [\Psi(x)]^{1-\lambda}[\Psi(y)]^{\lambda} \tag{4.14}$$

for any $x, y \in I$ and $\lambda \in [0,1]$. If the inequality is reversed in (4.14) then the function is called *GG-concave*.

This concept was introduced in 1928 by P. Montel [24], however, the roots of the research in this area can be traced long before him [25]. It is easy to see that [25], the function $\Psi : [a,b] \subset (0,\infty) \to (0,\infty)$ is *GG-convex* if and only if the the function $\gamma : [\ln a, \ln b] \to \mathbb{R}$, $\gamma = \ln \circ \Psi \circ \exp$ is convex on $[\ln a, \ln b]$. This is equivalent to the fact that Ψ is δ-composite-γ^{-1} convex on $[a,b]$ with $\delta : (0,\infty) \to \mathbb{R}$, $\delta(t) = \ln t$ and $\gamma(t) = \ln t$, $t \in [a,b]$.

Following [1] we say that the function $\Psi : I \subset \mathbb{R} \setminus \{0\} \to (0,\infty)$ is *HG-convex* if

$$\Psi\left(\frac{xy}{tx + (1-t)y}\right) \leq [\Psi(x)]^{1-t}[\Psi(y)]^{t} \tag{4.15}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (4.2) is reversed, then Ψ is said to be *HG-concave*.

Let $\Psi : [a,b] \subset (0,\infty) \to (0,\infty)$ and define the associated functions $G_\Psi : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$ defined by $G_\Psi(t) = \ln \Psi\left(\frac{1}{t}\right)$. Then Ψ is *HG-convex* on $[a,b]$ iff G_Ψ is convex on $\left[\frac{1}{b}, \frac{1}{a}\right]$. This is equivalent to the fact that Ψ is δ-composite-γ^{-1} convex on $[a,b]$ with $\delta : (0,\infty) \to \mathbb{R}$, $\delta(t) = \ln t$ and $\gamma(t) = -\frac{1}{t}$, $t \in [a,b]$.

Following [27], we say that the function $\Psi : [a,b] \to (0,\infty)$ is r-convex, for $r \neq 0$, if

$$\Psi((1-\lambda)x + \lambda y) \leq [(1-\lambda)\Psi^r(y) + \lambda\Psi^r(x)]^{1/r} \tag{4.16}$$

for any $x, y \in [a,b]$ and $\lambda \in [0,1]$.

If $r > 0$, then the condition (4.16) is equivalent to

$$\Psi^r((1-\lambda)x + \lambda y) \leq (1-\lambda)\Psi^r(y) + \lambda\Psi^r(x)$$

namely Ψ is δ-composite convex on $[a,b]$ where $\delta(t) = t^r$, $t \geq 0$.

If $r < 0$, then the condition (4.16) is equivalent to

$$\Psi^r((1-\lambda)x + \lambda y) \geq (1-\lambda)\Psi^r(y) + \lambda\Psi^r(x)$$

namely Ψ is δ-composite concave on $[a,b]$ where $\delta(t) = t^r$, $t > 0$.

For some results related to these concepts of convexity, see [9]-[15].

We assume in the following that $w:[a,b]\to(0,\infty)$ is a continuous function on $[a,b]$ and $g:[a,b]\to[m,M]$ is absolutely continuous on $[a,b]$.

If Ψ is *log convex* on $[m,M]$, then Ψ is δ-composite-γ^{-1} convex on $[a,b]$ with $\delta:(0,\infty)\to\mathbb{R}$, $\delta(t)=\ln t$ and $\gamma(t)=\ell(t)=t$, $t\in[a,b]$. If we use the inequality (2.10) and assume that Ψ is twice differentiable on (m,M), then we have

$$0\le\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\ln(\Psi\circ g)(t)\,dt-\ln\left[\Psi\left(\frac{\int_a^b w(t)\,g(t)\,dt}{\int_a^b w(s)\,ds}\right)\right] \tag{4.17}$$
$$\le\frac{1}{\pi^2}\left\|\frac{g'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{\Delta(\ln\Psi,g)}{w^{1/2}}\right\|_{[a,b],2}\int_a^b w(s)\,ds,$$

where

$$\Delta(\ln\Psi,g)(t)=\frac{(\Psi''\circ g)(t)(\Psi\circ g)(t)-((\Psi'\circ g)(t))^2}{((\Psi\circ g)(t))^2},\ t\in[a,b]$$

and provided that $\frac{g'}{w^{1/2}}\in L_2[a,b]$ and $\frac{\Delta(\ln\Psi,g)}{w^{1/2}}\in L_2[a,b]$.

If Ψ is *GA-convex* on $[a,b]\subset(0,\infty)$, then Ψ is δ-composite-γ^{-1} convex on $[a,b]$ with $\gamma:(0,\infty)\to\mathbb{R}$, $\gamma(t)=\ln t$ and $\delta(t)=\ell(t)=t$, $t\in[a,b]$. If we use the inequality (2.10) and assume that Ψ is twice differentiable on (m,M), then we have

$$0\le\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Psi\circ g)(t)\,dt-\Psi\left[\exp\left(\frac{\int_a^b w(t)\ln g(t)\,dt}{\int_a^b w(s)\,ds}\right)\right] \tag{4.18}$$
$$\le\frac{1}{\pi^2}\left\|\frac{g'}{gw^{1/2}}\right\|_{[a,b],2}\left\|\frac{\Delta(\Psi,\ln,g)}{w^{1/2}}\right\|_{[a,b],2}\int_a^b w(s)\,ds,$$

where

$$\Delta(\Psi,\ln,g)(t)=(\Psi''\circ g)(t)\,g(t)+(\Psi'\circ g)(t)$$

and provided that $\frac{g'}{gw^{1/2}},\ \frac{\Delta(\Psi,\ln,g)}{w^{1/2}}\in L_2[a,b]$.

The function $\Psi:[a,b]\to(0,\infty)$ is *AH-convex* on $[a,b]$ if and only if Ψ is δ-composite-γ^{-1} concave on $[a,b]$ with $\delta:(0,\infty)\to(0,\infty)$, $\delta(t)=\frac{1}{t}$ and and $\gamma(t)=\ell(t)=t$, $t\in[a,b]$. If we use the inequality (2.10) for the convex function $-\Psi^{-1}$ and assume that Ψ is twice differentiable on (m,M), then we have

$$0\le\left[\Psi\left(\frac{\int_a^b w(t)\,g(t)\,dt}{\int_a^b w(s)\,ds}\right)\right]^{-1}-\frac{1}{\int_a^b w(s)\,ds}\int_a^b\frac{w(t)}{(\Psi\circ g)(t)}dt \tag{4.19}$$
$$\le\frac{1}{\pi^2}\left\|\frac{g'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{\Delta(-\Psi^{-1},g)}{w^{1/2}}\right\|_{[a,b],2}\int_a^b w(s)\,ds,$$

where

$$\Delta(-\Psi^{-1},g)(t):=\frac{(\Psi''\circ g)(t)(\Psi\circ g)(t)-2((\Psi'\circ g)(t))^2}{((\Psi\circ g)(t))^3}$$

and provided that $\frac{g'}{w^{1/2}},\ \frac{\Delta(-\Psi^{-1},g)}{w^{1/2}}\in L_2[a,b]$.

If the function Ψ is *HA-convex* $[a,b]$, then Ψ is δ-composite-γ^{-1} convex on $[a,b]$ with $\gamma:(0,\infty)\to\mathbb{R}$, $\gamma(t)=-t^{-1}$ and $\delta(t)=\ell(t)=t$, $t\in[a,b]$. If we use the inequality (2.16) and assume that Ψ is twice differentiable on (m, M), then we have

$$0 \leq \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)(\Psi \circ g)(t)\,dt - \Psi\left(\frac{\int_a^b w(s)\,ds}{\int_a^b \frac{w(t)}{g(t)}dt}\right) \leq \frac{1}{\pi^2}\left\|\frac{g'}{g^2 w^{1/2}}\right\|_{[a,b],2}\left\|\frac{\Delta\left(\Psi, -\ell^{-1}, g\right)}{w^{1/2}}\right\|_{[a,b],2}\int_a^b w(s)\,ds, \tag{4.20}$$

where

$$\Delta\left(\Psi, -\ell^{-1}, g\right)(t) := (\Psi'' \circ g)(t)\, g^2(t) + 2g(t)(\Psi' \circ g)(t)$$

and provided that $\frac{g'}{g^2 w^{1/2}}, \frac{\Delta\left(\Psi, -\ell^{-1}, g\right)}{w^{1/2}} \in L_2[a,b]$.

Similar results may be stated for the other concepts of convexity as presented above, however the details are omitted.

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